Karlsruhe Institute of Technology

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Inverse Problems
Winter Semester 2011/2012

Problem Set 13 – January 27, 2012

Exercise 1:

(a) Let $f \in L^1(\mathbb{R}^m)$ and $g \in L^1(\mathbb{R}^n)$. Show that:

(i) $\Big(F\big(f(x+a)\big)\Big)(\xi) = \mathrm{e}^{i\xi\cdot a}\Big(F\big(f(x)\big)\Big)(\xi)$ for all $a \in \mathbb{R}^m$,

(ii) $\Big(F\big(f(\lambda x)\big)\Big)(\xi) = \dfrac{1}{|\lambda|^m}\Big(F\big(f(x)\big)\Big)\Big(\dfrac{\xi}{\lambda}\Big)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$,

(iii) $\Big(F\big(f(x)g(y)\big)\Big)(\xi,\eta) = \Big(F\big(f(x)\big)\Big)(\xi)\Big(F\big(g(y)\big)\Big)(\eta)$.

(b) Let $f, g \in \mathcal{S}(\mathbb{R}^m)$. Show that:

(i) $(FD^\alpha f)(\xi) = i^{|\alpha|}\xi^\alpha (Ff)(\xi)$ with $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_m^{\alpha_m}$,

(ii) $\displaystyle\int_{\mathbb{R}^m} \widehat{f}(\xi)g(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^m} f(x)\widehat{g}(x)\,\mathrm{d}x$,

(iii) $\displaystyle\int_{\mathbb{R}^m} f(x)\overline{g(x)}\,\mathrm{d}x \int_{\mathbb{R}^m} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,\mathrm{d}\xi$.

Exercise 2:
Let X, Y be Banach spaces and U a dense subspace of X. Prove that: If $A\colon U \to Y$ is a linear operator with

$$\|Ax\|_Y \le C\|x\|_X \quad \text{for all } x \in U,$$

then there exists a unique bounded linear operator $\overline{A}\colon X \to Y$ such that

$$\overline{A}x = Ax \quad \text{for all } x \in U \quad \text{and} \quad \|\overline{A}\| = \sup_{x\in U\setminus\{0\}} \frac{\|Ax\|_Y}{\|x\|_X}.$$

This operator is called the continuous extension of A.

Exercise 3:
Let $f \in \mathcal{S}(\mathbb{R}^m)$. Prove the Poisson summation formula

$$\sum_{j\in\mathbb{Z}^m} \widehat{f}\big(\xi - \frac{2\pi}{h}j\big) = \frac{h^m}{\sqrt{2\pi}^m}\sum_{j\in\mathbb{Z}^m} f(jh)\mathrm{e}^{-ih\xi\cdot j} \quad \text{for } \xi \in \mathbb{R}^m,$$

where both series converge uniformly.
Hint: Consider the function given by

$$\varphi(\xi) = \sum_{j\in\mathbb{Z}^m} \widehat{f}\big(\xi - \frac{2\pi}{h}j\big)$$

and calculate its Fourier series.

Please hand in your solutions in the problem class on **Friday, February 3, 2012.**