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Inverse Problems
Winter Semester 2011/2012

Problem Set 10 – December 23, 2011

Exercise 1:
Let $K\colon X \to Y$ be an injective compact operator with singular system (σ_n, u_n, v_n).

(a) Show that for $\alpha > 0$ the Tikhonov approximation x_α, i.e. the solution of (13.5), can be written in the form

$$x_\alpha = \sum_{n=1}^{\infty} \frac{q(\alpha, \sigma_n)}{\sigma_n} \langle y^\delta, u_n \rangle v_n$$

with a suitable function $q\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$.

(b) Use exercise 1 of problem set 8 to show that the method of Tikhonov is a regularization for $K^\dagger$ with

$$\|R_\alpha\| \le \frac{1}{2\sqrt{\alpha}}.$$

Exercise 2:
Let $K\colon X \to Y$ be a compact operator, $y^\delta \in Y$ and x_α^δ the Tikhonov approximation. By $\phi(\alpha) = \|y^\delta - Kx_\alpha^\delta\|^2$ we denote the discrepancy function. If $\|y^\delta\| \ge \delta$, then the regularization parameter $\alpha = \alpha(\delta)$ given by the discrepancy principle is the unique solution of the equation $\phi(\alpha) = \delta^2$. Now Newton's method can be applied to the equation

$$\psi(\beta) = \phi(1/\beta) = \delta^2.$$

(a) Determine the iteration sequence $\alpha_n = 1/\beta_n$, $n \in \mathbb{N}$, of this approach.
Hint: Show first that ψ can be written in the form

$$\psi(\beta) = \sum_{n=1}^{\infty} \frac{\langle y^\delta, u_n \rangle^2}{(\beta\sigma_n^2 + 1)^2},$$

where (σ_n, u_n, v_n) is a singular system of K. Use this representation to calculate the Newton iteration.

(b) Show: If α_0 is chosen such that $\alpha_0 \ge \alpha(\delta) > 0$, then the sequence $\{\alpha_n\}$ is monotonically decreasing and converges quadratically to $\alpha(\delta)$.

Exercise 3:
Let X and Y be Hilbert spaces and $A\colon X \to Y$ a bounded linear operator. Show that:

(a) If $x_n \rightharpoonup x$, then the weak limit x is uniquely determined.

(b) If $x_n \rightharpoonup x$, then the sequence $\{x_n\}_n$ is bounded.

(c) If $x_n \rightharpoonup x$, then $\|x_n\| \to \|x\|$ implies $x_n \to x$.

(d) If $x_n \rightharpoonup x$, then $\|x\| \le \liminf\limits_{n\to\infty} \|x_n\|$.

(e) If $x_n \rightharpoonup x$, then $Ax_n \rightharpoonup Ax$.

(f) A is compact if and only if for any weakly convergent sequence $x_n \rightharpoonup x$ follows that $Ax_n \to Ax$.

Please hand in your solutions in the problem class on **Friday, January 13, 2012**.

We wish you a merry Christmas and a happy New Year 2012 !