Karlsruhe Institute of Technology

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Inverse Problems
Winter Semester 2011/2012

Problem Set 4 – November 11, 2011

Exercise 1:
Prove the Riemann-Lebesgue lemma:

$$\int_0^\pi K(\cdot,y)\sin(ny)\,\mathrm{d}y \to 0 \quad \text{as } n\to\infty$$

with respect to the L^2 norm for any kernel $K \in L^2([0,\pi]\times[0,\pi])$.
Use this result to show that operators of the form

$$A\colon L^2[0,\pi]\to L^2[0,\pi], \quad \varphi\mapsto A\varphi = \int_0^\pi K(\cdot,y)\varphi(y)\,\mathrm{d}y,$$

do not have a bounded inverse.
Hint: Consider first continuously differentiable kernels and integrate by parts. Use then the fact that L^2 functions can be approximated by C^1 functions to show the general case.

Exercise 2:

(a) Prove that each of the integral equations

$$\varphi(x) - \frac{1}{2}\int_0^1 (x+1)\mathrm{e}^{-xy}\varphi(y)\,\mathrm{d}y = \mathrm{e}^{-x} - \frac{1}{2} + \frac{1}{2}\mathrm{e}^{-(x+1)}, \quad 0\le x\le 1,$$

and

$$\int_0^1 (x+1)\mathrm{e}^{-xy}\varphi(y)\,\mathrm{d}y = 1 - \mathrm{e}^{-(x+1)}, \quad 0\le x\le 1,$$

has a unique solution in $L^2[0,1]$.
Hint: The function $x\to \mathrm{e}^{-x}$ solves both equations. Give reasons for the continuity of solutions to the first equation and use the Banach fixed-point theorem to prove uniqueness. For the second equation differentiate the homogeneous equation with respect to x.

(b) Show that there does not exist a solution of the integral equation

$$\int_0^1 (x+1)\mathrm{e}^{-xy}\varphi(y)\,\mathrm{d}y = 1, \quad 0\le x\le 1,$$

in $L^2[0,1]$.
Hint: Suppose there exists a solution and deduce a contradiction differentiating the integral equation with respect to x.

Exercise 3:
Prove Theorem 7.3, i.e.:
Let X be a normed space and Y a Banach space. Suppose $A_n\colon X\to Y$ is a compact operator for each $n\in\mathbb{N}$ and there exists a linear operator A such that

$$\|A_n - A\| \to 0 \quad \text{as } n\to\infty.$$

Then A is a compact operator.
Hint: Use a diagonalization procedure.

Please hand in your solutions in the problem class on **Friday, November 18, 2011**.