**A4** Lösen Sie

$$x_1 x_2 \frac{\partial u}{\partial x_1}(x_1, x_2) - x_2^2 \frac{\partial u}{\partial x_2}(x_1, x_2) = x_1$$

$$u(1, x_2) = \frac{3}{2x_2}$$

mit dem Charakteristikenverfahren.

Lösung: Charakteristisches System (mit $w(s) = u(k_1, k_2)$)

$$k_1'(s) = k_1(s)\, k_2(s) \qquad (1)$$

$$k_2'(s) = -k_2^2(s) \qquad (2)$$

$$w'(s) = k_1(s). \qquad (3)$$

Aus (2): $-\int \frac{k_2'}{k_2^2} ds = \int 1 \, ds$ Subst: $k_2(s) = t$, $k_2'(s) = \frac{dt}{ds}$

$$\Rightarrow -\int \frac{dt}{t^2} = s + C_2$$

$$\Rightarrow \frac{1}{t} = s + C_2 \Rightarrow k_2(s) = \frac{1}{s + C_2}$$

In (1): $k_1'(s) = \frac{k_1(s)}{s + C_2}$

$$\Rightarrow \int \frac{k_1'(s)}{k_1(s)} ds = \int \frac{1}{s + C_2}$$

$$\Rightarrow \ln|k_1(s)| = \ln|s + C_2| + C_3$$

$$\Rightarrow k_1(s) = C_1(s + C_2)$$

In (3) $w'(s) = C_1(s + C_2)$

$$\Rightarrow w(s) = \frac{C_1}{2}(s + C_2)^2 + C_3$$

Anfangskurve: $\tilde{\Gamma} = \{(1, t, \frac{3}{2t}), t \in \mathbb{R}\}$

Verknüpfung char. System $\leftrightarrow$ Anfangsbed.:

$$(k_1(0), k_2(0), w(0)) = (1, t, \frac{3}{2t})$$

$$\Rightarrow \left( c_1 c_2, \frac{1}{c_2}, \frac{c_1}{2} c_2^2 + c_3 \right) = \left(1, t, \frac{3}{2t}\right)$$

$$\Rightarrow c_1 c_2 = 1, \quad \frac{1}{c_2} = t, \quad \frac{c_1}{2} c_2^2 + c_3 = \frac{3}{2t}$$

$$\Rightarrow c_1 = t \quad c_2 = \frac{1}{t}, \quad c_3 = \frac{3}{2t} - \frac{1}{2t} = \frac{1}{t}$$

$$\Rightarrow \begin{aligned} k_1(s) &= t\left(s + \frac{1}{t}\right) \quad \left( = t \cdot \frac{st+1}{t} \right) \\ k_2(s) &= \frac{1}{s + \frac{1}{t}} \quad \left( = \frac{t}{1+st} \right) \\ w(s) &= \frac{t}{2}\left(s + \frac{1}{t}\right)^2 + \frac{1}{t} \end{aligned}$$

s,t eleminieren: $x_1 = t\left(s + \frac{1}{t}\right) = (st+1) \quad (1)$

$$x_2 = \frac{t}{1+st}$$

$$\Rightarrow x_1 x_2 = t$$

$$\text{in } (1): \quad x_1 = s\, x_1 x_2 + 1$$

$$\Rightarrow s = \frac{x_1 - 1}{x_1 x_2}$$

Also $u(x_1, x_2) = w(s) = \frac{x_1 x_2}{2}\left(\frac{x_1 - 1}{x_1 x_2} + \frac{1}{x_1 x_2}\right)^2 + \frac{1}{x_1 x_2}$

$$= \frac{x_1 x_2}{2}\left(\frac{(x_1-1)^2}{x_1^2 x_2^2} + \frac{2(x_1-1)}{x_1^2 x_2^2} + \frac{1}{x_1^2 x_2^2}\right) + \frac{1}{x_1 x_2}$$

$$= \frac{(x_1-1)^2}{2 x_1 x_2} + \frac{x_1 - 1}{x_1 x_2} + \frac{1}{2 x_1 x_2} + \frac{1}{x_1 x_2}$$

$$= \frac{(x_1^2 + 2)}{2 x_1 x_2}$$