Exercise 3

(a) Let $x, y \in L^2[0,1]$:

$$\langle A_f x, y\rangle_{L^2} = \int_0^1 \overline{f(s)x(s)}\, y(s)\,ds = \int_0^1 \overline{x(s)}\;\overline{f(s)}\,y(s)\,ds = \langle x, \bar{f} y\rangle_{L^2}$$

$$\Rightarrow A_f^* = A_{\bar{f}}$$

(b) We suppose that A is compact. Since $f \in C[0,1]$ and $f \neq 0$, there exists $t \in]0,1[$ and $\eta > 0$ such that $|f|_{[t-\eta, t+\eta]}| \geq c$ for a $c > 0$.

Now let $\{x_n\} \subseteq L^2[t-\eta, t+\eta]$ be a bounded sequence. Then we can extend x_n by 0 to a function $\tilde{x}_n \in L^2[0,1]$.

As A is compact, there exists a subsequence $\{\tilde{x}_{n_k}\}$ s.t.

$$\{A\tilde{x}_{n_k}\} \to y \in L^2[0,1], \; k \to \infty.$$

Thus,

$$\left\|x_{n_k} - \frac{y}{f}\right\|_{L^2[t-\eta, t+\eta]} \overset{\frac{|f|}{c} \geq 1}{\leq} \left\|\frac{f x_{n_k} - y}{c}\right\|_{L^2[t-\eta, t+\eta]} \leq \frac{1}{c}\|A\tilde{x}_{n_k} - y\|_{L^2[0,1]} \to 0$$

as $k \to \infty$.

Since $\{x_n\}$ was chosen arbitrary, this implies that $L^2[t-\eta, t+\eta]$ is compact. That's a contradiction.

(c) It is sufficient to consider $\lambda = 0$. Using the result for $g = f - \lambda$ we obtain the general result.

"$\Leftarrow$" Let $0 \notin R(f)$. Then, there exists $c > 0$ s.t. $|f| \geq c$.

Moreover, $A_{\frac{1}{f}}$ is the inverse of A_f as

$$A_{\frac{1}{f}} A_f x = \frac{1}{f} f x = x = f \frac{1}{f} x = A_f A_{\frac{1}{f}} x.$$

and A_f^{-1} is bounded observing

$$\|A_f^{-1} x\|_{L^2}^2 = \|A_{\frac{1}{f}} x\|_{L^2} = \|\frac{x}{f}\|_{L^2}^2 \leq \frac{1}{c}\|x\|_{L^2} !$$

„$\Rightarrow$" Let $0 \in \Omega(f)$. Then, there exists $t \in [0,1]$ with $f(t) = 0$ and for all $\varepsilon > 0$ we find a $\delta = \delta(\varepsilon) > 0$ s.t. $|f|_{[t-\delta, t+\delta]}| < \varepsilon$.

W.l.o.g we assume $t \in]0,1[$ and $[t-\delta, t+\delta] \subseteq [0,1]$.

Thus, $\|A_f \chi_{[t-\delta,t+\delta]}\|_{L^2}^2 = \int_0^1 |f(s)|^2 \chi_{[t-\delta,t+\delta]}^2(s)\, ds \leq 2\varepsilon^2\delta$.

We suppose that A_f $\underbrace{\text{is}}_{\text{continuously}}$ invertible. Then,

$$2\delta = \|\chi_{[t-\delta,t+\delta]}\|_{L^2}^2 = \|A_f^{-1} A_f \chi_{[t-\delta,t+\delta]}\|_{L^2}^2$$
$$\leq \|A_f^{-1}\| \; \|A_f \chi_{[t-\delta,t+\delta]}\|_{L^2}^2$$
$$\leq \|A_f^{-1}\|_{L^2}^2 \; 2\varepsilon^2 \delta .$$

This implies $\frac{1}{\varepsilon^2} \leq \|A_f^{-1}\|$ for all $\varepsilon > 0$, i.e. that A_f^{-1} is unbounded. That contradicts the continuity of A_f^{-1}.

(d) Let f be injective and $\lambda \in \mathbb{K}$. Then, $f - \lambda \neq 0$ a.e.
Moreover, let $x \in L^2[0,1]$ be a solution of $A_f x = \lambda x$. In other words

$$A_f x - \lambda x = \underbrace{(f - \lambda)}_{\neq 0 \text{ a.e.}} x = 0$$

Thus, $x = 0$ a.e., i.e. $x = 0$ in $L^2[0,1]$.