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Inverse Problems
Winter Semester 2011/2012

Problem Set 12 – January 20, 2012

Exercise 1:
Let $\{x_k\}_{k\in\mathbb{N}}$ be the Landweber sequence for solving $Kx = y^\delta$ with initial guess $x_0 \neq 0$ and $\{\tilde{x}_k\}_{k\in\mathbb{N}}$ the Landweber sequence for solving $Kx = y^\delta - Kx_0$ with initial guess $\tilde{x}_0 = 0$, i.e.

$$x_k = x_{k-1} + K^*(y^\delta - Kx_{k-1}),\ x_0 \neq 0 \quad \text{and} \quad \tilde{x}_k = \tilde{x}_{k-1} + K^*(y^\delta - Kx_0 - K\tilde{x}_{k-1}),\ \tilde{x}_0 = 0.$$

Show that $x_k = \tilde{x}_k + x_0$ for $k \in \mathbb{N}_0$.

Exercise 2:
Let $K\colon X \to Y$ be a compact operator with $\|K\| < 1$ and singular system (σ_n, u_n, v_n). We consider the Landweber iteration for solving $Kx = y^\delta$ with initial guess $x_0 = 0$ given by

$$x_k = x_{k-1} + K^*(y^\delta - Kx_{k-1}), \quad k \in \mathbb{N}.$$

(a) Show that there exists a so-called filter polynomial q_k of degree k such that

(i) $x_k = \displaystyle\sum_{n=1}^{\infty} \frac{q_k(\sigma_n^2)}{\sigma_n} \langle y^\delta, u_n\rangle v_n,$

(ii) $\lim_{k\to\infty} q_k(\lambda) = 1$ for all $0 < \lambda < \|K\|^2$.

(b) Let $y^\delta \in \mathcal{D}(K^\dagger)$. Show that there exists a so-called residual polynomial r_k of degree k with the following properties:

(i) $K^\dagger y^\delta - x_k = \displaystyle\sum_{n=1}^{\infty} \frac{r_k(\sigma_n^2)}{\sigma_n} \langle y^\delta, u_n\rangle v_n,$

(ii) $r_k(0) = 1,$

(iii) $|r_k| \leq 1$ on $[0,1]$ for every $k \in \mathbb{N}$,

(iv) $\lim_{k\to\infty} r_k(\lambda) = 0$ for all $0 < \lambda < \|K\|^2$.

Exercise 3:
Let $\{r_k\}_{k\in\mathbb{N}_0}$ be a family of residual polynomials, i.e. polynomials of degree k satisfying properties (ii) to (iv) of part (b) of the last exercise, with $r_0 = 1$ and satisfying the recursive three-term formula

$$r_k(\lambda) = r_{k-1}(\lambda) + \mu_k(r_{k-1}(\lambda) - r_{k-2}(\lambda)) - \omega_k \lambda r_{k-1}(\lambda) \quad \text{for } k \geq 2.$$

We consider the accelerated Landweber method for solving $Kx = y^\delta$ given by

$$x_k = \sum_{n=1}^{\infty} \frac{q_k(\sigma_n^2)}{\sigma_n} \langle y^\delta, u_n\rangle v_n, \quad \text{for } k \geq 1$$

with initial guess $x_0 = 0$. Herein $q_k = 1 - r_k,\ k \in \mathbb{N}_0$.
Show that the accelerated Landweber sequence satisfies the three-term recursive relation

$$x_k = x_{k-1} + \mu_k(x_{k-1} - x_{k-2}) + \omega_k K^*(y^\delta - Kx_{k-1}) \quad \text{for } k \geq 2.$$

Please hand in your solutions in the problem class on **Friday, January 27, 2012**.