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Inverse Problems
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Problem Set 3 – November 4, 2011

Exercise 1:
Consider the one-dimensional diffusion equation

$$-\big(au'\big)'(x) = f(x), \quad 0 < x < 1,$$

with boundary conditions

$$a(0)u'(0) = b_0, \quad a(1)u'(1) = b_1.$$

The reconstruction of the diffusion coefficient a given u and f defines an inverse problem.

(a) Let $f(x) = -1$ for $x \in [0,1]$ and $b_0 = 0$, $b_1 = 1$. Then, for $u(x) = x$ the diffusion coefficient $a(x) = x$ solves this inverse problem. Show that for the perturbed solution

$$u_\delta(x) = \delta \sin\Big(\frac{x}{\delta^2}\Big) + x$$

the corresponding diffusion coefficient is given by

$$a_\delta(x) = \frac{\delta x}{\delta + \cos(\frac{x}{\delta^2})}.$$

Discuss the asymptotic behavior of u_δ and a_δ for $\delta \to 0$.

(b) Let a denote the diffusion coefficient corresponding to the solution u and a_δ the one corresponding to the perturbed solution u_δ. Show that

$$a(x) - a_\delta(x) = \frac{u'_\delta(x) - u'(x)}{u'_\delta(x)u'(x)}\Big(b_0 - \int_0^x f(t)\,\mathrm{d}t\Big)$$

holds.

Exercise 2:
Consider the Abel's integral equation

$$(A\varphi)(\zeta) = \frac{1}{\sqrt{\pi}}\int_0^\zeta \frac{\varphi(z)}{\sqrt{\zeta - z}}\,\mathrm{d}z = \psi(\zeta) \quad \text{for } \zeta > 0$$

with $\psi \in C^1[0,S],\ S > 0.$

(a) Let D denote the differentiation operator and I the identity. Prove that

$$DA^2 = I.$$

(b) Show that the (unique solution) φ is given by

$$\varphi(\zeta) = \frac{1}{\sqrt{\pi}}\left[\frac{\psi(0)}{\sqrt{\zeta}} + \int_0^\zeta \frac{\psi'(z)}{\sqrt{\zeta - z}}\,\mathrm{d}z\right] \quad \text{for } \zeta > 0.$$

Exercise 3:
Let a ball of mass m move along a smooth, fixed and monotone curve Γ from a variable point P to a fixed point P_0. The only force acting on the element is the gravitational force mg and its initial velocity is zero. The time T which the ball needs to go from the point P to P_0 depends on the curve Γ. For a fixed curve Γ it is a function of the height difference h between the points P and P_0. If the time T is independent of h, i.e. constant with respect to P, then the curve Γ is called tautochrone. The *direct problem* is to determine the time $T = T(h)$ given the curve Γ. In the *inverse problem* we want to reconstruct the curve Γ given the time $T = T(h)$ for all heights h.
Let the curve Γ be parametrized by

$$x = f(y)$$

in the (x, y)-plane and P_0 and P have the coordinates $(f(0), 0)$ and $(f(h), h)$, respectively. The conservation of energy, i.e.

$$\frac{m}{2}v^2 + mgy = mgh,$$

implies for the velocity v that

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v = \sqrt{2g(h-y)}.$$

Therefore,

$$T = T(h) = \int_{P_0}^{P} \frac{\mathrm{d}s}{v} = \int_0^h \sqrt{\frac{1+f'(y)^2}{2g(h-y)}}\,\mathrm{d}y \quad \text{for } h > 0.$$

Defining

$$\varphi(y) = \sqrt{1+f'(y)^2} \quad \text{and} \quad \psi(h) = \sqrt{2g}\,T(h)$$

the inverse problem can be reformulated as: Find a solution φ to Abel's integral equation

$$\int_0^h \frac{\varphi(y)}{\sqrt{h-y}}\,\mathrm{d}y = \psi(h) \quad \text{for } h > 0.$$

Show that the tautochrone with constant falling time

$$T_0 = \pi\sqrt{\frac{a}{2g}}, \quad a > 0,$$

is a cycloid

$$x(t) = \frac{a}{2}(t + \sin t), \quad y(t) = \frac{a}{2}(1 - \cos t), \quad 0 \le t \le \pi,$$

i.e. the path traced by the point $(0, 0)$ as the ball of radius $a/2$ and with midpoint $(0, a/2)$ rolls along the line $y = a$.
Hint: Solve Abel's integral equation and show that the arc length of the solution is given by $s = 2\sqrt{ay}$. Substitute $y(t) = a\sin^2(t/2)$ in order to obtain a parametrization.

Please hand in your solutions in the problem class on **Friday, November 11, 2011**.