Karlsruhe Institute of Technology

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Inverse Problems
Winter Semester 2011/2012

Problem Set 5 – November 18, 2011

Exercise 1:

(a) Show that every compact operator is continuous.

(b) Let X, Y, Z be normed space and $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,Z)$. Prove that BA is compact if one of the operators A or B is compact.

(c) Let X, Y be normed spaces, Y infinite-dimensional and $A\colon X \to Y$ compact. Show that A cannot have a continuous inverse.

Exercise 2:
We consider the integral operator $K\colon L^2[0,1] \to L^2[0,1]$ given by

$$(Kx)(s) = \int_0^1 k(s,t)x(t)\,\mathrm{d}t, \quad s \in [0,1],$$

with

$$k(s,t) = \begin{cases} t(1-s), & 0 \le t \le s, \\ s(1-t), & s \le t \le 1. \end{cases}$$

(a) Show that $Kx = g$ if and only if $x = -g'' \in L^2[0,1]$ and $g(0) = g(1) = 0$. Here g is a two times differentiable function on $[0,1]$.

(b) Determine all eigenvalues and eigenfunctions of K.

Exercise 3:
For $f \in C[0,1]$, $f \neq 0$, consider the operator $A_f\colon L^2[0,1] \to L^2[0,1]$ defined by $(A_f x)(s) = f(s)x(s)$.

(a) Determine the adjoint operator A_f^*.

(b) Show that A_f is not compact.

(c) Show that $A_f - \lambda I$ is continuously invertible if and only if $\lambda \notin \mathcal{R}(f)$. Here $\mathcal{R}(f)$ denotes the range of f.

(d) Show that for injective f the equation $A_f x = \lambda x$ does not have a solution $x \neq 0$ for any $\lambda \in \mathbb{K}$.

Please hand in your solutions in the problem class on **Friday, November 25, 2011**.