Karlsruhe Institute of Technology

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Inverse Problems
Winter Semester 2011/2012

Problem Set 6 – November 25, 2011

Exercise 1:
Let X and Y be Hilbert spaces and $A\colon X \to Y$ a bounded linear operator. Show that:

(a) There exists a uniquely determined linear operator $A^*\colon Y \to X$ such that
$$\langle Ax, y\rangle_Y = \langle x, A^*y\rangle_X \quad \text{for all } x \in X, y \in Y.$$

(b) The operator A^* is bounded and $\|A^*\| = \|A\|$.

(c) A is compact if and only if A^* is compact.

(d) The following relations between the range of A, $\mathcal{R}(A) = \{y \in Y \mid y = Ax \text{ for some } x \in X\}$, and the null space of A^*, $\mathcal{N}(A^*) = \{y \in Y \mid A^*y = 0\}$, hold:
$$\mathcal{R}(A)^\perp = \mathcal{N}(A^*) \quad \text{and} \quad \mathcal{N}(A^*)^\perp = \overline{\mathcal{R}(A)}.$$

Exercise 2:
Determine the singular system (σ_n, u_n, v_n) of the Volterra integral operator $K\colon L^2[0,1] \to L^2[0,1]$ given by
$$(Kx)(s) = \int_0^s x(t)\,\mathrm{d}t, \quad s \in [0,1].$$

Exercise 3:
Let X, Y be real Hilbert spaces and $P_A\colon Y \to Y$ the orthogonal projection from Y onto $\overline{\mathcal{R}(A)}$, the closure of the range of the bounded operator $A\colon X \to Y$. Furthermore, let $x \in X$ and $y \in Y$. Show the equivalence of the following statements:

(a) $Ax = P_A y$,

(b) $\|Ax - y\| \le \|Av - y\| \quad$ for all $v \in X$,

(c) $A^*Ax = A^*y$.

Please hand in your solutions in the problem class on **Friday, December 02, 2011**.