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Inverse Problems
Winter Semester 2011/2012

Problem Set 9 – December 16, 2011

Exercise 1:
Let $K\colon X \to Y$ be an injective bounded operator and let $X_1 \subset X_2 \subset \cdots \subset X$ be subspaces with $\dim X_n = n$ and $\overline{\bigcup_{n\in\mathbb{N}} X_n} = X$. In the *Least Square Method* the approximation $x_n \in X_n$ to the solution of $Kx = y$ is chosen such that

$$\|Kx_n - y\| \le \|Kz_n - y\| \quad \text{for all } z_n \in X_n. \qquad (*)$$

Show that this projection method is not convergent in general:
Start with an orthonormal basis $\{v_1, v_2, \dots\}$ of X and set $X_n = \operatorname{span}\{v_1, \dots, v_n\}$. Then, consider the operator $K\colon X \to X$ defined by

$$Kv_k = \begin{cases} v_1 + \displaystyle\sum_{j=2}^{\infty} \frac{1}{j} v_j, & k = 1, \\ \dfrac{1}{k} v_k, & k > 1,\ k \text{ even}, \\ \dfrac{1}{k^3} v_k, & k > 1,\ k \text{ odd}, \end{cases}$$

and the right-hand side

$$y = Kx = K \sum_{k=1}^{\infty} \frac{1}{k} v_k.$$

Show that the sequence $(x_n)_{n\in\mathbb{N}} \subset X_n$ of solutions to $(*)$ is unbounded.
Hint: Writing $x_n = \sum_{k=1}^{n} \xi_k v_k$ the solution to $(*)$ has to satisfy

$$\frac{\partial}{\partial \xi_k} \|Kx_n - y\| = 0 \quad \text{for } k = 1, \dots, n.$$

Exercise 2:
Apply the moment collocation method to the Volterra integral operator,

$$K\colon L^2[0,1] \to C[0,1], \quad (Kx)(s) = \int_0^s x(t)\,\mathrm{d}t,$$

with an equidistant mesh $s_j = jh$, $j = 1, \dots, n$ where $h = 1/n$. Show that the inverse matrix for the linear system (12.5) in this case is given by the tridiagonal matrix

$$n \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}.$$

Exercise 3:
Consider the same situation as in exercise 2. Moreover, let $y \in \mathcal{R}(K)$ be given such that the exact solution $K^{-1}y$ is continuously differentiable and let x_n^δ denote the moment collocation solution to inexact data $\eta^\delta \in \mathbb{R}^n$ with

$$\sum_{j=1}^{n} |\eta_j^\delta - y(s_j)|^2 \leq \delta^2.$$

Show that the reconstruction error can be estimated by

$$\|x_n^\delta - K^{-1}y\|_{L^2} \leq c_1 n^{3/2}\delta + \frac{c_2}{n}$$

with some constants c_1 and c_2.

Please hand in your solutions in the problem class on **Friday, December 23, 2011**.