The Mathematical Theory of Maxwell's Equations

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Contents

Preface

This book arose from a lecture on Maxwell's equations given by the authors between ?? and 2009.

The emphasis is put on three topics which are clearly structured into Chapters 2, **??**, and **??**. In each of these chapters we study first the simpler scalar case where we replace the Maxwell system by the scalar Helmholtz equation. Then we investigate the time harmonic Maxwell's equations.

In Chapter 1 we start from the (time dependent) Maxwell system in integral form and derive ...

Chapter 1

Introduction

1.1 Maxwell's Equations

Electromagnetic wave propagation is described by particular equations relating five vector fields $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{J}}$ and the scalar field ρ, where $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{D}}$ denote the **electric field** (in V/m) and **electric displacement** (in As/m^2) respectively, while $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{B}}$ denote the **magnetic field** (in A/m) and **magnetic flux density** (in $Vs/m^2 = T$ =Tesla). Likewise, $\boldsymbol{\mathcal{J}}$ and ρ denote the **current density** (in A/m^2) and **charge density** (in As/m^3) of the medium. Here and throughout the lecture we use the **rationalized MKS-system**, i.e. V, A, m and s. All fields will be assumed to depend both on the space variable $x \in \mathbb{R}^3$ and on the time variable $t \in \mathbb{R}$.

The actual equations that govern the behavior of the electromagnetic field, first completely formulated by Maxwell, may be expressed easily in integral form. Such a formulation has the advantage of being closely connected to the physical situation. The more familiar differential form of Maxwell's equations can be derived very easily from the integral relations as we will see below.

In order to write these integral relations, we begin by letting S be a connected smooth surface with boundary ∂S in the interior of a region Ω_0 where electromagnetic waves propagate. In particular, we require that the unit normal vector $\nu(x)$ for $x \in S$ be continuous and directed always into "one side" of S, which we call the positive side of S. By $\tau(x)$ we denote the unit vector tangent to the boundary of S at $x \in \partial S$. This vector, lying in the tangent plane of S together with a vector $n(x)$, $x \in \partial S$, normal to ∂S is oriented so as to form a mathematically positive system (i.e. τ is directed counterclockwise when we sit on the positive side of S, and $n(x)$ is directed to the outside of S). Furthermore, let $\Omega \in \mathbb{R}^3$ be an open set with boundary $\partial\Omega$ and outer unit normal vector $\nu(x)$ at $x \in \partial\Omega$. Then Maxwell's equations in integral form state:

$$\int_{\partial S} \boldsymbol{\mathcal{H}} \cdot \tau \, d\ell = \frac{d}{dt} \int_{S} \boldsymbol{\mathcal{D}} \cdot \nu \, ds + \int_{S} \boldsymbol{\mathcal{J}} \cdot \nu \, ds \quad \text{(Ampère's Law)} \tag{1.1a}$$

$$\int_{\partial S} \boldsymbol{\mathcal{E}} \cdot \tau \, d\ell = -\frac{d}{dt} \int_{S} \boldsymbol{\mathcal{B}} \cdot \nu \, ds \quad \text{(Law of Induction)} \tag{1.1b}$$

$$\int_{\partial \Omega} \boldsymbol{\mathcal{D}} \cdot \nu \, ds = \iint_{\Omega} \rho \, dx \quad \text{(Gauss' Electric Law)} \tag{1.1c}$$

$$\int_{\partial \Omega} \boldsymbol{\mathcal{B}} \cdot \nu \, ds = 0 \quad \text{(Gauss' Magnetic Law)} \tag{1.1d}$$

To derive the Maxwell's equations in differential form we consider a region Ω_0 where μ and ε are constant (**homogeneous medium**) or at least continuous. In regions where the vector fields are smooth functions we can apply the Stokes and Gauss theorems for surfaces S and solids Ω lying completely in Ω_0:

$$\int_{S} \operatorname{curl} \mathbf{F} \cdot \nu \, ds = \int_{\partial S} \mathbf{F} \cdot \tau \, d\ell \quad \text{(Stokes)}, \tag{1.2}$$

$$\iint_{\Omega} \operatorname{div} \mathbf{F} \, dx = \int_{\partial \Omega} \mathbf{F} \cdot \nu \, ds \quad \text{(Gauss)}, \tag{1.3}$$

where $\mathbf{F}$ denotes one of the fields $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{B}}$ or $\boldsymbol{\mathcal{D}}$. We recall the **differential operators** (in cartesian coordinates):

$$\operatorname{div} \mathbf{F}(x) = \sum_{j=1}^{3} \frac{\partial F_j}{\partial x_j}(x) \qquad \text{(divergenz, “Divergenz”)}$$

$$\operatorname{curl} \mathbf{F}(x) = \begin{pmatrix} \frac{\partial F_3}{\partial x_2}(x) - \frac{\partial F_2}{\partial x_3}(x) \\ \frac{\partial F_1}{\partial x_3}(x) - \frac{\partial F_3}{\partial x_1}(x) \\ \frac{\partial F_2}{\partial x_1}(x) - \frac{\partial F_1}{\partial x_2}(x) \end{pmatrix} \qquad \text{(curl, “Rotation”)}\,.$$

With these formulas we can eliminate the boundary integrals in (1.1a-1.1d). We then use the fact that we can vary the surface S and the solid Ω in D arbitrarily. By equating the integrands we are led to Maxwell's equations in **differential form** so that Ampère's Law, the Law of Induction and Gauss' Electric and Magnetic Laws, respectively, become:

$$\begin{aligned}
\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \operatorname{curl}_x \boldsymbol{\mathcal{E}} &= 0 && \text{(Faraday's Law of Induction, “Induktionsgesetz”)}\\
\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} - \operatorname{curl}_x \boldsymbol{\mathcal{H}} &= -\boldsymbol{\mathcal{J}} && \text{(Ampere's Law, “Durchflutungsgesetz”)}\\
\operatorname{div}_x \boldsymbol{\mathcal{D}} &= \rho && \text{(Gauss' Electric Law, “Coulombsches Gesetz”)}\\
\operatorname{div}_x \boldsymbol{\mathcal{B}} &= 0 && \text{(Gauss' Magnetic Law)}
\end{aligned}$$

We note that the differential operators are always taken w.r.t. the spacial variable x (not w.r.t. time t!). Therefore, in the following we often drop the index x.

Physical remarks:

- The law of induction describes how a time-varying magnetic field effects the electric field.
- Ampere's Law describes the effect of the current (external and induced) on the magnetic field.
- Gauss' Electric Law describes the sources of the electric displacement.
- The forth law states that there are no magnetic currents.
- Maxwell's equations imply the existence of electromagnetic waves (as ligh, X-rays, etc) in vacuum and explain many electromagnetic phenomena.
- Literature wrt physics: J.D. Jackson, Klassische Elektrodynamik, de Gruyter Verlag

Historical Remark:

- Dates: André Marie Ampère (1775–1836), Charles Augustin de Coulomb (1736–1806), Michael Faraday (1791–1867), James Clerk Maxwell (1831–1879)
- It was the ingeneous idea of Maxwell to modify Ampere's Law which was known up to that time in the form curl $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{J}}$ for stationary currents. Furthermore, he collected the four equations as a consistent theory to describe the electromagnetic fields. (James Clerk Maxwell, Treatise on Electricity and Magnetism, 1873).

Conclusion 1.1 *Gauss' Electric Law and Ampere's Law imply the* ***equation of continuity***

$$\frac{\partial \rho}{\partial t} = \operatorname{div} \frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} = \operatorname{div}\left(\operatorname{curl} \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{J}}\right) = -\operatorname{div} \boldsymbol{\mathcal{J}}$$

since div curl = 0.

1.2 The Constitutive Equations

In this general setting the equation are not yet consistent (more unknown than equations). The **Constitutive Equations** couple them:

$$\boldsymbol{\mathcal{D}} \;=\; \boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{H}}) \quad \text{and} \quad \boldsymbol{\mathcal{B}} \;=\; \boldsymbol{\mathcal{B}}(\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{H}})$$

The electric properties of material are complicated. In general, they not only depend on the molecular character but also on macroscopic quantities as density and temperature of the material. Also, there are time-dependent dependencies as, e.g., the hysteresis effect, i.e. the fields at time t depend also on the past.

As a first approximation one starts with representations of the form

$$\boldsymbol{\mathcal{D}} \;=\; \boldsymbol{\mathcal{E}} + 4\pi\boldsymbol{\mathcal{P}} \quad \text{and} \quad \boldsymbol{\mathcal{B}} \;=\; \boldsymbol{\mathcal{H}} - 4\pi\boldsymbol{\mathcal{M}}$$

where $\boldsymbol{\mathcal{P}}$ denotes the electric polarisation vector and $\boldsymbol{\mathcal{M}}$ the magnetization of the material. These can be interpreted as mean values of microscopic effects in the material. Analogously, ρ and $\boldsymbol{\mathcal{J}}$ are macroskopic mean values of the free charge and current densities in the medium.

If we ignore ferro-electric and ferro-magnetic media and if the fields are small one can model the dependencies by linear equations of the form

$$\boldsymbol{\mathcal{D}} \;=\; \varepsilon\boldsymbol{\mathcal{E}} \quad \text{and} \quad \boldsymbol{\mathcal{B}} \;=\; \mu\boldsymbol{\mathcal{H}}$$

with matrix-valued functions $\varepsilon : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ (**dielectric tensor**), and $\mu : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ (**permeability tensor**). In this case we call the media **inhomogenous** and **anisotropic**.

The special case of an **isotropic medium** means that polarization and magnetisation do not depend on the directions. In this case they are just real valued functions, and we have

$$\boldsymbol{\mathcal{D}} \;=\; \varepsilon\boldsymbol{\mathcal{E}} \quad \text{and} \quad \boldsymbol{\mathcal{B}} \;=\; \mu\boldsymbol{\mathcal{H}}$$

with functions $\varepsilon, \mu : \mathbb{R}^3 \to \mathbb{R}$.

In the simplest case these functions ε and μ are constant. This is the case, e.g., in vacuum.

We indicated already that also ρ and $\boldsymbol{\mathcal{J}}$ can depend on the material and the fields. Therefore, we need a further equation. In conducting media the electric field induces a current. In a linear approximation this is described by **Ohm's Law**:

$$\boldsymbol{\mathcal{J}} \;=\; \sigma\boldsymbol{\mathcal{E}} \;+\; \boldsymbol{\mathcal{J}}_e$$

where $\boldsymbol{\mathcal{J}}_e$ is the external current density. For isotropic media the function $\sigma : \mathbb{R}^3 \to \mathbb{R}$ is called the **conductivity**.

Remark: If $\sigma = 0$ the the material is called **dielectric**. In vacuum we have $\sigma = 0$, $\varepsilon = \varepsilon_0 \approx 8.854 \cdot 10^{-12} AS/Vm$, $\mu = \mu_0 = 4\pi \cdot 10^{-7} Vs/Am$. In anisotropic media, also the function σ is matrix valued.

1.3 Special Cases

Vacuum

In regions of vacuum with no charge distributions and (external) currents (i.e. ($\rho = 0, \boldsymbol{\mathcal{J}}_e = 0$) the law of induction takes the form

$$\mu_0 \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} \;+\; \operatorname{curl} \boldsymbol{\mathcal{E}} \;=\; 0\,.$$

Differentiation wrt time t and use of Ampere's Law yields

$$\mu_0 \frac{\partial^2 \boldsymbol{\mathcal{H}}}{\partial t^2} \;+\; \frac{1}{\varepsilon_0} \operatorname{curl}\operatorname{curl} \boldsymbol{\mathcal{H}} \;=\; 0\,,$$

i.e.

$$\varepsilon_0\mu_0 \frac{\partial^2 \boldsymbol{\mathcal{H}}}{\partial t^2} \;+\; \operatorname{curl}\operatorname{curl} \boldsymbol{\mathcal{H}} \;=\; 0\,.$$

$1/\sqrt{\varepsilon_0\mu_0}$ has the dimension of velocity and is called the **speed of light**: $c_0 = \sqrt{\varepsilon_0\mu_0}$.
From $\operatorname{curl}\operatorname{curl} = \nabla \operatorname{div} - \Delta$ it follows that the components of $\boldsymbol{\mathcal{H}}$ are solutions of the linear **wave equation**

$$c_0^2 \frac{\partial^2 \boldsymbol{\mathcal{H}}}{\partial t^2} \;-\; \Delta \boldsymbol{\mathcal{H}} \;=\; 0\,.$$

Analogously, one derives the same equation for the electric field:

$$c_0^2 \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} \;-\; \Delta \boldsymbol{\mathcal{E}} \;=\; 0\,.$$

Remark: Heinrich Rudolf Hertz (1857–1894) showed also experimentally the existence of electromagenetic waves about 20 years after Maxwell's paper (in Karlsruhe!).

Electrostatics

If $\boldsymbol{\mathcal{E}}$ is in some region Ω constant wrt time t (i.e. in the static case) the law of induction reduces to

$$\operatorname{curl} \boldsymbol{\mathcal{E}} \;=\; 0 \quad \text{in } \Omega\,.$$

Therefore, if Ω is simply connected there exists a potential $u : \Omega \to \mathbb{R}$ with $\boldsymbol{\mathcal{E}} = -\nabla u$ in Ω. Gauss' Electric Law yields in homogeneous media the **Poisson equation**

$$\rho \;=\; \operatorname{div} \boldsymbol{\mathcal{D}} \;=\; -\operatorname{div}(\varepsilon_0 \boldsymbol{\mathcal{E}}) \;=\; -\varepsilon_0 \Delta u$$

for the potential u. The electrostatics is described by this basic elliptic partial differential equation $\Delta u = -\rho/\varepsilon_0$. Mathematically, this is the subject of **potential theory**.

Magnetostatics

The same technique does not work in magnetostatics since, in general, $\operatorname{curl}\boldsymbol{\mathcal{H}} \neq 0$. However, since

$$\operatorname{div}\boldsymbol{\mathcal{B}} \;=\; 0$$

we conclude the existence of a *vector potential* $\boldsymbol{\mathcal{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ with $\boldsymbol{\mathcal{B}} = -\operatorname{curl}\boldsymbol{\mathcal{A}}$ in D. Substituting this into Ampere's Law yields (for homogeneous media Ω) after multiplication with μ_0

$$-\mu_0\boldsymbol{\mathcal{J}} \;=\; \operatorname{curl}\operatorname{curl}\boldsymbol{\mathcal{A}} \;=\; \nabla\operatorname{div}\boldsymbol{\mathcal{A}} \;-\; \Delta\boldsymbol{\mathcal{A}}.$$

Since $\operatorname{curl}\nabla = 0$ we can add gradients ∇u to $\boldsymbol{\mathcal{A}}$ without changing $\boldsymbol{\mathcal{B}}$. We will see later that we can choose u such that the resulting potential $\boldsymbol{\mathcal{A}}$ satisfies $\operatorname{div}\boldsymbol{\mathcal{A}} = 0$. This choice of normalization is called *Coulomb gauge*.

With this normalization we also get in the magnetostatic case the Poisson equation

$$\Delta\boldsymbol{\mathcal{A}} \;=\; -\mu_0\boldsymbol{\mathcal{J}}\,.$$

We note that in this case the Laplacian has to be taken component wise.

Time Harmonic Fields

Under the assumptions that the fields allow a Fourier transformation w.r.t. time we set

$$\begin{aligned} E(x;\omega) &= (\mathcal{F}_t\boldsymbol{\mathcal{E}})(x;\omega) = \int_{\mathbb{R}} \boldsymbol{\mathcal{E}}(x,t)\, e^{i\omega t}\, dt\,, \\ H(x;\omega) &= (\mathcal{F}_t\boldsymbol{\mathcal{H}})(x;\omega) = \int_{\mathbb{R}} \boldsymbol{\mathcal{H}}(x,t)\, e^{i\omega t}\, dt\,, \end{aligned}$$

etc. We note that the fields E, H etc are now complex valued, i.e, $E(\cdot;\omega), H(\cdot;\omega) : \mathbb{R}^3 \to \mathbb{C}^3$ (and also the other fields). Although they are vector fields we denote them by capital Latin letters only. Maxwell's equations transform into (since $\mathcal{F}_t(u') = -i\omega\mathcal{F}_t u$) the **time harmonic Maxwell's equations**

$$\begin{aligned} -i\omega B \;+\; \operatorname{curl}E &= 0\,, \\ i\omega D \;+\; \operatorname{curl}H &= \sigma E \;+\; J_e\,, \\ \operatorname{div}D &= \rho\,, \\ \operatorname{div}B &= 0\,. \end{aligned}$$

Remark: The time harmonic Maxwell's equation can also be derived from the assumption that all fields behave periodically w.r.t. time with the same frequency ω. Then the forms $\boldsymbol{\mathcal{E}}(x,t) = e^{-i\omega t}E(x)$, $\boldsymbol{\mathcal{H}}(x,t) = e^{-i\omega t}H(x)$, etc satisfy the time harmonic Maxwell's equations.

With the constitutive equations $D = \varepsilon E$ and $B = \mu H$ we arrive at

$$-i\omega\mu H \;+\; \operatorname{curl} E \;=\; 0\,, \tag{1.4a}$$

$$i\omega\varepsilon E \;+\; \operatorname{curl} H \;=\; \sigma E \;+\; J_e\,, \tag{1.4b}$$

$$\operatorname{div}(\varepsilon E) \;=\; \rho\,, \tag{1.4c}$$

$$\operatorname{div}(\mu H) \;=\; 0\,. \tag{1.4d}$$

Eliminating H or E, respectively, from (1.4a) and (1.4b) yields

$$\operatorname{curl}\left(\frac{1}{i\omega\mu}\operatorname{curl} E\right) \;+\; (i\omega\varepsilon - \sigma)\,E \;=\; J_e\,. \tag{1.5}$$

and

$$\operatorname{curl}\left(\frac{1}{i\omega\varepsilon - \sigma}\operatorname{curl} H\right) \;+\; i\omega\mu\,H \;=\; \operatorname{curl}\left(\frac{1}{i\omega\varepsilon - \sigma} J_e\right), \tag{1.6}$$

respectively. Usually, one writes these equations in a slightly different way by introducing the constant values $\varepsilon_0 > 0$ and $\mu_0 > 0$ in vacuum and *relative* values (dimensionless!) $\mu_r, \varepsilon_r \in \mathbb{R}$ and $\varepsilon_c \in \mathbb{C}$, defined by

$$\mu_r \;=\; \frac{\mu}{\mu_0}\,, \qquad \varepsilon_r \;=\; \frac{\varepsilon}{\varepsilon_0}\,, \qquad \varepsilon_c \;=\; \varepsilon_r \;+\; i\,\frac{\sigma}{\omega\varepsilon_0}\,.$$

Then equations (1.5) and (1.6) take the form

$$\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E\right) \;-\; k^2\varepsilon_c\,E \;=\; i\omega\mu_0 J_e\,, \tag{1.7}$$

$$\operatorname{curl}\left(\frac{1}{\varepsilon_c}\operatorname{curl} H\right) \;-\; k^2\mu_r\,H \;=\; \operatorname{curl}\left(\frac{1}{\varepsilon_c} J_e\right), \tag{1.8}$$

with the **wave number** $k = \omega\sqrt{\varepsilon_0\mu_0}$. In vacuum we have $\varepsilon_c = 1$, $\mu_r = 1$ and thus

$$\operatorname{curl}\operatorname{curl} E \;-\; k^2\,E \;=\; i\omega\mu_0 J_e\,, \tag{1.9}$$

$$\operatorname{curl}\operatorname{curl} H \;-\; k^2\,H \;=\; \operatorname{curl} J_e\,. \tag{1.10}$$

Example 1.2 In the case $J_e = 0$ and in vacuum the fields

$$E(x) \;=\; p\,e^{ik\,d\cdot x} \quad \text{and} \quad H(x) \;=\; (p \times d)\,e^{ik\,d\cdot x}$$

are solutions of the time harmonic Maxwell's equations (1.9), (1.10) provided d is a unit vector in $\mathbb{R}^3$ and $p \in \mathbb{C}^3$ with $p \cdot d = 0$[1]. Such fields are called **plane time harmonic fields** with *polarization vector* $p \in \mathbb{C}^3$ and *direction* d.

[1] We set $p \cdot d = \sum_{j=1}^{3} p_j d_j$ even for $p \in \mathbb{C}^3$

We make the **assumption** $\varepsilon_c = 1$, $\mu_r = 1$ for the rest of Section 1.3. Taking the divergence of these equations yield div $H = 0$ and k^2 div $E = -i\omega\mu_0$ div J_e, i.e. div $E = -(i/\omega\varepsilon_0)$ div J_e. Comparing this to (1.4c) yields the time harmonic version of the **equation of continuity**

$$\text{div } J_e \;=\; i\omega\rho\,.$$

With the vector identity curl curl $= -\Delta + \text{div } \nabla$ equations (1.10) and (1.9) can be written as

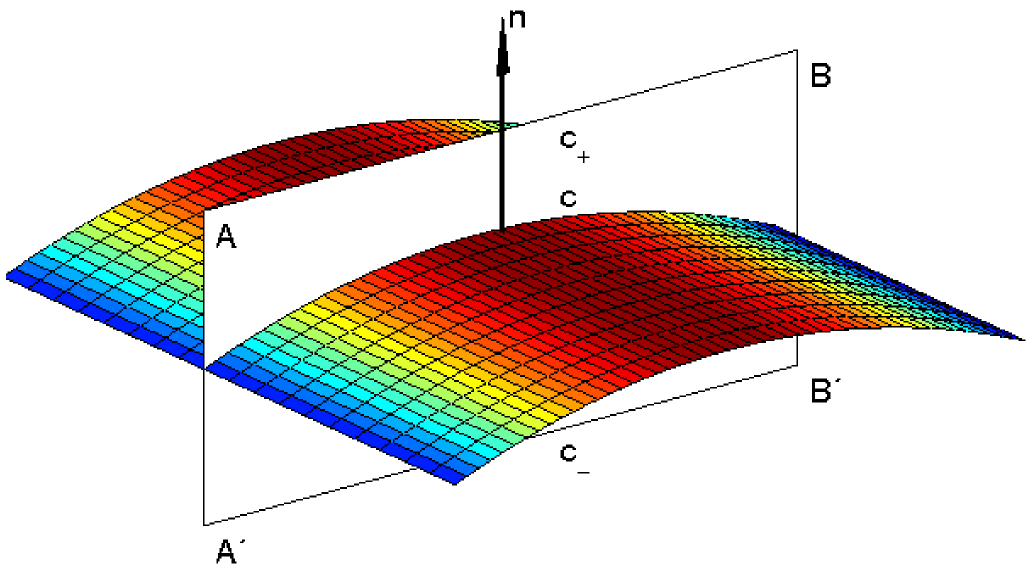
$$\Delta E \;+\; k^2 E \;=\; -i\omega\mu_0 J_e \;+\; \frac{1}{\varepsilon_0}\nabla\rho\,, \tag{1.11}$$

$$\Delta H \;+\; k^2 H \;=\; -\,\text{curl}\, J_e\,. \tag{1.12}$$

1.4 Boundary and Radiation Conditions

Maxwell's equations hold only in regions with smooth parameter functions ε_r, μ_r and σ. If we consider a situation in which a surface S separates two homogeneous media from each other, the constitutive parameters ε, μ and σ are no longer continuous but piecewise continuous with finite jumps on S. While on both sides of S Maxwell's equations (1.4a)–(1.4d) hold, the presence of these jumps implies that the fields satisfy certain conditions on the surface.

To derive the mathematical form of this behaviour (the boundary conditions) we apply the law of induction (1.1b) to a narrow rectangle-like surface R, containing the normal n to the surface S and whose long sides C_+ and C_- are parallel to S and are on the opposite sides of it, see the following figure.



When we let the height of the narrow sides, AA' and BB', approach zero then C_+ and C_- approach a curve C on S, the surface integral $\frac{\partial}{\partial t}\int_R \boldsymbol{\mathcal{B}} \cdot \nu\, ds$ will vanish in the limit since the field remains finite (note, that the normal ν is the normal to R lying in the tangential plane of S). Hence, the line integrals $\int_C \boldsymbol{\mathcal{E}}_+ \cdot \tau\, d\ell$ and $\int_C \boldsymbol{\mathcal{E}}_- \cdot \tau\, d\ell$ must be equal. Since the curve C is arbitrary the integrands $\boldsymbol{\mathcal{E}}_+ \cdot \tau$ and $\boldsymbol{\mathcal{E}}_- \cdot \tau$ coincide on every arc C, i.e.

$$n \times \boldsymbol{\mathcal{E}}_+ \; - \; n \times \boldsymbol{\mathcal{E}}_- \; = \; 0 \quad \text{on } S\,. \tag{1.13}$$

A similar argument holds for the magnetic field in (1.1a) if the current distribution $\boldsymbol{\mathcal{J}} = \sigma\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{J}}_e$ remains finite. In this case, the same arguments lead to the boundary condition

$$n \times \boldsymbol{\mathcal{H}}_+ \; - \; n \times \boldsymbol{\mathcal{H}}_- \; = \; 0 \quad \text{on } S\,. \tag{1.14}$$

If, however, the external current distribution is a surface current, i.e. if $\boldsymbol{\mathcal{J}}_e$ is of the form $\boldsymbol{\mathcal{J}}_e(x + \tau n(x)) = \boldsymbol{\mathcal{J}}_s(x)\delta(\tau)$ for small τ and $x \in S$ and with tangential surface field $\boldsymbol{\mathcal{J}}_s$ and σ is finite, then the surface integral $\int_R \boldsymbol{\mathcal{J}}_e \cdot \nu\, ds$ will tend to $\int_C \boldsymbol{\mathcal{J}}_s \cdot \nu\, d\ell$, and so the boundary condition is

$$n \times \boldsymbol{\mathcal{H}}_+ \; - \; n \times \boldsymbol{\mathcal{H}}_- \; = \; \boldsymbol{\mathcal{J}}_s \quad \text{on } S\,. \tag{1.15}$$

We will call (1.13) and (1.14) or (1.15) the **transmission boundary conditions**.

A special and very important case is that of a **perfectly conducting medium** with boundary S. Such a medium is characterized by the fact that the electric field vanishes inside this medium, and (1.13) reduces to

$$n \times \boldsymbol{\mathcal{E}} \; = \; 0 \quad \text{on } S \tag{1.16}$$

Another important case is the **impedance-** or **Leontovich boundary condition**

$$n \times \boldsymbol{\mathcal{H}} \; = \; \lambda\, n \times (\boldsymbol{\mathcal{E}} \times n) \quad \text{on } S \tag{1.17}$$

which, under appropriate conditions, may be used as an approximation of the transmission conditions.

The same kind of boundary occur also in the time harmonic case (where we denote the fields by capital Latin letters).

Finally, we specify the boundary conditions to the E- and H-modes derived above. We assume that the surface S is an infinite cylinder in x_3-direction with constant cross section. Furthermore, we assume that the volume current density J vanishes near the boundary S and that the surface current densities take the form $J_s = j_s\hat{z}$ for the E-mode and $J_s = j_s\big(\nu \times \hat{z}\big)$ for the H-mode. We use the notation $[v] := v|_+ - v|_-$ for the jump of the function v at the boundary. Also, we abbreviate (only for this table) $\sigma' = \sigma - i\omega\varepsilon$. We list the boundary conditions in the following table.

Boundary condition	E-mode	H-mode
transmission	$[u] = 0$ on S, $\left[\sigma' \frac{\partial u}{\partial \nu}\right] = -j_s$ on S,	$\left[\mu \frac{\partial u}{\partial \nu}\right] = 0$ on S, $[u] = j_s$ on S,
impedance	$\lambda k^2 u + \sigma' \frac{\partial u}{\partial \nu} = -j_s$ on S,	$k^2 u - \lambda\, i\omega\mu \frac{\partial u}{\partial \nu} = j_s$ on S,
perfect conductor	$u = 0$ on S,	$\frac{\partial u}{\partial \nu} = 0$ on S.

The situation is different for the normal components. We consider Gauss' Electric and Magnetic Laws and choose Ω to be a box which is separated by S into two parts Ω_1 and Ω_2. We apply (1.1c) first to all of Ω and then to Ω_1 and Ω_2 separately. The addition of the last two formulas and the comparison with the first yields that the normal component $\boldsymbol{\mathcal{D}} \cdot n$ has to be continuous as well as (application of (1.1d)) $\boldsymbol{\mathcal{B}} \cdot n$. With the constitutive equations one gets

$$n \cdot (\varepsilon_{r,1}\mathcal{E}_1 - \varepsilon_{r,2}\mathcal{E}_2) \;=\; 0 \text{ on } S \quad \text{and} \quad n \cdot (\mu_{r,1}\mathcal{H}_1 - \mu_{r,2}\mathcal{H}_2) \;=\; 0 \text{ on } S\,.$$

Conclusion 1.3 *The normal components of $\boldsymbol{\mathcal{E}}$ and/or $\boldsymbol{\mathcal{H}}$ are not continuous at interfaces where ε_c and/or μ_r have jumps.*

The Silver-Müller radiation condition

Reference Problems

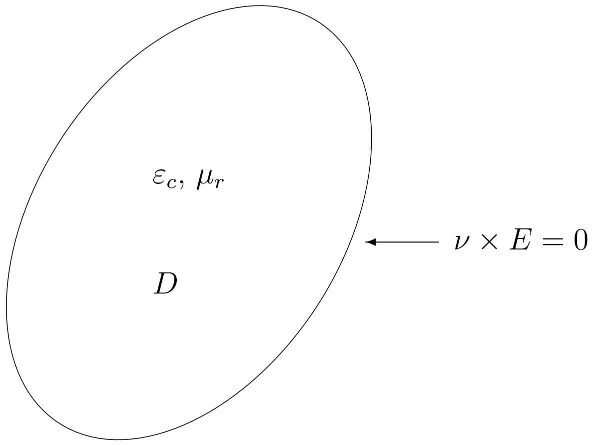
During our course we will consider two classical boundary value problems.

- **(Cavity with an ideal conductor as boundary)** Let $D \subseteq \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary ∂D and exterior unit normal vector $\nu(x)$ at $x \in \partial D$. Let $J_e : D \to \mathbb{C}^3$ be a vector field. Determine a solution (E, H) of the time harmonic Maxwell system

$$\operatorname{curl} E - i\omega\mu H \;=\; 0 \quad \text{in } D\,, \tag{1.18a}$$

$$\operatorname{curl} H + (i\omega\varepsilon - \sigma)E \;=\; J_e \quad \text{in } D\,, \tag{1.18b}$$

$$\nu \times E \;=\; 0 \quad \text{on } \partial D\,. \tag{1.18c}$$



- **(Scattering by an ideal conductor)** Given a bounded region D and some solution E^i and H^i of the "unperturbed" time harmonic Maxwell system

$$\operatorname{curl} E^i \;-\; i\mu_0 H^i \;=\; 0 \text{ in } \mathbb{R}^3\,, \quad \operatorname{curl} H^i \;+\; i\varepsilon_0 E^i \;=\; 0 \text{ in } \mathbb{R}^3\,,$$

determine E, H of the Maxwell system

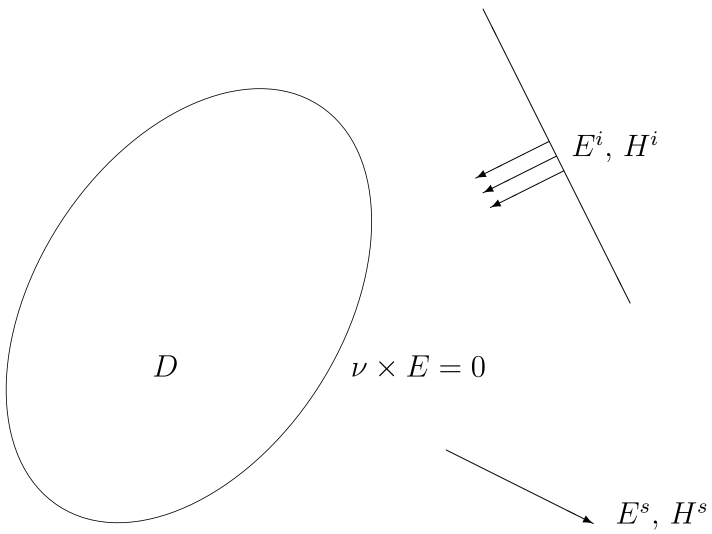
$$\operatorname{curl} E \;-\; i\mu_0 H \;=\; 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}\,, \quad \operatorname{curl} H \;+\; i\varepsilon_0 E \;=\; 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}\,,$$

such that E satisfies the boundary condition $\nu \times E = 0$ on ∂D, and E and H have the decompositions into $E = E^s + E^i$ and $H = H^s + H^i$ in $\mathbb{R}^3 \setminus \overline{D}$ with some scattered field E^s, H^s which satisfy the **Silver-Müller radiation condition**

$$\lim_{|x| \to \infty} |x| \left(H^s(x) \times \frac{x}{|x|} \;-\; \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E^s(x) \right) \;=\; 0$$
$$\lim_{|x| \to \infty} |x| \left(E^s(x) \times \frac{x}{|x|} \;+\; \sqrt{\frac{\mu_0}{\varepsilon_0}}\, H^s(x) \right) \;=\; 0$$

uniformly with respect to all directions $x/|x|$.

Remark: For general $\mu_r, \varepsilon_c \in L^\infty(\mathbb{R}^3)$ we have to give a correct interpretation of the differential equations ("variational or weak formulation") and transmission conditions ("trace theorems").



1.5 Vector Calculus

In this subsection we collect the most important formulas from vector calculus.

Table of Differential Operators and Their Properties

We assume that all functions are sufficiently smooth. In **cartesian coordinates**:

Operator	**Application** to function
∇	$\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3})^\top$
div $= \nabla\cdot$	$\nabla \cdot A = \sum_{j=1}^{3} \frac{\partial A_j}{\partial x_j}$
curl $= \nabla\times$	$\nabla \times A = \begin{pmatrix} \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \\ \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \\ \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \end{pmatrix}$
$\Delta = \text{div } \nabla = \nabla \cdot \nabla$	$\Delta u = \sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2}$

The following **formulas**, which can be obatined from straightforward calculations, will be used often and have been used already:

For $x, y, z \in \mathbb{C}^3$, $\lambda : \mathbb{C}^3 \to \mathbb{C}$ und $A, B : \mathbb{C}^3 \to \mathbb{C}^3$ we have

$$x \cdot (y \times z) = y \cdot (z \times x) = z \cdot (x \times y) \tag{1.19}$$

$$x \times (y \times z) = (x \cdot z)y - (x \cdot y)z \tag{1.20}$$

$$\operatorname{curl} \nabla u = 0 \tag{1.21}$$

$$\operatorname{div} \operatorname{curl} A = 0 \tag{1.22}$$

$$\operatorname{curl} \operatorname{curl} A = \nabla \operatorname{div} A - \Delta A \tag{1.23}$$

$$\operatorname{div} (\lambda A) = A \cdot \nabla \lambda + \lambda \operatorname{div} A \tag{1.24}$$

$$\operatorname{curl}(\lambda A) = \nabla \lambda \times A + \lambda \operatorname{curl} A \tag{1.25}$$

$$\nabla(A \cdot B) = (A \cdot \nabla)B + (B \cdot \nabla)A + A \times (\operatorname{curl} B) + B \times (\operatorname{curl} A) \tag{1.26}$$

$$\operatorname{div} (A \times B) = B \cdot \operatorname{curl} A - A \cdot \operatorname{curl} B \tag{1.27}$$

$$\operatorname{curl}(A \times B) = A \operatorname{div} B - B \operatorname{div} A + (B \cdot \nabla)A - (A \cdot \nabla)B \tag{1.28}$$

For completeness we add the expressions of the differential operators ∇, div , curl, and Δ in other coordinate systems. Let $f : \mathbb{R}^3 \to \mathbb{C}$ be a scalar function and $F : \mathbb{R}^3 \to \mathbb{C}^3$ a vector field.

Cylindrical Coordinates

$$x = \begin{pmatrix} r \cos\varphi \\ r \sin\varphi \\ z \end{pmatrix}$$

Let $\hat{z} = (0, 0, 1)^\top$ and $\hat{r} = (\cos\varphi, \sin\varphi, 0)^\top$ and $\hat{\varphi} = (-\sin\varphi, \cos\varphi, 0)^\top$ be the coordinate unit vectors. Let $F = F_r\hat{r} + F_\varphi\hat{\varphi} + F_z\hat{z}$. Then

$$\nabla f(r, \varphi, z) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{z},$$

$$\operatorname{div} F(r, \varphi, z) = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z},$$

$$\operatorname{curl} F(r, \varphi, z) = \left(\frac{1}{r}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right)\hat{r} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\varphi} + \frac{1}{r}\left(\frac{\partial(rF_\varphi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \varphi}\right)\hat{z},$$

$$\Delta f(r, \varphi, z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Spherical Coordinates

$$x = \begin{pmatrix} r \sin\theta \cos\varphi \\ r \sin\theta \sin\varphi \\ r \cos\theta \end{pmatrix}$$

Let $\hat{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^\top$ and $\hat{\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)^\top$ and $\hat{\varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)^\top$ be the coordinate unit vectors. Let $F = F_r\hat{r} + F_\theta\hat{\theta} + F_\varphi\hat{\varphi}$.

Then

$$\begin{aligned}
\nabla f(r,\theta,\varphi) &= \frac{\partial f}{\partial r}\,\hat{r} \;+\; \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\theta} \;+\; \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\,\hat{\varphi}\,,\\
\operatorname{div} F(r,\theta,\varphi) &= \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} \;+\; \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,F_\theta)}{\partial\theta} \;+\; \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}\,,\\
\operatorname{curl} F(r,\theta,\varphi) &= \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,F_\varphi)}{\partial\theta} - \frac{\partial F_\theta}{\partial\varphi}\right)\hat{r} \;+\; \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{\partial(rF_\varphi)}{\partial r}\right)\hat{\theta} \;+\\
&\quad + \frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)\hat{\varphi}\,,\\
\Delta f(r,\theta,\varphi) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) \;+\; \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \;+\; \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\,.
\end{aligned}$$

Elementary Facts from Differential Geometry

Before we recall the basic integral identity of Gauss and Green we have to define rigourously the notion of domain with C^n-boundaries. We denote by $B(x,r) := B_j(x,r) := \{y \in \mathbb{R}^j : |y-x| < r\}$ and $B[x,r] := B_j[x,r] := \{y \in \mathbb{R}^j : |y-x| \le r\}$ the open and closed ball, respectively, of radius $r > 0$ centered at x in $\mathbb{R}^j$ for $j = 2$ or $j = 3$.

Definition 1.4 *We call a region $D \subset \mathbb{R}^3$ to be C^n-smooth (i.e. $D \in C^n$), if there exists a finite number of open sets $U_j \subset \mathbb{R}^3$, $j = 1,\dots,m$, and bijective mappings $\tilde{\Psi}_j$ from the closed unit ball $B_3[0,1] := \{u \in \mathbb{R}^3 : |u| \le 1\}$ onto $\overline{U}_j$ such that*

(i) $\partial D \subset \bigcup_{j=1}^{m} U_j$,

(ii) $\tilde{\Psi}_j \in C^n(B[0,1])$ *and* $\tilde{\Psi}_j^{-1} \in C^n(\overline{U}_j)$ *for all* $j = 1,\dots,m$,

(iii) $\det \tilde{\Psi}_j'(u) \neq 0$ *for all* $|u| \le 1$, $j = 1,\dots,m$, *where* $\tilde{\Psi}_j'(u) \in \mathbb{R}^{3\times 3}$ *denotes the Jacobian of* $\tilde{\Psi}_j$ *at* u,

(iv) it holds that

$$\begin{aligned}
\tilde{\Psi}_j^{-1}(U_j \cap D) &= B_3^+(0,1) \;:=\; \{u \in \mathbb{R}^3 : |u| < 1,\, u_3 > 0\}\,,\\
\tilde{\Psi}_j^{-1}(U_j \cap \partial D) &= \{u \in \mathbb{R}^3 : |u| < 1,\, u_3 = 0\}\,.
\end{aligned}$$

We call $\{U_j, \tilde{\Psi}_j : j = 1,\dots,m\}$ a ***coordinate system*** *of ∂D.*

The restriction Ψ_j of the mapping $\tilde{\Psi}_j$ to $B_2[0,1] \times \{0\} \subset B_3[0,1]$ yields a parametrization of $\partial D \cap \overline{U}_j$ in the form $x = \Psi_j(u) = \tilde{\Psi}_j(u_1,u_2,0)$, $u \in B_2[0,1]$.
If Ψ is one of the mappings Ψ_j then $\frac{\partial\Psi}{\partial u_1}(u)$ and $\frac{\partial\Psi}{\partial u_2}(u)$, are tangential vectors at $x = \Psi(u)$.

They are linearly independent since $\det \tilde{\Psi}'(u_1, u_2, 0) \neq 0$ and, therefore, span the **tangent plane** at $x = \Psi(u)$. The unit vectors

$$\nu(x) \;=\; \pm \frac{\frac{\partial \Psi}{\partial u_1}(u) \times \frac{\partial \Psi}{\partial u_2}(u)}{\left|\frac{\partial \Psi}{\partial u_1}(u) \times \frac{\partial \Psi}{\partial u_2}(u)\right|}$$

for $x = \Psi(u) \in \partial D \cap U_j$ are orthogonal to the the tangent plane, thus normal vectors. The sign is chosen such that $\nu(x)$ is directed into the exterior of D (i.e. $x + t\nu(x) \in U_j \setminus D$ for small $t > 0$). The unit vector $\nu(x)$ is called the **exterior unit normal vector**. For such domains and continuous functions $f : \partial D \to \mathbb{C}$ the **surface integral** $\int_{\partial D} f(x)\, ds$ exists. First, we need the following tool:

Lemma 1.5 *For every covering $\{U_j : j = 1, \ldots, m\}$ of ∂D there exist $\phi_j \in C^\infty(\mathbb{R}^3)$ with* $\operatorname{supp}(\phi_j) \subset U_j$ *for all j and $\sum_{j=1}^m \phi_j(x) = 1$ for all $x \in \partial D$ (Partition of Unity).*

With such a partion of unity we write $\int_{\partial D} f(x)\, ds$ in the form $\int_{\partial D} f\, ds = \sum_{j=1}^m \int_{\partial D \cap U_j} \phi_j\, f\, ds = \sum_{j=1}^m \int_{\partial D \cap U_j} \tilde{f}_j(x)\, ds$ with $\tilde{f}_j = \phi_j f$. Using a coordinate system $\{U_j, \Psi_j : j = 1, \ldots, m\}$ of ∂D as in Definition 1.4 the integral over the surface patch $U_j \cap \partial D$ is given by

$$\int_{U_j \cap \partial D} \tilde{f}_j(x)\, ds \;=\; \int_{B_2(0,1)} \tilde{f}_j\big(\Psi_j(u)\big) \left|\frac{\partial \Psi_j}{\partial u_1}(u) \times \frac{\partial \Psi_j}{\partial u_2}(u)\right| du\,.$$

We collect important properties of the smooth domain D in the following lemma.

Lemma 1.6 *Let $D \in C^2$. Then there exists $c_0 > 0$ such that*

(a) $\big|\nu(y) \cdot (y - z)\big| \le c_0 |z - y|^2$ *for all $y, z \in \partial D$,*

(b) $\big|\nu(y) - \nu(z)\big| \le c_0 |y - z|$ *for all $y, z \in \partial D$.*

(c) Define

$$H_\rho \;:=\; \big\{ z + t\nu(z) : z \in \partial D\,,\ |t| < \rho \big\}\,.$$

Then there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$ and every $x \in H_\rho$ there exist unique (!) $z \in \partial D$ and $|t| \le \rho$ with $x = z + t\nu(z)$. The set H_ρ is an open neighborhood of ∂D for every $\rho \le \rho_0$. Furthermore, $z - t\nu(z) \in D$ and $z + t\nu(z) \notin \overline{D}$ for $0 < t < \rho$ and $z \in \partial D$.

One can choose ρ_0 such that for all $\rho \le \rho_0$ the following holds:

- $|z - y| \le 2|x - y|$ *for all $x \in H_\rho$ and $y \in \partial D$, and*
- $|z_1 - z_2| \le 2|x_1 - x_2|$ *for all $x_1, x_2 \in H_\rho$.*

If $U_\delta := \big\{ x \in \mathbb{R}^3 : \inf_{z \in \partial D} |x - z| < \delta \big\}$ denotes the strip around ∂D then there exists $\delta > 0$ with

$$\overline{U_\delta} \;\subset\; H_{\rho_0} \;\subset\; U_{\rho_0} \tag{1.29}$$

(d) There exists $r_0 > 0$ such that the surface area of $\partial B(z,r) \cap D$ for $z \in \partial D$ can be estimated by

$$\Big| |\partial B(z,r) \cap D| - 2\pi r^2 \Big| \;\leq\; 4\pi c_0\, r^3 \quad \textit{for all } r \leq r_0\,. \tag{1.30}$$

Proof: We make use of a finite covering $\bigcup U_j$ of ∂D, i.e. we write $\partial D = \bigcup (U_j \cap \partial D)$ and use local coordinates $\Psi_j : \mathbb{R}^2 \supset B_2[0,1] \to \mathbb{R}^3$ which yields the parametrization of $\partial D \cap U_j$. First, it is easy to see (proof by contradiction) that there exists $\delta > 0$ with the property that for every pair $(z,x) \in \partial D \times \mathbb{R}^3$ with $|z-x| < \delta$ there exists U_j with $z, x \in U_j$. Let $diam(D) = \sup\big\{|x_1 - x_2| : x_1, x_2 \in D\big\}$ be the diameter of D.
(a) Let $x, y \in \partial D$ and assume first that $|y-x| \geq \delta$. Then

$$\big|\nu(y)\cdot(y-x)\big| \;\leq\; |y-x| \;\leq\; \frac{diam(D)}{\delta^2}\,\delta^2 \;\leq\; \frac{diam(D)}{\delta^2}\,|y-x|^2\,.$$

Let now $|y-x| < \delta$. Then there exists U_j with $y, x \in U_j$. Let $x = \Psi_j(u)$ and $y = \Psi_j(v)$. Then

$$\nu(x) \;=\; \pm \frac{\frac{\partial \Psi_j}{\partial u_1}(u) \times \frac{\partial \Psi_j}{\partial u_2}(u)}{\left|\frac{\partial \Psi_j}{\partial u_1}(u) \times \frac{\partial \Psi_j}{\partial u_2}(u)\right|}$$

and, by the definition of the derivative,

$$y - x \;=\; \Psi_j(v) - \Psi_j(u) \;=\; \sum_{k=1}^{2} (v_k - u_k)\, \frac{\partial \Psi_j}{\partial u_k}(u) \;+\; a(v,u)$$

with $\big|a(v,u)\big| \leq c|u-v|^2$ for all $u, v \in U_j$ and some $c > 0$. Therefore,

$$\begin{aligned}
\big|\nu(x)\cdot(y-z)\big| \;&\leq\; \frac{1}{\left|\frac{\partial \Psi_j}{\partial u_1}(u) \times \frac{\partial \Psi_j}{\partial u_2}(u)\right|} \sum_{k=1}^{2} (v_k - u_k) \underbrace{\left|\left(\frac{\partial \Psi_j}{\partial u_1}(u) \times \frac{\partial \Psi_j}{\partial u_2}(u)\right)\cdot \frac{\partial \Psi_j}{\partial u_k}(u)\right|}_{=\,0} \\
&\quad + \frac{1}{\left|\frac{\partial \Psi_j}{\partial u_1}(u) \times \frac{\partial \Psi_j}{\partial u_2}(u)\right|} \left|\left(\frac{\partial \Psi_j}{\partial u_1}(u) \times \frac{\partial \Psi_j}{\partial u_2}(u)\right)\cdot a(v,u)\right| \\
&\leq\; c\,|u-v|^2 \;=\; c\left|\Psi_j^{-1}(x) - \Psi_j^{-1}(y)\right|^2 \;\leq\; c_0\,|x-y|^2\,.
\end{aligned}$$

This proves part (a). The proof of (b) follows analogously from the differentiability of $u \mapsto \nu$.

(c) Choose $\rho_0 > 0$ such that

(i) $\rho_0\, c_0 < 1/16$ and

(ii) $\nu(x_1)\cdot\nu(x_2) \geq 0$ for $x_1, x_2 \in \partial D$ with $|x_1 - x_2| \leq 2\rho_0$ and

(iii) $H_{\rho_0} \subset \bigcup U_j$.

Assume that x has two representation as $x = z_1 + t_1\nu_1 = z_2 + t_2\nu_2$ where we write ν_j for $\nu(z_j)$. Then

$$|z_1 - z_2| \;=\; \big|(t_2 - t_1)\,\nu_2 \,+\, t_1\,(\nu_2 - \nu_1)\big| \;\leq\; |t_1 - t_2| \,+\, \rho\, c_0 |z_1 - z_2| \;\leq\; |t_1 - t_2| \,+\, \frac{1}{16}\,|z_1 - z_2|\,,$$

thus $|z_1 - z_2| \le \frac{16}{15}|t_1 - t_2| \le 2|t_1 - t_2|$. Furthermore, since $\nu_1 \cdot \nu_2 \ge 0$,

$$(\nu_1 + \nu_2) \cdot (z_1 - z_2) \;=\; (\nu_1 + \nu_2) \cdot (t_2\nu_2 - t_1\nu_1) \;=\; (t_2 - t_1)\underbrace{(\nu_1 \cdot \nu_2 + 1)}_{\ge 1},$$

thus

$$|t_2 - t_1| \;\le\; \big|(\nu_1 + \nu_2) \cdot (z_1 - z_2)\big| \;\le\; 2c_0|z_1 - z_2|^2 \;\le\; 8c_0|t_1 - t_2|^2,$$

i.e. $|t_2 - t_1|\big(1 - 8c_0|t_2 - t_1|\big) \le 0$. This yields $t_1 = t_2$ since $1 - 8c_0|t_2 - t_1| \ge 1 - 16c_0\rho > 0$ and thus also $z_1 = z_2$.

Let U be one of the sets U_j and $\Psi : \mathbb{R}^2 \supset B_2(0,1) \to U \cap \partial D$ the corresponding bijective mapping. We define the new mapping $F : \mathbb{R}^2 \supset B_2(0,1) \times (-\rho, \rho) \to H_\rho$ by

$$F(u,t) \;=\; \Psi(u) \;+\; t\,\nu(u)\,, \quad (u,t) \in B_2(0,1) \times (-\rho, \rho)\,.$$

For sufficiently small ρ the mapping F is one-to-one and satisfies $\big|\det F'(u,t)\big| \ge \tilde{c} > 0$ on $B_2(0,1) \times (-\rho, \rho)$ for some $\tilde{c} > 0$. Indeed, this follows from

$$F'(u,t) \;=\; \left(\frac{\partial \Psi}{\partial u_1}(u) + t\frac{\partial \nu}{\partial u_1}(u)\,,\; \frac{\partial \Psi}{\partial u_2}(u) + t\frac{\partial \nu}{\partial u_2}(u)\,,\; \nu(u)\right)^\top$$

and the fact that for $t = 0$ the matrix $F'(u,0)$ has full rank 3. Therefore, F is a bijective mapping from $B_2(0,1) \times (-\rho, \rho)$ onto $U \cap H_\rho$. Therefore, $H_\rho = \bigcup(H_\rho \cap U_j)$ is an open neighborhood of ∂D. This proves also that $x = z - t\nu(z) \in D$ and $x = z + t\nu(z) \notin \overline{D}$ for $0 < t < \rho$.

For $x = z + t\nu(z)$ and $y \in \partial D$ we have

$$\begin{aligned} |x - y|^2 &= \big|(z - y) + t\nu(z)\big|^2 \;\ge\; |z - y|^2 + 2t(z - y) \cdot \nu(z) \\ &\ge |z - y|^2 - 2\rho c_0|z - y|^2 \\ &\ge \frac{1}{4}|z - y|^2 \quad \text{since} \quad 2\rho c_0 \le \frac{3}{4}\,. \end{aligned}$$

Therefore, $|z - y| \le 2|x - y|$. Finally,

$$\begin{aligned} |x_1 - x_2|^2 &= \big|(z_1 - z_2) + (t_1\nu_1 - t_2\nu_2)\big|^2 \;\ge\; |z_1 - z_2|^2 \;-\; 2\,\big|(z_1 - z_2) \cdot (t_1\nu_1 - t_2\nu_2)\big| \\ &\ge |z_1 - z_2|^2 \;-\; 2\,\rho\big|(z_1 - z_2) \cdot \nu_1\big| \;-\; 2\,\rho\big|(z_1 - z_2) \cdot \nu_2\big| \\ &\ge |z_1 - z_2|^2 \;-\; 4\,\rho\,c_0\,|z_1 - z_2|^2 \;=\; (1 - 4\rho c_0)\,|z_1 - z_2|^2 \;\ge\; \frac{1}{4}\,|z_1 - z_2|^2 \end{aligned}$$

since $1 - 4\rho c_0 \ge 1/4$.
The proof of (1.29) is simple and left as an exercise.

(d) Let c_0 and ρ_0 as in parts (a) and (c). Choose r_0 such that $B[z,r] \subset H_{\rho_0}$ for all $r \le r_0$ (which is possible by (1.29)) and $\nu(z_1) \cdot \nu(z_1) > 0$ for $|z_1 - z_2| \le 2r_0$. For fixed $r \le r_0$ and arbitrary $z \in \partial D$ and $\sigma > 0$ we define

$$Z(\sigma) \;=\; \big\{x \in \partial B(z,r) : (x - z) \cdot \nu(z) \le \sigma\big\}$$

We show that

$$Z(-2c_0r^2) \;\subset\; \partial B(z,r)\cap D \;\subset\; Z(+2c_0r^2)$$

Let $x \in Z(-2c_0r^2)$ have the form $x = x_0 + t\nu(x_0)$. Then

$$(x-z)\cdot\nu(z) \;=\; (x_0-z)\cdot\nu(z) + t\,\nu(x_0)\cdot\nu(z) \;\le\; -2c_0r^2\,,$$

i.e.

$$\begin{aligned} t\,\nu(x_0)\cdot\nu(z) \;&\le\; -2c_0r^2 \;+\; \big|(x_0-z)\cdot\nu(z)\big| \;\le\; -2c_0r^2 \;+\; c_0|x_0-z|^2 \\ &\le\; -2c_0r^2 \;+\; 2c_0|x-z|^2 \;=\; 0\,, \end{aligned}$$

i.e. $t \le 0$ since $|x_0 - z| \le 2r$ and thus $\nu(x_0)\cdot\nu(z) > 0$. This shows $x = x_0 + t\nu(x_0) \in D$. Analogously, for $x = x_0 - t\nu(x_0) \in \partial B(z,r)\cap D$ we have $t > 0$ and thus

$$(x-z)\cdot\nu(z) \;=\; (x_0-z)\cdot\nu(z) - t\,\nu(x_0)\cdot\nu(z) \;\le\; c_0|x_0-z|^2 \;\le\; 2c_0|x-z|^2 \;=\; 2c_0r^2\,.$$

Therefore, the surface area of $\partial B(z,r)\cap D$ is bounded from below and above by the surface areas of $Z(-2c_0r^2)$ and $Z(+2c_0r^2)$, respectively. Since the surface area of $Z(\sigma)$ is $2\pi r(r+\sigma)$ we have

$$-4\pi c_0\,r^3 \;\le\; |\partial B(z,r)\cap D| - 2\pi r^2 \;\le\; 4\pi c_0\,r^3\,.$$

□

Integral identities

Now we can formulate the mentioned integral identities. We do it only in $\mathbb{R}^3$. By $C^n(D)^3$ we denote the space of vector fields $F : D \to \mathbb{C}^3$ which are $n-$times continuously differentiable. By $C^n(\overline{D})^3$ we denote the subspace of $C^n(D)^3$ that consists of those functions F which, together with all derivatives up to order n, have continuous extentions to the closure $\overline{D}$ of D.

Theorem 1.7 *(Theorem of Gauss, Divergence Theorem)*

Let $D \subset \mathbb{R}^3$ be a bounded domain which is C^2-smooth. For $F \in C^1(D)^3 \cap C(\overline{D})^3$ the identity

$$\iint_D \operatorname{div}\, F(x)\,dx \;=\; \int_{\partial D} F_\nu(x)\,ds$$

holds. In particular, the integral on the left hand side exists.

As a conclusion one derives the theorems of Green.

Theorem 1.8 *(Green's first and second theorem)*

Let $D \subset \mathbb{R}^3$ be a bounded domain which is C^2-smooth. Furthermore, let $u, v \in C^2(D) \cap C^1(\overline{D})$. Then

$$\iint_D (u\,\Delta v + \nabla u\cdot\nabla v)\,dx \;=\; \int_{\partial D} u\,\frac{\partial v}{\partial\nu}\,ds\,,$$

$$\iint_D (u\,\Delta v - \Delta u\,v)\,dx \;=\; \int_{\partial D} \left(u\,\frac{\partial v}{\partial\nu} - v\,\frac{\partial u}{\partial\nu}\right)\,ds\,.$$

Here, $\partial u(x)/\partial\nu = \nu(x)\cdot\nabla u(x)$ for $x\in\partial D$.

Proof: The first identity is derived from the divergence theorem be setting $F = u\nabla v$. Then F satisfies the assumption of Theorem 1.7 and div $F = u\,\Delta v + \nabla u\cdot\nabla v$.
The second identity is derived by interchanging the roles of u and v in the first identity and taking the difference of the two formulas. □

We will also need their vector valued analoga.

Theorem 1.9 *(Integral identities for vector fields)*
Let $D\subset\mathbb{R}^3$ be a bounded domain which is C^2-smooth. Furthermore, let $A, B\in C^1(D)^3\cap C(\overline{D})^3$ and let $u\in C^2(D)\cap C^1(\overline{D})$. Then

$$\iint_D \operatorname{curl} A\,dx = \int_{\partial D}\nu\times A\,ds\,, \tag{1.31a}$$

$$\iint_D (B\cdot\operatorname{curl} A - A\cdot\operatorname{curl} B)\,dx = \int_{\partial D}(\nu\times A)\cdot B\,ds\,, \tag{1.31b}$$

$$\iint_D (u\ \operatorname{div}\ A + A\cdot\nabla u)\,dx = \int_{\partial D} u\,(\nu\cdot A)\,ds\,. \tag{1.31c}$$

Proof: For the first identity we consider the components separately. For the first one we have

$$\begin{aligned}\iint_D (\operatorname{curl} A)_1\,dx &= \iint_D\left(\frac{\partial A_3}{\partial x_2}-\frac{\partial A_2}{\partial x_3}\right)dx = \iint_D \operatorname{div}\begin{pmatrix}0\\ A_3\\ -A_2\end{pmatrix}dx\\ &= \int_{\partial D}\nu\cdot\begin{pmatrix}0\\ A_3\\ -A_2\end{pmatrix}ds = \int_{\partial D}(\nu\times A)_1\,ds\,.\end{aligned}$$

For the other components it is proven in the same way.
For the second equation we set $F = A\times B$. Then div $F = B\cdot\operatorname{curl} A - A\cdot\operatorname{curl} B$ and $\nu\cdot F = \nu\cdot(A\times B) = (\nu\times A)\cdot B$.
For the third identity we set $F = uA$ and have div $F = u$ div $A + A\cdot\nabla u$ and $\nu\cdot F = u(\nu\cdot A)$.
□

Surface Gradient and Surface Divergence

We have to introduce two more notions from differential geometry, the **surface gradient** and **surface divergence**.

To continue we do need a decomposition of the vector Laplace operator. To this end we define some tangential differential operators on the boundary ∂D of a C^2 smooth domain $D\subseteq\mathbb{R}^3$ (). As an abbreviation we introduce the notation

$$E = E_\nu\nu + E_\tau$$

for the normal component $E_\nu = E \cdot \nu$ and the projection $E_\tau = \nu \times (E \times \nu)$ on the tangential plane of a vectorfield E on a surface ∂D with unit normal vector field ν.

Definition 1.10 *Let $\varphi \in C^1(U)$ be defined in a neighborhood U of the C^2-boundary boundary ∂D of the domain D and $h \in C^1(U)$ be a tangential vector field, i.e. $h_\nu = 0$ for the unit normal field ν on $\partial D \cap U$.*
(a) The **surface gradient** *of φ is defined by*

$$\nabla_\tau \varphi = (\nabla\varphi)_\tau = \nabla\varphi - \frac{\partial\varphi}{\partial\nu}\nu. \tag{1.32}$$

(b) The **surface divergence** *of h is given by*

$$\mathrm{Div}(h) = \mathrm{div}(h) - \nu \cdot J_h \nu. \tag{1.33}$$

where J_h denotes the Jacobian matrix of h.

Example 1.11 *We parametrize the boundary of the unit sphere $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ by spherical coordinates*

$$\Psi(\theta,\phi) \;=\; \big(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\big)^\top .$$

Then the surface gradient and surface divergence on S^2 are given by

$$\begin{aligned}
\mathrm{Grad}\, f(\theta,\phi) \;&=\; \frac{\partial f}{\partial\theta}(\theta,\phi)\,\hat\theta \;+\; \frac{1}{\sin\theta}\frac{\partial f}{\partial\phi}(\theta,\phi)\,\hat\phi\,,\\
\mathrm{Div}\, F(\theta,\phi) \;&=\; \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\, F_\theta(\theta,\phi)\big) \;+\; \frac{1}{\sin\theta}\frac{\partial F_\phi}{\partial\phi}(\theta,\phi)\,,
\end{aligned}$$

where $\hat\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)^\top$ and $\hat\phi = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)^\top$ are the tangential vectors which span the tangent plane and F_θ, F_ϕ are the components of F w.r.t. these vectors, i.e. $F = F_\theta\hat\theta + F_\phi\hat\phi$. From this we note that

$$\mathrm{Div}\,\mathrm{Grad}\; f(\theta,\phi) \;=\; \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}(\theta,\phi)\right) \;+\; \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}(\theta,\phi)\,.$$

This differential operator is called the Laplace-Beltrami operator *and will be denoted by* $\Delta_S = \mathrm{Div}\,\mathrm{Grad}\,.$

Since ∂D is the closed boundary of a bounded domain applying the divergence theorem leads to teh surface divergence theorem.

Lemma 1.12 *The surface divergence defined in (1.33) satisfies*

$$\int_{\partial D} \mathrm{Div}(h)\, ds = 0.$$

Proof Let $\nu \in C^2(D) \cap C^1(\overline{D})$ be an extension of the normal vector ν to the bounded domain $D \subseteq \mathbb{R}^3$. The identity (1.28) leads to

$$\nu \cdot \mathrm{curl}(\nu \times h) = \mathrm{div}(h) - h_\nu \mathrm{div}(\nu) + \nu \cdot J_\nu h - \nu \cdot J_h \nu.$$

By differentiation of $\nu \cdot \nu = 1$ on ∂D we obtain $\nu \cdot J_\nu h = 0$. And $h \cdot \nu = 0$ on ∂D yields

$$\mathrm{Div}(h) = \nu \cdot \mathrm{curl}(\nu \times h). \tag{1.34}$$

Finally the divergence theorem implies

$$\int_{\partial D} \mathrm{Div}(h)\, ds = \int_D \mathrm{div}(\mathrm{curl}(\nu \times h))\, dx = 0\,.$$

□

From the product rule it follows that

$$\mathrm{Div}(\varphi h) = (\nabla_\tau \varphi) \cdot h + \varphi\, \mathrm{Div}(h).$$

The previous Lemma implies the partial integration formula

$$\int_{\partial D} \mathrm{Div}(h) \varphi\, ds = -\int_{\partial D} (\nabla_\tau \varphi) \cdot h\, ds. \tag{1.35}$$

Lemma 1.13 *The tangential gradient and the tangential divergence depend only on the values $\varphi|_{\partial D}$ and the tangential field $h|_{\partial D}$ on ∂D, respectively.*

Proof Let $\varphi_1, \varphi_2 \in C^\infty(U)$ be such that $\varphi = \varphi_1 - \varphi_2$ vanishes on ∂D. The identity (1.35) shows

$$\int_{\partial D} (\nabla_\tau \varphi) \cdot h\, ds = 0$$

for all $h \in C^1(U)$. Thus, choosing $h \in C^1(U)$ with $h|_{\partial D} = \nabla_\tau \varphi$ yields $\nabla_\tau \varphi = 0$ on ∂D. By a density argument follows the equation for continuously differentiable functions φ. Similarly the assertion for the tangential divergence is obtained. □

...

Corollary 1.14 *Let $w \in C^1(D)^3$ such that $w, \mathrm{curl}\, w \in C(\overline{D})^3$. Then the surface divergence of $\nu \times w$ exists and is given by*

$$\mathrm{Div}(\nu \times w) \;=\; -\nu \cdot \mathrm{curl}\, w \quad \text{on } \partial D\,. \tag{1.36}$$

Proof: Let $\varphi \in C^1(\overline{D})$ be arbitrary. By Gauss' theorem:

$$\begin{aligned}
\int_{\partial D} \varphi\, \nu \cdot \mathrm{curl}\, w\, ds &= \iint_D \mathrm{div}\,(\varphi\ \mathrm{curl}\, w)\, dx \;=\; \iint_D \nabla\varphi \cdot \mathrm{curl}\, w\, dx \\
&= \iint_D \mathrm{div}\,(w \times \nabla\varphi)\, dx \;=\; \int_{\partial D} \nu \cdot (w \times \nabla\varphi)\, ds \\
&= \int_{\partial D} (\nu \times w) \cdot \mathrm{Grad}\, \varphi\, ds \;=\; -\int_{\partial D} \mathrm{Div}(\nu \times w)\, \varphi\, ds\,.
\end{aligned}$$

The assertion follows since φ is arbitrary. □