Exercise 3

Let $\{x_m\} \subset X$ be bounded. We will use a diagonalization procedure to show that $(Ax_m)$ has a convergent subsequence in $Y$.
Since $A_1$ is compact, $\{x_m\}$ has a subsequence $\{x_{1,m}\}$ such that $\{A_1 x_{1,m}\}$ is convergent. Similarly, $\{x_{1,m}\}$ has a subsequence $\{x_{2,m}\}$ such that $\{A_2 x_{2,m}\}$ is convergent.
Continuing in this manner, we see that the diagonal sequence $\{x_{m,m}\}$ is a subsequence of $\{x_m\}$ such that, for every fixed positive integer $n$, the sequence $\{A_n x_{m,m}\}$ is convergent.
Since $\{x_m\}$ is bounded, say $\|x_m\| \leq C$ for all $m$, $\|x_{m,m}\| \leq C$ for all $m$.
Now we use the fact that $\|A_n - A\| \to 0$ as $n \to \infty$ to conclude that for each $\varepsilon > 0$ there exists an integer $n_0 > 0$ such that

$$\|A_{n_0} - A\| < \frac{\varepsilon}{3C}.$$

And since $\{A_{n_0} x_{m,m}\}$ is convergent, there exists $N > 0$ such that

$$\|A_{n_0} x_{j,j} - A_{n_0} x_{k,k}\| < \frac{\varepsilon}{3} \qquad \text{for } j,k > N.$$

Hence, for $j,k > N$, we have that

$$\|Ax_{j,j} - Ax_{k,k}\| \leq \|Ax_{j,j} - A_{n_0}x_{j,j}\| + \underbrace{\|A_{n_0}x_{j,j} - A_{n_0}x_{k,k}\|}_{< \frac{\varepsilon}{3}} + \|A_{n_0}x_{k,k} - Ax_{k,k}\|$$

$$< \underbrace{\|A - A_{n_0}\|}_{< \frac{\varepsilon}{3C}} \underbrace{\left(\|x_{j,j}\| + \|x_{k,k}\|\right)}_{\leq 2C} + \frac{\varepsilon}{3} < \varepsilon.$$

Thus, $\{Ax_{m,m}\}$ is a Cauchy sequence and therefore convergent in the Banach space $Y$.