Karlsruhe Institute of Technology

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Inverse Problems
Winter Semester 2011/2012

Problem Set 14 – February 3, 2012

Exercise 1:
Denote by $\mathbf{R}$ the Radon transform.

(a) Prove that the L^2 adjoint of $\mathbf{R}$ is given by

$$(\mathbf{R}^*g)(x) = \int_{S^1} g(\theta, x\cdot\theta)\,\mathrm{d}\theta \quad \text{for } g \in \mathcal{S}(S^1\times\mathbb{R}).$$

(b) Let $f \in \mathcal{S}(\mathbb{R}^2)$ and $\phi\colon \mathbb{R}^2\setminus\{0\}\to\mathbb{R}$, $\phi(x) = |x|^{-1}$. Show that

$$\mathbf{R}^*\mathbf{R}f = 2\phi * f.$$

Exercise 2:
Let $f \in \mathcal{S}(\mathbb{R}^2)$ with compact support. Suppose $\mathbf{R}f$ is zero on every line not intersecting a fixed compact convex set $K \subset \mathbb{R}^2$. Prove that f is zero outside K.
Hint: First give reasons that it is sufficient to show that f vanishes on the line parametrized by $\gamma_0(t) = (s,t)$, $t\in\mathbb{R}$, for $s > 0$ such that this line lies to the right of K. In a second step show that

$$\int_{-\infty}^{\infty} t f(s,t)\,\mathrm{d}t = 0.$$

In order to see this, differentiate

$$\mathbf{R}f(\theta(\vartheta), r) = \int_{-\infty}^{\infty} f(r\cos\vartheta + t\sin\vartheta, -r\sin\vartheta + t\cos\vartheta)\,\mathrm{d}t, \quad \theta(\vartheta) = \begin{pmatrix}\cos\vartheta\\ -\sin\vartheta\end{pmatrix}, r \geq s,$$

with respect to ϑ, evaluate the derivative at $\vartheta = 0$ and integrate with respect to r over $[s,\infty[$. In the last step show inductively that

$$\int_{-\infty}^{\infty} p(t) f(s,t)\,\mathrm{d}t = 0$$

for every polynomial p. Conclude, that $f(s,\cdot) = 0$.

Exercise 3:
Consider the function $f\colon \mathbb{R}^2\setminus\{0\}\to\mathbb{C}$ defined by

$$f(x) = \frac{1}{(x_1 + ix_2)^n}, \quad x = (x_1,x_2) \neq 0,$$

for a natural number $n \geq 2$. Show that

$$\mathbf{R}f(\cdot, s) = 0 \quad \text{for all } s \neq 0.$$

Hint: Use Cauchy's integral formula.

Please hand in your solutions in the problem class on **Friday, February 10, 2012**.