Exercise 3

For $t \in [0,2\pi]$ we consider the (linear!) operator

$$s_n(\cdot,t) : C[0,2\pi] \to \mathbb{R}$$
$$x \mapsto s_n(x,t) = \frac{a_0(x)}{2} + \sum_{k=1}^{n}\left(a_k(x)\cos(kt) + b_k(x)\sin(kt)\right)$$
$$= \frac{1}{\pi}\int_{-t}^{2\pi-t} x(s+t)\,\frac{\sin((n+\frac{1}{2})s)}{2\sin\frac{s}{2}}\,ds$$

Since
$$|s_n(x,t)| \le \frac{|a_0(x)|}{2} + \sum_{k=1}^{n}\left(|a_k(x)| + |b_k(x)|\right)$$
$$\le (2n+1)2\,\|x\|_{[0,2\pi]},$$

the operators $s_n(\cdot,t)$ are bounded.

We recall that $C[0,2\pi]$ is a Banach space w.r.t. $\|\cdot\|_{[0,2\pi]}$.

Suppose that the Fourier series converges for all $x \in C[0,2\pi]$ and all $t \in [0,2\pi]$.

Then by the Principle of Uniform Boundedness

$$\sup_{n\in\mathbb{N}} \|s_n(\cdot,t)\| < \infty \quad \text{for all } t \in [0,2\pi].$$

Let now $t=0$. Then by exercise 4 of problem set 1

$$\|s_n(\cdot,0)\| = \frac{1}{\pi}\int_0^{2\pi}\left|\frac{\sin((n+\frac{1}{2})s)}{2\sin\frac{s}{2}}\right|ds$$
$$\ge \frac{1}{\pi}\int_0^{\pi}\left|\frac{\sin((n+\frac{1}{2})s)}{s}\right|ds \qquad \boxed{\tau = (n+\tfrac{1}{2})s}$$
$$= \frac{1}{\pi}\int_0^{(n+\frac{1}{2})\pi}\left|\frac{\sin\tau}{\tau}\right|d\tau$$
$$\ge \frac{1}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi}|\sin\tau|\,d\tau$$
$$= \frac{2}{\pi^2}\sum_{k=1}^{n}\frac{1}{k}$$

Thus, $\|s_n(\cdot,0)\| \to \infty$, $n \to \infty$. This contradicts the uniform boundedness.

We conclude that there exists a continuous function x such that $s_n(x,0)$ does not converge.