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Inverse Problems
Winter Semester 2011/2012

Problem Set 8 – December 09, 2011

Exercise 1:
Let $K\colon X \to Y$ be an injective, compact and non-degenerate operator with singular system (σ_n, u_n, v_n) and $q\colon]0,\infty[\times]0,\|K\|] \to \mathbb{R}$ with the following properties:

(1) $|q(\alpha,\sigma)| \le 1$ for all $\alpha > 0$ and $0 < \sigma \le \|K\|$.

(2) For every $\alpha > 0$ there exists $c(\alpha)$ such that

$$|q(\alpha,\sigma)| \le c(\alpha)\sigma \quad \text{for all } 0 < \sigma \le \|K\|.$$

(3) $\lim_{\alpha\to 0} q(\alpha,\sigma) = 1$ for every $0 < \sigma \le \|K\|$.

Prove that the family of operators $R_\alpha\colon Y \to X$, $\alpha > 0$, defined by

$$R_\alpha y = \sum_{n=1}^{\infty} \frac{q(\alpha,\sigma_n)}{\sigma_n}\langle y, u_n\rangle v_n \quad \text{for } y \in Y$$

is a regularization for $K^\dagger$ with $\|R_\alpha\| \le c(\alpha)$.

Exercise 2:
Consider the central difference quotient

$$R_h y(s) = \frac{1}{h}\left(y\left(s+\frac{h}{2}\right) - y\left(s-\frac{h}{2}\right)\right)$$

for functions y that are odd with respect to $s = 0$ and even with respect to $s = 1$.
Show that $\{R_h\}_{h>0}$ is a regularization for $K^\dagger$ with K being the Volterra-integral operator, i.e.

$$K\colon L^2[0,1] \to L^2[0,1], \quad (Kx)(s) = \int_0^s x(t)\,\mathrm{d}t.$$

Hint: Apply R_h to the singular value decomposition of K (see exercise 2 on problem set 6) and use exercise 1.

Exercise 3:
Let $K\colon X \to Y$ be a compact and non-degenerate operator. Moreover, let $\{R_\alpha\}_{\alpha>0}$ be a regularization for $K^\dagger$ and $\alpha\colon]0,\infty[\times Y \to]0,\infty[$ a parameter choice rule. Prove that there does not exist a function $h\colon [0,\infty[\to [0,\infty[$ with $\lim_{\delta\to 0} h(\delta) = 0$ such that

$$\sup\{\|R_{\alpha(\delta,y^\delta)}y^\delta - K^\dagger y\| \mid y \in \mathcal{R}(K), \|y\| \le 1, y^\delta \in Y, \|y - y^\delta\| \le \delta\} \le h(\delta).$$

Hint: Show that if such a function h would exist, then $K^\dagger$ would be continuous.

Please hand in your solutions in the problem class on **Friday, December 16, 2011**.