The proof of $\big|(K_2\varphi)(x)\big| \leq c\|\varphi\|_{C^\alpha(\partial D)}$ is again simpler and is left again to the reader. □

We need compactness properties of boundary operators in Hölderspaces. This follows from the previous theorem and the compact imbedding of $C^\alpha(\partial D)$ in $C(\partial D)$.

Lemma 3.17 *The imbedding $C^\alpha(\partial D) \to C(\partial D)$ is compact for every $\alpha \in (0,1)$.*

Proof: We have to prove that the unit ball $B = \big\{\varphi \in C^\alpha(\partial D) : \|\varphi\|_{C^\alpha(\partial D)} \leq 1\big\}$ is relatively compact[5] in $C(\partial D)$. This follows directly from the theorem of Arcela-Ascoli (see [?]). Indeed, B is equi-continuous since

$$\big|\varphi(x_1) - \varphi(x_2)\big| \;\leq\; \|\varphi\|_{C^\alpha(\partial D)}|x_1 - x_2|^\alpha \;\leq\; |x_1 - x_2|^\alpha$$

for all $x_1, x_2 \in \partial D$. Furthermore, B is bounded. □

Corollary 3.18 *Under the assumptions of Theorem 3.16 the operator K_1 is compact from $C^\alpha(\partial D)$ into itself for every $\alpha \in (0,1)$.*

Proof: This follows immediately from the boundedness of K_1 from $C(\partial D)$ into $C^\alpha(\partial D)$ and the compactness of the imbedding $C^\alpha(\partial D)$ into $C(\partial D)$. □

We apply this result to the **boundary integral operators** which appear in the traces of the single and double layer potentials of Theorems 3.11, 3.13, and 3.15.

Theorem 3.19 *The operators $S, D, D' : C^\alpha(\partial D) \to C^\alpha(\partial D)$, defined by*

$$(S\varphi)(x) \;=\; \int_{\partial D} \varphi(y)\, \Phi_k(x,y)\, ds(y)\,, \quad x \in \partial D\,, \tag{3.25a}$$

$$(D\varphi)(x) \;=\; \int_{\partial D} \varphi(y)\, \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\, ds(y)\,, \quad x \in \partial D\,, \tag{3.25b}$$

$$(D'\varphi)(x) \;=\; \int_{\partial D} \varphi(y)\, \frac{\partial \Phi_k}{\partial \nu(x)}(x,y)\, ds(y)\,, \quad x \in \partial D\,, \tag{3.25c}$$

are well defined and compact. The operator S is bounded from $C^\alpha(\partial D)$ into $C^{1,\alpha}(\partial D)$

Proof: We have to check the assumptions (3.24a) and (3.24b) of Theorem 3.16. For $x, y \in \partial D$ we have by the definition of the fundamental solution Φ and part (a) of Lemma 1.6 that

$$\begin{aligned}
\big|\Phi(x,y)\big| &= \frac{1}{4\pi|x-y|}\,, \\
\left|\frac{\partial}{\partial\nu(y)}\Phi(x,y)\right| &= \frac{1}{4\pi|x-y|}\left|ik - \frac{1}{|x-y|}\right| \frac{\big|(y-x)\cdot\nu(y)\big|}{|x-y|} \\
&\leq \frac{c}{4\pi|x-y|}\left|ik - \frac{1}{|x-y|}\right| |x-y| \\
&\leq \frac{c}{4\pi|x-y|}\big[k|x-y| + 1\big] \;\leq\; \frac{c(kd+1)}{4\pi|x-y|}
\end{aligned}$$

[5] i.e. its closure is compact

where $d = \sup\{|x-y| : x, y \in \partial D\}$. The same stimate holds for $\partial\Phi(x,y)/\partial\nu(x)$. This proves (3.24a) with $\alpha = 1$. Furthermore, we will prove (3.24b) with $\alpha = 1$. Let $x_1, x_2, y \in \partial D$ such that $|x_1 - y| \geq 3|x_1 - x_2|$. Then, for any $t \in [0,1]$, we conclude that $|x_1 + t(x_2 - x_1) - y| \geq |x_1 - y| - |x_2 - x_1| \geq |x_1 - y| - |x_1 - y|/3 = 2|x_1 - y|/3$.
First we consider Φ and apply the mean value theorem:

$$\begin{aligned} \big|\Phi(x_1,y) - \Phi(x_2,y)\big| &\leq |x_1 - x_2| \sup_{0\leq t\leq 1} \big|\nabla_x \Phi\big(x_1 + t(x_2 - x_1), y\big)\big| \\ &\leq c \sup_{0\leq t\leq 1} \frac{|x_1 - x_2|}{|x_1 + t(x_2 - x_1) - y|^2} \leq c\,\frac{9}{4}\,\frac{|x_1 - x_2|}{|x_1 - y|^2}\,. \end{aligned}$$

To show the corresponding estimate for the normal derivative of the fundamental solution we can restrict ourselves to the case $k = 0$. Indeed, from the representation $\Phi_k(x,y) - \Phi_0(x,y) = A\big(|x-y|^2\big) + |x-y|\, B\big(|x-y|^2\big)$ with analytic functions A and B we observe that $\partial(\Phi_k - \Phi_0)/\partial\nu$ is continuous.
Let again $x_1, x_2, y \in \partial D$ such that $|x_1 - y| \geq 3|x_1 - x_2|$. Then

$$\begin{aligned} &\left|\frac{\partial}{\partial\nu(y)}\Phi_0(x_1,y) - \frac{\partial}{\partial\nu(y)}\Phi_0(x_2,y)\right| \\ &\leq \frac{1}{4\pi|x_1 - y|^3}\,\big|\underbrace{\nu(y)\cdot(y - x_1) - \nu(y)\cdot(y - x_2)}_{=\nu(y)\cdot(x_2 - x_1)}\big| \\ &\quad + \frac{1}{4\pi}\left|\frac{1}{|x_1 - y|^3} - \frac{1}{|x_2 - y|^3}\right| \big|\nu(y)\cdot(y - x_2)\big| \\ &\leq \frac{1}{4\pi|x_1 - y|^3}\Big[\big|(\nu(y) - \nu(x_1))\cdot(x_2 - x_1)\big| + \big|\nu(x_1)\cdot(x_2 - x_1)\big|\Big] \\ &\quad + \frac{1}{4\pi}\left|\frac{1}{|x_1 - y|^3} - \frac{1}{|x_2 - y|^3}\right| \big|\nu(y)\cdot(y - x_2)\big| \end{aligned}$$

Now we use estimates (a) and (b) of Lemma 1.6 for the first term and the mean value theorem for the second term. Using again $|x_1 + t(x_2 - x_1) - y| \geq 2|x_1 - y|/3$ we have

$$\left|\frac{\partial}{\partial\nu(y)}\Phi_0(x_1,y) - \frac{\partial}{\partial\nu(y)}\Phi_0(x_2,y)\right| \leq c\,\frac{|y - x_1||x_2 - x_1| + |x_1 - x_2|^2}{|x_1 - y|^3} + c\,\frac{|y - x_2|^2|x_1 - x_2|}{|x_1 - y|^4}$$

Estimate (3.24b) now follows from the estimates $|x_1 - x_2| \leq |x_1 - y|/3$ and $|y - x_2| \leq |y - x_1| + |x_1 - x_2| \leq 4|y - x_1|/3$. The proof for the normal derivative with respect to x follows the same arguments. Finally, we have to show that $\operatorname{Grad} S$ is bounded from $C^\alpha(\partial D)$ into $C^\alpha(\partial D)^3$. But this follows from the representation (3.23). □

3.1.5 Uniqueness and Existence

Now we come back to the scattering problem (3.1), (3.2) from the beginning of this section. First we study the question of uniqueness. The following lemma is fundamental for proving uniqueness and tells us, that a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for real and positive (!) k cannot decay faster than $1/|x|$ as x tends to infinity. We will give two proofs of this result. The first - and simpler - one uses the expansion arguments from the previous chapter. In particular, properties of the spherical Bessel- and Hankel functions are used. The second proof which goes back to the original work by Rellich (see [?]) avoids the use of these special functions but is far more technical and also needs a stronger assumption on the field. For completeness, we present both versions. We begin with the first form.

Lemma 3.20 *(Rellich's Lemma, first form) Let $u \in C^2(\mathbb{R}^3 \setminus B[0,R_0])$ be a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for $|x| > R_0$ and wave number $k \in \mathbb{R}_{>0}$ such that*

$$\lim_{R\to\infty} \int_{|x|=R} |u|^2 \, ds \;=\; 0\,.$$

Then u vanishes for $|x| > R_0$.

Proof: The general solution of the Helmholtz equation in the exterior of $B(0,R_0)$ is given by (2.37); that is,

$$u(r\hat{x}) \;=\; \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[a_n^m \, h_n^{(1)}(kr) + b_n^m \, j_n(kr)\right] Y_n^m(\hat{x})\,, \quad \hat{x} \in S^2\,,\; r > R\,,$$

for some $a_n^m, b_n^m \in \mathbb{C}$. The spherical harmonics $\left\{Y_n^m : |m| \le n,\; n \in \mathbb{N}_0\right\}$ form an orthogonal system. Therefore, Parceval's theorem yields

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m \, h_n^{(1)}(kr) + b_n^m \, j_n(kr)\right|^2 \;=\; \int_{S^2} \left|u(r\hat{x})\right|^2 ds(\hat{x})\,,$$

and from the assumption on u we note that $r^2 \int_{S^2} \left|u(r\hat{x})\right|^2 ds(\hat{x})$ tends to zero as r tends to infinity. Therefore, for every fixed $n \in \mathbb{N}_0$ and m with $|m| \le n$ we conclude that

$$r^2 \left|a_n^m \, h_n^{(1)}(kr) + b_n^m \, j_n(kr)\right|^2 \;\longrightarrow\; 0$$

as r tends to infinity. Defining $c_n^m = a_n^m + b_n^m$ we can write this as $(kr)\, i\, a_n^m \, y_n(kr) + (kr)\, c_n^m \, j_n(kr) \to 0$. Now we use the asymptotic behaviour of $j_n(kr)$ and $y_n(kr)$ as r tends to infinity. From Theorem 2.28 we conclude that

$$i\, a_n^m \,\mathrm{Im}\left[e^{ikr}(-i)^{n+1}\right] \;+\; c_n^m \,\mathrm{Re}\left[e^{ikr}(-i)^{n+1}\right] \;\longrightarrow\; 0\,.$$

The term $(-i)^{n+1}$ can take the values ± 1 and $\pm i$. Therefore, we have that (depending on n)

$$i\, a_n^m \sin(kr) \;+\; c_n^m \cos(kr) \;\longrightarrow\; 0 \quad \text{or} \quad i\, a_n^m \cos(kr) \;-\; c_n^m \sin(kr) \;\longrightarrow\; 0\,.$$

In any case, a_n^m and c_n^m have to vanish by taking particular sequences $r_j \to \infty$. This shows that also $b_n^m = 0$. Since this holds for all n and m we conclude that u vanishes. □

The second proof avoids the use of the Bessel and Hankel functions but needs, however, a stronger assumption on u.

Lemma 3.21 *(Rellich's Lemma, second form) Let $u \in C^2(\mathbb{R}^3 \setminus B[0,R_0])$ be a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for $|x| > R_0$ with wave number $k \in \mathbb{R}_{>0}$ such that*

$$\lim_{r\to\infty} \int_{|x|=r} |u|^2\, ds \;=\; 0 \quad \textit{and} \quad \lim_{r\to\infty} \int_{|x|=r} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 ds \;=\; 0\,.$$

Then u vanishes for $|x| > R_0$.

Proof:[6] The proof is lengthy, and we will structure it. We assume that u is real valued (take real and imaginary parts separately).

1st step: Transforming the integral onto the unit sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ we conclude that

$$\int_{|\hat{x}|=1} \big|u(r\hat{x})|^2\, r^2\, ds(\hat{x}) = \int_{|x|=r} \big|u(x)\big|^2\, ds(x) \quad \text{and} \quad \int_{|\hat{x}|=1} \left|\frac{\partial u}{\partial r}(r\hat{x})\right|^2 r^2\, ds(\hat{x}) \tag{3.26}$$

tend to zero as r tends to infinity. We transform the partial differential equation into an ordinary differential equation (not quite!) for the function $v(r,\hat{x}) = r\,u(r,\hat{x})$ w.r.t. r. We write $v(r)$ and $v'(r)$ and $v''(r)$ for $v(r,\cdot)$ and $\partial v(r,\cdot)/\partial r$ and $\partial^2 v(r,\cdot)/\partial r^2$, respectively. Then (3.26) yields that $\|v(r)\|_{L^2(S^2)} \to 0$ and $\|v'(r)\|_{L^2(S^2)} \to 0$ as $r \to \infty$. The latter follows from $\frac{\partial}{\partial r}\big(ru(r,\cdot,\cdot)\big) = \frac{1}{r}\,\big(ru(r,\cdot)\big) + r\frac{\partial u}{\partial r}(r,\cdot)$ and the triangle inequality.
We observe that $u = \frac{1}{r}v$, thus $r^2\frac{\partial u}{\partial r} = -v + r\frac{\partial v}{\partial r}$ and $\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = r\frac{\partial^2 v}{r^2}$, thus

$$\begin{aligned} 0 \;&=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}(r,\theta,\phi)\right) \;+\; \frac{1}{r^2}\,\Delta_S u(r,\theta,\phi) \;+\; k^2 u(r,\theta,\phi) \\ &=\; \frac{1}{r}\left[\frac{\partial^2 v}{\partial r^2}(r,\theta,\phi) \;+\; k^2 v(r,\theta,\phi) \;+\; \frac{1}{r^2}\,\Delta_S v(r,\theta,\phi)\right], \end{aligned}$$

i.e.

$$v''(r) \;+\; k^2 v(r) \;+\; \frac{1}{r^2}\,\Delta_S v(r) \;=\; 0 \quad \text{for } r \geq R_0\,, \tag{3.27}$$

where again $\Delta_S = \operatorname{Div}\operatorname{Grad}$ denotes the Laplace-Beltrami operator; that is, in polar coordinates $\hat{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^\top$

$$(\Delta_S w)(\theta,\phi) \;=\; \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial w}{\partial\theta}(\theta,\phi)\right) \;+\; \frac{1}{\sin^2\theta}\,\frac{\partial^2 w}{\partial\phi^2}(\theta,\phi)$$

for any $w \in C^2(S^2)$. It is easily seen either by direct integration or by application of Theorem **??** that Δ_S is selfadjoint and negative definite, i.e.

$$\big(\Delta_S v, w\big)_{L^2(S^2)} = \big(v, \Delta_S w\big)_{L^2(S^2)} \quad \text{and} \quad \big(\Delta_S v, v\big)_{L^2(S^2)} \leq 0 \quad \text{for all } v, w \in C^2(S^2)\,.$$

[6] We took the proof from the monograph *Partielle Differentialgleichungen zweiter Ordnung* by R. Leis

2nd step: We introduce the functions E, v_m and F by

$$\begin{aligned} E(r) &:= \|v'(r)\|^2_{L^2(S^2)} \;+\; k^2\|v(r)\|^2_{L^2(S^2)} \;+\; \frac{1}{r^2}\big(\Delta_S v(r), v(r)\big)_{L^2(S^2)}\,, \quad r \geq R_0\,, \\ v_m(r) &:= r^m v(r)\,, \quad r \geq R_0\,,\; m \in \mathbb{N}\,, \\ F(r,m,c) &:= \|v_m'(r)\|^2_{L^2(S^2)} + \left(k^2 + \frac{m(m+1)}{r^2} - \frac{2c}{r}\right)\|v_m(r)\|^2_{L^2(S^2)} \\ &\quad + \frac{1}{r^2}\big(\Delta_S v_m(r), v_m(r)\big)_{L^2(S^2)}\,, \end{aligned}$$

for $r \geq R_0$, $m \in \mathbb{N}$, $c \geq 0$. In the following we write $\|\cdot\|$ and $(\cdot,\cdot)$ for $\|\cdot\|_{L^2(S^2)}$ and $(\cdot,\cdot)_{L^2(S^2)}$, respectively. We show:

(a) E satisfies $E'(r) \geq 0$ for all $r \geq R_0$.

(b) The functions v_m solve the differential equation

$$v_m''(r) \;-\; \frac{2m}{r}\,v_m'(r) \;+\; \left(\frac{m(m+1)}{r^2} + k^2\right) v_m(r) \;+\; \frac{1}{r^2}\,\Delta_S v_m(r) \;=\; 0\,. \tag{3.28}$$

(c) For every $c > 0$ there exist $r_0 = r_0(c) \geq R_0$ and $m_0 = m_0(c) \in \mathbb{N}$ such that

$$\frac{\partial}{\partial r}\big[r^2 F(r,m,c)\big] \;\geq\; 0 \quad \text{for all } r \geq r_0,\; m \geq m_0\,.$$

(d) Expressed in terms of v the function F has the forms

$$\begin{aligned} F(r,m,c) &= r^{2m}\left\{\left\|v'(r) + \frac{m}{r}v(r)\right\|^2 + \left(k^2 + \frac{m(m+1)}{r^2} - \frac{2c}{r}\right)\|v(r)\|^2 \right. \\ &\quad \left. + \;\frac{1}{r^2}\big(\Delta_S v(r), v(r)\big)\right\} & (3.29a) \\ &= r^{2m}\left\{E(r) \;+\; \frac{2m}{r}\big(v(r), v'(r)\big) \;+\; \left(\frac{m(2m+1)}{r^2} - \frac{2c}{r}\right)\|v(r)\|^2\right\} & (3.29b) \end{aligned}$$

Proof of these statements:
(a) We just differentiate E and substitute the second derivative from (3.27). Note that $\frac{d}{dr}\|v(r)\|^2 = 2(v,v')$ and $\frac{d}{dr}\big(\Delta_S v, v\big) = 2\big(\Delta_S v, v'\big)$:

$$\begin{aligned} E'(r) &= 2\big(v'(r), v''(r)\big) \;+\; 2k^2\big(v(r), v'(r)\big) \;-\; \frac{1}{r^3}\big(\Delta_S v(r), v(r)\big) \;+\; \frac{2}{r^2}\big(\Delta_S v(r), v'(r)\big) \\ &= 2\left(v'(r)\,,\, \left[v''(r) + k^2 v(r) + \frac{1}{r^2}\,\Delta_S v(r)\right]\right) \;-\; \frac{1}{r^3}\big(\Delta_S v(r), v(r)\big) \\ &= -\frac{1}{r^3}\big(\Delta_S v(r), v(r)\big) \;\geq\; 0\,. \end{aligned}$$

(b) We substitute $v(r) = r^{-m} v_m(r)$ into (3.27) and obtain directly (3.28). We omit the calculation.

(c) Again we differentiate $r^2 F(r,m,c)$ w.r.t. r, substitute the form of v_m'' from (3.28) and obtain

$$\begin{aligned}
\frac{\partial}{\partial r}\big[r^2 F(r,m,c)\big] &= 2r\|v_m'(r)\|^2 \;+\; 2r^2\big(v_m'(r), v_m''(r)\big) \;+\; 2(k^2 r - c)\|v_m(r)\|^2 \\
&\quad + 2r^2\left(k^2 + \frac{m(m+1)}{r^2} - \frac{2c}{r}\right)\big(v_m(r), v_m'(r)\big) \;+\; 2\big(\Delta_S v_m(r), v_m'(r)\big) \\
&= \cdots = 2r(1+2m)\|v_m'(r)\|^2 - 4cr\big(v_m'(r), v_m(r)\big) \;+\; 2(k^2 r - c)\|v_m(r)\|^2 \\
&= 2r\left[\left\|\sqrt{1+2m}\, v_m'(r) - \frac{c}{\sqrt{1+2m}}\, v_m(r)\right\|^2 + \left(k^2 - \frac{c}{r} - \frac{c^2}{1+2m}\right)\|v_m(r)\|^2\right].
\end{aligned}$$

From this the assertion (c) follows if r_0 and m_0 are chosen such that the bracket $(\cdots)$ is positive.
(d) The first equation is easy to see by just inserting the form of v_m. For the second form one uses simply the binomial theorem for the first term and the definition of $E(r)$.

3rd step: We begin with the actual proof of the lemma and show first that there exists $R_1 \geq R_0$ such that $\|v(r)\| = 0$ for all $r \geq R_1$. Assume, on the contrary, that this is not the case. Then, for every $R \geq R_0$ there exists $\hat{r} \geq R$ such that $\|v(\hat{r})\| > 0$.
We choose the constants $\hat{c} > 0$, r_0, m_0, r_1, m_1 in the following order:

- Choose $\hat{c} > 0$ with $k^2 - \frac{2\hat{c}}{R_0} > 0$.
- Choose $r_0 = r_0(\hat{c}) \geq R_0$ and $m_0 = m_0(\hat{c}) \in \mathbb{N}$ according to property (c) above, i.e. such that $\frac{\partial}{\partial r}\big[r^2 F(r,m,\hat{c})\big] \geq 0$ for all $r \geq r_0$ and $m \geq m_0$.
- Choose $r_1 > r_0$ such that $\|v(r_1)\| > 0$.
- Choose $m_1 \geq m_0$ such that $m_1(m_1+1)\|v(r_1)\|^2 + \big(\Delta_S v(r_1), v(r_1)\big) > 0$.

Then, by (3.29a) and since $k^2 - \frac{2\hat{c}}{r_1} \geq k^2 - \frac{2\hat{c}}{R_0} > 0$, it follows that $F(r_1, m_1, \hat{c}) > 0$ and thus, by the monotonicity of $r \mapsto r^2 F(r, m_1, \hat{c})$ that also $F(r, m_1, \hat{c}) > 0$ for all $r \geq r_1$. Therefore, from (3.29b) we conclude that, for $r \geq r_1$,

$$\begin{aligned}
0 \;<\; r^{-2m_1} F(r, m_1, \hat{c}) &= E(r) \;+\; \frac{2m_1}{r}\big(v(r), v'(r)\big) \;+\; \left(\frac{m_1(2m_1+1)}{r^2} - \frac{2\hat{c}}{r}\right)\|v(r)\|^2 \\
&= E(r) \;+\; \frac{m_1}{r}\frac{d}{dr}\|v(r)\|^2 \;+\; \frac{1}{r}\left(\frac{m_1(2m_1+1)}{r} - 2\hat{c}\right)\|v(r)\|^2 .
\end{aligned}$$

Choose now $r_2 \geq r_1$ such that $\frac{m_1(2m_1+1)}{r_2} - 2\hat{c} < 0$. Finally, choose $\hat{r} \geq r_2$ such that $\frac{d}{dr}\|v(\hat{r})\|^2 \leq 0$. (This is possible since $\|v(r)\|^2 \to 0$ as $r \to \infty$.) We finally have

$$0 \;<\; p \;:=\; \hat{r}^{-2m_1} F(\hat{r}, m_1, \hat{c}) \;\leq\; E(\hat{r}) .$$

By the monotonicity of E we conclude that $E(r) \geq p$ for all $r \geq \hat{r}$. On the other hand, by the definition of $E(r)$ we have that $E(r) \leq \|v'(r)\|^2 + k^2 \|v(r)\|^2$ and this tends to zero as r tends to infinity. This is a contradiction. Therefore, there exists $R_1 \geq R_0$ with $v(r) = 0$ for all $r \geq R_1$ and thus also Re $u = 0$ for $|x| \geq R_1$. The same holds for Im u and thus $u = 0$ for $|x| \geq R_1$. □

We can now prove uniqueness of the scattering problem.

Theorem 3.22 *For any incident field u^{inc} there exists at most one solution $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ of the scattering problem (3.1), (3.2).*

Proof: Let u be the difference of two solutions. Then u satisfies (3.1) and also the radiation condition (3.2). From the radiation condition we conclude that

$$\int\limits_{|x|=R} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds \;=\; \int\limits_{|x|=R} \left\{ \left| \frac{\partial u}{\partial r} \right|^2 + k^2 |u|^2 \right\} ds \;+\; 2k \,\mathrm{Im} \int\limits_{|x|=R} u \, \frac{\partial \overline{u}}{\partial r} \, ds$$

tends to zero as R tends to infinity. Green's theorem, applied in $B_R \setminus D$ to the function u yields that

$$\int\limits_{|x|=R} u \, \frac{\partial \overline{u}}{\partial r} \, ds \;=\; \int\limits_{\partial D} u \, \frac{\partial \overline{u}}{\partial r} \, ds \;+\; \iint\limits_{B_R \setminus D} \left[|\nabla u|^2 - k^2 |u|^2 \right] dx \;=\; \iint\limits_{B_R \setminus D} \left[|\nabla u|^2 - k^2 |u|^2 \right] dx$$

since the surface integral over ∂D vanishes by the boundary condition. The volume integral is real valued. Therefore, its imaginary part vanishes and we conclude that

$$\int\limits_{|x|=R} \left| \frac{\partial u}{\partial r} \right|^2 + k^2 |u|^2 ds$$

tends to zero as R tends to infinity. Rellich's lemma (in the form Lemma 3.20 or Lemma 3.21) implies that u vanishes outside of every ball which encloses ∂D. Finally, we note that u is an analytic function in the exterior of D. Since the exterior of D is connected we conclude that u vanishes in $\mathbb{R}^3 \setminus D$. □

We turn to the question of **existence** and choose the integral equation method for its treatment. We follow again the approach of [**?**] but prefer to work in the space $C^\alpha(\partial D)$ of Hölder continuous functions rather that in the space of merely continuous functions. This avoids the necessity to introduce the class of continuous functions for which the normal derivatives exist "in the uniform sense along the normal".

We recall the notion of the single layer potentials of (3.4a), see also (3.12), and make the ansatz for the scattered field in the form of a single layer potential. We remark already here that we will face some difficulties with this ansatz. Before we modify the ansatz below we try the single layer ansatz for the scattered field in the form

$$u^s(x) \;=\; \int_{\partial D} \varphi(y) \, \Phi(x,y) \, ds(y) \, , \quad x \in \mathbb{R}^3 \setminus \partial D \, , \tag{3.30}$$

where again $\Phi(x,y) = \exp(ik|x-y|)/(4\pi|x-y|)$ denotes the fundamental solution of the Helmholtz equation, and $\varphi \in C^\alpha(\partial D)$ is some density to be determined. First we note that u^s solves the Helmholtz equation in the exterior of $\overline{D}$ and also the radiation condition. This follows from the corresponding properties of the fundamental solution $\Phi(\cdot, y)$, uniformly with respect to y on the compact surface ∂D. Furthermore, by Theorems 3.11 and 3.15 the function u^s and its derivatives can be extended continuously (from the exterior) into $\mathbb{R}^3 \setminus D$ with limiting values

$$u^s(x)\big|_+ \;=\; \int_{\partial D} \varphi(y)\, \Phi(x,y)\, ds(y) \;=\; (S\varphi)(x)\,, \quad x \in \partial D\,, \tag{3.31a}$$

$$\begin{aligned} \frac{\partial u^s}{\partial \nu}(x)\bigg|_+ \;&=\; -\frac{1}{2}\,\varphi(x) \;+\; \int_{\partial D} \varphi(y)\, \frac{\partial}{\partial \nu(x)} \Phi(x,y)\, ds(y) \\ &=\; -\frac{1}{2}\,\varphi(x) \;+\; (D'\varphi)(x)\,, \quad x \in \partial D\,, \end{aligned} \tag{3.31b}$$

where we used the notations of the boundary integral operators from Theorem 3.19. Therefore, in order that $u = u^{inc} + u^s$ satisfies the boundary condition $\partial u/\partial \nu = 0$ on ∂D the density φ has to satisfy the boundary integral equation

$$-\frac{1}{2}\varphi \;+\; D'\varphi \;=\; -\frac{\partial u^{inc}}{\partial \nu} \quad \text{in } C^\alpha(\partial D)\,. \tag{3.32}$$

By Theorem 3.19 the operator D' is compact. Therefore, we can apply the Fredholm theory. In particular, existence follows from uniqueness. To prove uniqueness we assume that $\varphi \in C^\alpha(\partial D)$ satisfies the homogeneous equation $-\frac{1}{2}\varphi + D'\varphi = 0$. Define v to be the single layer potential with density φ just as in (3.30), but for arbitrary $x \notin \partial D$. Then, again from the jump contions of the normal derivative of the single layer, $\partial v/\partial \nu\big|_+ = -\frac{1}{2}\varphi + D'\varphi = 0$. Therefore, v is the solution of the exterior Neumann problem with vanishing boundary data. The uniqueness result of Theorem 3.22 yields that v vanishes in the exterior of D. Furthermore, v is continuous in $\mathbb{R}^3$, thus v is a solution of the Helmholtz equation in D with vanishing boundary data. This is the point where we wish to conclude that v vanishes also in D. However, this is not always the case. Indeed, this is not the case if, and only if, k^2 is an eigenvalue of $-\Delta$ in D with respect to Dirichlet boundary conditions. This is the reason why we have to modify the ansatz (3.30). There are several ways how to do it, see the discussion in [**?**]. We choose a modification which we have not found in the literature. It avoids the use of double layer potentials. We assume for simplicity that D is connected although this is not necessary as one observes from the following arguments.

We choose an open ball B with boundary Γ such that $\Gamma \subset D$ and such that k^2 is not an eigenvalue of $-\Delta$ inside B with respect to Dirichlet boundary conditions. By Theorem 2.30 from the previous chapter we observe that we have to choose the radius ρ of B such that $k\rho$ is not a zero of any of the Bessel functions j_n.

Now we make an ansatz for u^s as a sum of two single layer potentials in the form

$$u^s(x) \;=\; (\tilde{S}_{\partial D}\varphi)(x) + (\tilde{S}_\Gamma \psi)(x) \;=\; \int_{\partial D} \varphi(y)\, \Phi(x,y)\, ds(y) + \int_\Gamma \psi(y)\, \Phi(x,y)\, ds(y)\,, \quad x \notin \overline{D}\,, \tag{3.33}$$

where $\phi \in C^\alpha(\partial D)$ and $\psi \in C^\alpha(\Gamma)$ are two densities to be determined from the system of two boundary integral equations

$$-\frac{1}{2}\varphi + D'\varphi + \frac{\partial}{\partial\nu}\tilde{S}_\Gamma\psi = -\frac{\partial u^{inc}}{\partial\nu} \quad \text{on } \partial D\,, \tag{3.34a}$$

$$\left(\frac{\partial}{\partial\nu} + ik\right)\tilde{S}_{\partial D}\varphi - \frac{1}{2}\psi + D'_\Gamma\psi + ik\,S_\Gamma\psi = 0 \quad \text{on } \Gamma\,. \tag{3.34b}$$

The operators S_Γ and D'_Γ denote the boundary operators S and D', respectively, on the boundary Γ instead of ∂D. These two equations can be written in matrix form as

$$-\frac{1}{2}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \begin{pmatrix} D' & \partial\tilde{S}_\Gamma/\partial\nu \\ (\partial/\partial\nu + ik)\tilde{S}_{\partial D} & D'_\Gamma + ik\,S_\Gamma\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = -\begin{pmatrix}\partial u^{inc}/\partial\nu \\ 0\end{pmatrix}$$

in $C^\alpha(\partial D) \times C^\alpha(\Gamma)$. The operators D', D'_Γ, $\partial\tilde{S}_\Gamma/\partial\nu$, and $(\partial/\partial\nu + ik)\tilde{S}_{\partial D}$ are all compact. Therefore, we can apply the Fredholm alternative to this system. Existence is assured if the homogeneous system admits only the trivial solution $\varphi = 0$ and $\psi = 0$. Therefore, let $(\varphi, \psi) \in C^\alpha(\partial D) \times C^\alpha(\Gamma)$ be a solution of the homogeneous system and define the v as the sum of the single layers with densities φ and ψ for all x in $\mathbb{R}^3 \setminus (\partial D \cup \Gamma)$. From the the jump condition for the normal derivative and the first (homogeneous) integral equation we conclude – just in the above case of only one single layer potential – that $\partial v/\partial\nu\big|_+ = -\frac{1}{2}\varphi + D'\varphi + \partial\tilde{S}_\Gamma\psi/\partial\nu = 0$. Again, v is a solution of the exterior Neumann problem with vanishing boundary data. Therefore, by the uniqueness theorem, v vanishes in the exterior of D. Furthermore, v is continuous in $\mathbb{R}^3$ and satisfies also the Helmholtz equation in $D \setminus \overline{B}$. From the jump conditions on the boundary Γ we conclude that

$$\frac{\partial v}{\partial\nu}\bigg|_+ + ikv = \left(\frac{\partial}{\partial\nu} + ik\right)\tilde{S}_{\partial D}\varphi - \frac{1}{2}\psi + D'_\Gamma\psi + ik\,S_\Gamma\psi = 0 \quad \text{on } \Gamma\,.$$

Therefore, $v = 0$ on ∂D and $\partial v/\partial\nu|_+ + ikv = 0$ on Γ. Application of Green's first theorem in $D \setminus \overline{B}$ yields

$$\iint\limits_{D\setminus\overline{B}} \left[|\nabla v|^2 - k^2|v|^2\right] dx = \int\limits_{\partial D} \overline{v}\,\frac{\partial v}{\partial\nu}\,ds - \int\limits_{\Gamma} \overline{v}\,\frac{\partial v}{\partial\nu}\bigg|_+ ds = ik\int\limits_{\Gamma} |v|^2\,ds\,.$$

Taking the imaginary part yields that v vanishes on Γ and therefore also $\partial v/\partial\nu|_+ = 0$ on Γ. Holmgren's uniqueness Theorem 3.5 implies that v vanishes in all of $D \setminus B$. The jump conditions on ∂D yield

$$0 = \frac{\partial v}{\partial\nu}\bigg|_- - \frac{\partial v}{\partial\nu}\bigg|_+ = \varphi \quad \text{on } \partial D\,.$$

Therefore, v is a single layer potential on Γ with density ψ and vanishes on Γ. The wave number k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B by the choice of the radius of B. Therefore, v vanishes also in B. The jump conditions on Γ yield

$$0 = \frac{\partial v}{\partial\nu}\bigg|_- - \frac{\partial v}{\partial\nu}\bigg|_+ = \psi \quad \text{on } \Gamma\,.$$

Therefore, $\varphi = 0$ on ∂D and $\psi = 0$ on Γ.

If D consists of several components $D = \bigcup_{m=1}^{M} D_m$ then one has to choose balls B_m in each of the domains D_m and make an ansatz as a sum of single layers on ∂D and ∂B_m for $m = 1, \ldots, M$.

Application of Fredholm's alternative yields the following result:

Theorem 3.23 *There exists a unique solution* $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ *of the scattering problem (3.1), (3.2).*

3.2 A Scattering Problem for the Maxwell System

3.2.1 Formulation of the Problem

For this chapter we make the following assumptions on the data:

Assumption: Let the wave number be given by $k = \omega\sqrt{\varepsilon_0\mu_0} > 0$ with constants $\varepsilon_0, \mu_0 > 0$. Let $D \subset \mathbb{R}^3$ be bounded and C^2-smooth such that the complement $\mathbb{R}^3 \setminus \overline{D}$ is connected.

Scattering Problem: Given a solution (E^i, H^i) of the Maxwell system

$$\operatorname{curl} E^i - i\omega\mu_0 H^i = 0\,, \quad \operatorname{curl} H^i + i\omega\varepsilon_0 E^i = 0 \text{ in some neighborhood of } D\,,$$

determine the total fields $E, H \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ such that

$$\operatorname{curl} E - i\omega\mu_0 H = 0 \quad \text{and} \quad \operatorname{curl} H + i\omega\varepsilon_0 E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}\,, \tag{3.35a}$$

E satisfies the boundary condition

$$\nu \times E \;=\; 0 \quad \text{on } \partial D\,, \tag{3.35b}$$

and the radiating parts $E^s = E - E^i$ and $H^s = H - H^i$ satisfy the Silver Müller radiation conditions

$$\sqrt{\varepsilon_0}\, E^s(x) \;-\; \sqrt{\mu_0}\, H^s(x) \times \frac{x}{|x|} \;=\; \mathcal{O}\left(\frac{1}{|x|^2}\right)\,, \tag{3.36a}$$

and

$$\sqrt{\mu_0}\, H^s(x) \;+\; \sqrt{\varepsilon_0}\, E^s(x) \times \frac{x}{|x|} \;=\; \mathcal{O}\left(\frac{1}{|x|^2}\right)\,, \tag{3.36b}$$

uniformly w.r.t. $x/|x|$. Clearly, after renaming the unknown fields, this is a special case of the following problem:

Exterior Boundary Value Problem: Given a tangential field[7] $c \in C^\alpha(\partial D)^3$ such that $\operatorname{Div} c \in C^\alpha(\partial D)$ determine radiating[8] solutions $E^s, H^s \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ of

$$\operatorname{curl} E^s - i\omega\mu_0 H^s = 0 \quad \text{and} \quad \operatorname{curl} H^s + i\omega\varepsilon_0 E^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}\,, \tag{3.37a}$$

[7] i.e. $\nu(x) \cdot c(x) = 0$ on ∂D

[8] i.e. E^s, H^s satisfy the radiating conditions (3.36a) and (3.36b)

such that E^s satisfies the boundary condition

$$\nu \times E^s \;=\; c \quad \text{on } \partial D\,. \tag{3.37b}$$

We note again that the assumption on the surface divergence of c is necessary by Corollary 1.16.

3.2.2 Representation Theorems

We have seen in the previous section (Theorem 3.3) that every sufficiently smooth function u can be written as a sum of a volume potential with density $\Delta u + k^2 u$, a single layer potential with density $\partial u/\partial\nu$, and a double layer potential with potential u. In this subsection we show the corresponding theorem for vector fields.

Theorem 3.24 *Let $k \in \mathbb{C}$ and $E \in C^1(D)^3 \cap C(\overline{D})^3$ such that $\operatorname{curl} E \in C(\overline{D})^3$ and $\operatorname{div} E \in C(\overline{D})$. Then we have for $x \in D$:*

$$\begin{aligned} E(x) \;&=\; \operatorname{curl} \iint_D \Phi(x,y)\, \operatorname{curl} E(y)\, dy - \nabla \iint_D \Phi(x,y)\, \operatorname{div} E(y)\, dy - k^2 \iint_D E(y)\, \Phi(x,y)\, dy \\ &\quad - \operatorname{curl} \int_{\partial D} \big[\nu(y) \times E(y)\big]\, \Phi(x,y)\, ds(y) \;+\; \nabla \int_{\partial D} \big[\nu(y) \cdot E(y)\big]\, \Phi(x,y)\, ds(y) \end{aligned}$$

where the domain integrals exist as improper integrals. The right hand side of this equation vanishes for $x \notin \overline{D}$ and is equal to $\frac{1}{2}\, E(x)$ for $x \in \partial D$.

Proof: Fix $z \in D$, choose $r > 0$ such that $B[z,r] \subset D$, and set $D_r = D \setminus B[z,r]$. For $x \in B(z,r)$ we set

$$\begin{aligned} I_r(x) \;&:=\; \operatorname{curl} \iint_D \Phi(x,y)\, \operatorname{curl} E(y)\, dy - \nabla \iint_D \Phi(x,y)\, \operatorname{div} E(y)\, dy - k^2 \iint_D E(y)\, \Phi(x,y)\, dy \\ &\quad - \operatorname{curl} \int_{\partial D} \big[\nu(y) \times E(y)\big]\, \Phi(x,y)\, ds(y) \;+\; \nabla \int_{\partial D} \big[\nu(y) \cdot E(y)\big]\, \Phi(x,y)\, ds(y) \\ &=\; \iint_{D_r} \nabla_x \Phi(x,y) \times \operatorname{curl} E(y)\, dy - \iint_{D_r} \nabla_x \Phi(x,y)\, \operatorname{div} E(y)\, dy - k^2 \iint_{D_r} E(y)\, \Phi(x,y)\, dy \\ &\quad - \operatorname{curl} \int_{\partial D_r} \big[\nu(y) \times E(y)\big]\, \Phi(x,y)\, ds(y) \;+\; \nabla \int_{\partial D_r} \big[\nu(y) \cdot E(y)\big]\, \Phi(x,y)\, ds(y)\,. \end{aligned}$$

We show that $I_r(x)$ vanishes. Indeed, we can interchange differentiation and integration and write

$$\begin{aligned}
I_r(x) &= \operatorname{curl}\left[\iint_{D_r} \Phi(x,y)\ \operatorname{curl} E(y)\, dy \;-\; \int_{\partial D_r} \big[\nu(y)\times E(y)\big]\,\Phi(x,y)\, ds(y)\right] \\
&\quad - \nabla\left[\iint_{D_r} \Phi(x,y)\ \operatorname{div} E(y)\, dy \;-\; \int_{\partial D_r} \big[\nu(y)\cdot E(y)\big]\,\Phi(x,y)\, ds(y)\right] \\
&\quad - k^2 \iint_{D_r} E(y)\,\Phi(x,y)\, dy \\
&= \operatorname{curl}\left[\iint_{D_r} \big\{\operatorname{curl}_y\big[E\,\Phi(x,\cdot)\big] - \nabla_y\Phi(x,\cdot)\times E\big\}\, dy \;-\; \int_{\partial D_r} \nu\times\big[E\,\Phi(x,\cdot)\big]\, ds\right] \\
&\quad - \nabla\left[\iint_{D_r} \big\{\operatorname{div}_y\big[E\,\Phi(x,\cdot)\big] - \nabla_y\Phi(x,\cdot)\cdot E\big\}\, dy \;-\; \int_{\partial D_r} \nu\cdot\big[E\,\Phi(x,\cdot)\big]\, ds\right] \\
&\quad - k^2 \iint_{D_r} E\,\Phi(x,\cdot)\, dy\,.
\end{aligned}$$

Now we use the divergence theorem in the forms:

$$\iint_{D_r} \operatorname{div} F\, dx \;=\; \int_{\partial D_r} \nu\cdot F\, ds\,, \qquad \iint_{D_r} \operatorname{curl} F\, dx \;=\; \int_{\partial D_r} \nu\times F\, ds\,.$$

Therefore,

$$\begin{aligned}
I_r(x) &= -\operatorname{curl}\iint_{D_r} \nabla_y\Phi(x,\cdot)\times E\, dy \;+\; \nabla\iint_{D_r} \nabla_y\Phi(x,\cdot)\cdot E\, dy \;-\; k^2\iint_{D_r} E\,\Phi(x,\cdot)\, dy \\
&= \iint_{D_r} \big\{\nabla_x\big[E\cdot\nabla_y\Phi(x,\cdot)\big] - \operatorname{curl}_x\big[\nabla_y\Phi(x,\cdot)\times E\big]\big\}\, dy \;-\; k^2\iint_{D_r} E\,\Phi(x,\cdot)\, dy \\
&= -\iint_{D_r} E\,\big[\Delta_y\Phi(x,\cdot) + k^2\Phi(x,\cdot)\big]\, dy \;=\; 0\,.
\end{aligned}$$

Hier we used the formula

$$\begin{aligned}
\operatorname{curl}_x\big[\nabla_y\Phi(x,\cdot)\times E\big] &= -E\ \operatorname{div}_x\nabla_y\Phi(x,\cdot) + (E\cdot\nabla_x)\nabla_y\Phi(x,\cdot) \\
&= E\,\Delta_y\Phi(x,\cdot) + \nabla_x\big[E\cdot\nabla_y\Phi(x,\cdot)\big]\,.
\end{aligned}$$

Therefore, $I_r(x)=0$ for all $x\in B(z,r)$, i.e.

$$0=\iint_{D_r} \nabla_x\Phi(x,y)\times\operatorname{curl} E(y)\, dy - \iint_{D_r} \nabla_x\Phi(x,y)\ \operatorname{div} E(y)\, dy - k^2\iint_{D_r} E(y)\,\Phi(x,y)\, dy$$

$$-\ \operatorname{curl}\int_{\partial D} \big[\nu(y)\times E(y)\big]\,\Phi(x,y)\, ds(y) \;+\; \nabla\int_{\partial D} \big[\nu(y)\cdot E(y)\big]\,\Phi(x,y)\, ds(y)$$

$$- \int\limits_{|y-z|=r} \nabla_x \Phi(x,y) \times \big[\nu(y) \times E(y)\big]\, ds(y) \;+\; \int\limits_{|y-z|=r} \nabla_x \Phi(x,y) \big[\nu(y) \cdot E(y)\big]\, ds(y)\,.$$

We set $x = z$ and compute the last two surface integrals explicitely. We recall that

$$\nabla_z \Phi(z,y) \;=\; \frac{\exp(ik|z-y|)}{4\pi|z-y|} \left(ik - \frac{1}{|z-y|} \right) \frac{z-y}{|z-y|}$$

and thus for $|z-y| = r$:

$$\begin{aligned}
& - \int\limits_{|y-z|=r} \nabla_z \Phi(z,y) \times \big[\nu(y) \times E(y)\big]\, ds(y) \;+\; \int\limits_{|y-z|=r} \nabla_z \Phi(z,y) \big[\nu(y) \cdot E(y)\big]\, ds(y) \\
&= \frac{\exp(ikr)}{4\pi r} \left(ik - \frac{1}{r} \right) \int\limits_{|y-z|=r} \underbrace{\left[\frac{z-y}{|z-y|} \left(\frac{z-y}{|z-y|} \cdot E(y) \right) - \frac{z-y}{|z-y|} \times \left(\frac{z-y}{|z-y|} \times E(y) \right) \right]}_{=\, E(y)} ds(y) \\
&= \frac{\exp(ikr)}{4\pi r} \left(ik - \frac{1}{r} \right) \int\limits_{|y-z|=r} E(y)\, ds(y) \\
&= -e^{ikr}\, E(z) \;+\; ik\, \frac{\exp(ikr)}{4\pi r} \int\limits_{|y-z|=r} E(y)\, ds(y) \;+\; \frac{\exp(ikr)}{4\pi r^2} \int\limits_{|y-z|=r} \big[E(z) - E(y)\big]\, ds(y)\,.
\end{aligned}$$

This term converges to $-E(z)$ as r tends to zero. This proves the formula for $x \in D$.
The same arguments (replacing D_r by D) which lead to $I_r(x) = 0$ yield that the expression vanishes if $x \notin \overline{D}$. The formula for $x \in \partial D$ follows from the same arguments as in the proof of Theorem 3.3. □

Theorem 3.25 *(Stratton-Chu formula)*
Let $k = \omega\sqrt{\varepsilon_0 \mu_0} > 0$ and $E, H \in C^1(D)^3 \cap C(\overline{D})^3$ satisfy Maxwell's equations

$$\operatorname{curl} E - i\omega\mu_0 H = 0 \;\text{ in } D\,, \qquad \operatorname{curl} H + i\omega\varepsilon_0 E = 0 \;\text{ in } D\,.$$

Then we have for $x \in D$:

$$\begin{aligned}
E(x) &= -\operatorname{curl} \int_{\partial D} \big[\nu(y) \times E(y)\big]\, \Phi(x,y)\, ds(y) \;+\; \nabla \int_{\partial D} \big[\nu(y) \cdot E(y)\big]\, \Phi(x,y)\, ds(y) \\
&\quad - i\omega\mu_0 \int_{\partial D} \big[\nu(y) \times H(y)\big]\, \Phi(x,y)\, ds(y) \\
&= -\operatorname{curl} \int_{\partial D} \big[\nu(y) \times E(y)\big]\, \Phi(x,y)\, ds(y) \;+\; \frac{1}{i\omega\varepsilon_0} \operatorname{curl}^2 \int_{\partial D} \big[\nu(y) \times H(y)\big]\, \Phi(x,y)\, ds(y)\,, \\
H(x) &= -\operatorname{curl} \int_{\partial D} \big[\nu(y) \times H(y)\big]\, \Phi(x,y)\, ds(y) \;-\; \frac{1}{i\omega\mu_0} \operatorname{curl}^2 \int_{\partial D} \big[\nu(y) \times E(y)\big]\, \Phi(x,y)\, ds(y)\,.
\end{aligned}$$

Proof: The second term in the representation of E in Theorem 3.24 vanishes since $\operatorname{div} E = 0$. The first term is rewritten as follows where now $D_r = D \setminus B[x,r]$.

$$\iint_{D_r} \nabla_x \Phi(x,y) \times \operatorname{curl} E(y)\, dy \;=\; i\omega\mu_0 \iint_{D_r} \nabla_x \Phi(x,y) \times H(y)\, dy$$

$$\begin{aligned}
&= -i\omega\mu_0 \iint_{D_r} \nabla_y \Phi(x,y) \times H(y)\, dy \\
&= -i\omega\mu_0 \iint_{D_r} \big\{\operatorname{curl}_y\big[H(y)\,\Phi(x,y)\big] - \Phi(x,y)\,\operatorname{curl} H(y)\big\}\, dy \\
&= -i\omega\mu_0 \iint_{D_r} \operatorname{curl}_y\big(H(y)\,\Phi(x,y)\big)\, dy \;+\; \omega^2\mu_0\varepsilon_0 \iint_{D_r} \Phi(x,y)\, E(y)\, dy \\
&= -i\omega\mu_0 \int_{\partial D_r} \big[\nu(y) \times H(y)\big]\, \Phi(x,y)\, dy \;+\; \underbrace{\omega^2\mu_0\varepsilon_0}_{=\,k^2} \iint_{D_r} \Phi(x,y)\, E(y)\, dy\,.
\end{aligned}$$

This proves the first formula by letting r tend to zero. For the second we continue with:

$$\int_{\partial D_r} \big[\nu(y)\cdot E(y)\big]\, \Phi(x,y)\, ds(y) \;=\; -\frac{1}{i\omega\varepsilon_0} \int_{\partial D_r} \big[\nu(y)\cdot \operatorname{curl} H(y)\big]\, \Phi(x,y)\, ds(y)$$

$$\begin{aligned}
&= -\frac{1}{i\omega\varepsilon_0} \underbrace{\int_{\partial D_r} \nu(y)\cdot \operatorname{curl}\big[H(y)\,\Phi(x,y)\big]\, ds(y)}_{=\,0 \text{ by the divergence theorem}} \;+\; \frac{1}{i\omega\varepsilon_0} \int_{\partial D_r} \nu(y)\cdot \big[\nabla_y \Phi(x,y) \times H(y)\,\big]\, ds(y) \\
&= \frac{1}{i\omega\varepsilon_0} \int_{\partial D_r} \nabla_y \Phi(x,y) \cdot \big[H(y) \times \nu(y)\,\big]\, ds(y)\,.
\end{aligned}$$

Now we let r tend to zero. We observe that the integral $\int_{|y-x|=r} \nabla_y \Phi(x,y) \cdot \big[H(y)\times \nu(y)\,\big]\, ds(y)$ vanishes since $\nabla_y \Phi(x,y)$ and $\nu(y)$ are parallel. Therefore,

$$\begin{aligned}
\int_{\partial D} \big[\nu(y)\cdot E(y)\big]\, \Phi(x,y)\, ds(y) &= \frac{1}{i\omega\varepsilon_0} \int_{\partial D} \nabla_y \Phi(x,y) \cdot \big[H(y) \times \nu(y)\,\big]\, ds(y) \\
&= \frac{1}{i\omega\varepsilon_0} \operatorname{div} \int_{\partial D} \Phi(x,y)\, \big[\nu(y) \times H(y)\,\big]\, ds(y)\,.
\end{aligned}$$

Taking the gradient and using $\operatorname{curl}\operatorname{curl} = \nabla \operatorname{div} - \Delta$ yields

$$\begin{aligned}
\nabla \int_{\partial D} \big[\nu(y)\cdot E(y)\big]\, \Phi(x,y)\, ds(y) &= \frac{1}{i\omega\varepsilon_0} \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x,y)\, \big[\nu(y) \times H(y)\,\big]\, ds(y) \\
&\quad - \frac{k^2}{i\omega\varepsilon_0} \int_{\partial D} \Phi(x,y)\, \big[\nu(y) \times H(y)\,\big]\, ds(y)\,.
\end{aligned}$$

This ends the proof for E. The representation for $H(x)$ follows directly by $H = \frac{1}{i\omega\mu_0} \operatorname{curl} E$ (note that $\operatorname{curl}\operatorname{curl}\operatorname{curl} = -\operatorname{curl}\Delta$!) □

Conclusion 3.26 *Solutions E, H of Maxwell's equations in vacuum $D \subset \mathbb{R}^3$ are*

- *analytic in every component*
- *divergence-free solutions of the vector-Helmholtz equation*

On the other side, every divergence-free solution E of the vector-Helmholtz equation is, combined with $H := \frac{1}{i\omega\mu_0}\operatorname{curl} E$, a solution of Maxwell's equations.

Remark: Fields of the form (for some $y \in \mathbb{R}^3$ and $p \in \mathbb{C}^3$)

$$\begin{aligned} E_{md}(x) &= \operatorname{curl}\big[p\,\Phi(x,y)\big]\,, \quad H_{md} = \frac{1}{i\omega\mu_0}\operatorname{curl} E_{md} = \frac{1}{i\omega\mu_0}\operatorname{curl}\operatorname{curl}\big[p\,\Phi(x,y)\big]\,, \\ H_{ed}(x) &= \operatorname{curl}\big[p\,\Phi(x,y)\big]\,, \quad E_{ed} = -\frac{1}{i\omega\varepsilon_0}\operatorname{curl} H_{ed} = -\frac{1}{i\omega\varepsilon_0}\operatorname{curl}\operatorname{curl}\big[p\,\Phi(x,y)\big]\,, \end{aligned}$$

are called *magnetic* and *electric dipols*, respectively, at y with polarization p. We assume here, that k is **real and positive!**

Lemma 3.27 *Let $k > 0$. The electromagnetic fields E_{md}, H_{md} and E_{ed}, H_{ed} of a magnetic or electric dipol, respectively, satisfy the* ***Silver-Müller radiation condition***

$$\sqrt{\varepsilon_0}\,E(x) \;-\; \sqrt{\mu_0}\,H(x) \times \frac{x}{|x|} \;=\; \mathcal{O}\left(\frac{1}{|x|^2}\right)\,, \tag{3.38a}$$

and

$$\sqrt{\mu_0}\,H(x) \;+\; \sqrt{\varepsilon_0}\,E(x) \times \frac{x}{|x|} \;=\; \mathcal{O}\left(\frac{1}{|x|^2}\right)\,, \tag{3.38b}$$

uniformly w.r.t. $x/|x|$ and $|y| \le R$ for any fixed $R > 0$.

Proof: Direct computation yields

$$\begin{aligned} E_{md}(x) &= \operatorname{curl}\big[p\,\Phi(x,y)\big] = \Phi(x,y)\left(ik - \frac{1}{|x-y|}\right)\left(\frac{x-y}{|x-y|} \times p\right) \\ &= ik\,\Phi(x,y)\left(\frac{x-y}{|x-y|} \times p\right) + \mathcal{O}\left(\frac{1}{|x|^2}\right), \\ H_{md}(x) &= \frac{1}{i\omega\mu_0}\operatorname{curl}\operatorname{curl}\big[p\,\Phi(x,y)\big] = \frac{1}{i\omega\mu_0}\left(-\Delta + \nabla\operatorname{div}\right)\big[p\,\Phi(x,y)\big] \\ &= \cdots = \frac{k^2}{i\omega\mu_0}\Phi(x,y)\left[p - \frac{x-y}{|x-y|}\frac{(x-y)\cdot p}{|x-y|}\right] + \mathcal{O}\left(\frac{1}{|x|^2}\right). \end{aligned}$$

for $|x| \to \infty$. With $\frac{x-y}{|x-y|} = \frac{x}{|x|} + \mathcal{O}(1/|x|^2)$ and $k = \omega\sqrt{\mu_0\varepsilon_0}$ the first assertion follows. Analogously, the second can be proven. □

Theorem 3.28 *(Stratton-Chu formula in exterior domains)*
Let $k = \omega\sqrt{\varepsilon_0\mu_0} > 0$ and $E, H \in C^1(\mathbb{R}^3 \setminus \overline{D})^3 \cap C(\mathbb{R}^3 \setminus D)^3$ solutions of the homogeneous Maxwell's equations

$$\operatorname{curl} E - i\omega\mu_0 H \;=\; 0\,, \quad \operatorname{curl} H + i\omega\varepsilon_0 E \;=\; 0$$

in $\mathbb{R}^3 \setminus \overline{D}$ which satisfy also one of the Silver-Müller radiation conditions (3.38a) or (3.38b). Then

$$\operatorname{curl}\int_{\partial D}\big[\nu(y)\times E(y)\big]\,\Phi(x,y)\,ds(y) \;-\; \frac{1}{i\omega\varepsilon_0}\operatorname{curl}\operatorname{curl}\int_{\partial D}\big[\nu(y)\times H(y)\big]\,\Phi(x,y)\,ds(y) \;=$$

$$= \begin{cases} 0\,, & x \in D\,, \\ \frac{1}{2}\,E(x)\,, & x \in \partial D\,, \\ E(x)\,, & x \notin \overline{D}\,, \end{cases}$$

and

$$\operatorname{curl}\int_{\partial D}\big[\nu(y)\times H(y)\big]\,\Phi(x,y)\,ds(y) \;+\; \frac{1}{i\omega\mu_0}\operatorname{curl}\operatorname{curl}\int_{\partial D}\big[\nu(y)\times E(y)\big]\,\Phi(x,y)\,ds(y) \;=$$

$$= \begin{cases} 0\,, & x \in D\,, \\ \frac{1}{2}\,H(x)\,, & x \in \partial D\,, \\ H(x)\,, & x \notin \overline{D}\,. \end{cases}$$

Proof: Let us first assume the radiation condition (3.38a). One applies Theorem 3.25 in the region $D_R = \big\{x \notin \overline{D} : |y| < R\big\}$ for large values of R. Then the assertion follows if one can show that

$$I_R \;:=\; \operatorname{curl}\int_{|y|=R}\big[\nu(y)\times E(y)\big]\,\Phi(x,y)\,ds(y) - \frac{1}{i\omega\varepsilon_0}\operatorname{curl}\operatorname{curl}\int_{|y|=R}\big[\nu(y)\times H(y)\big]\,\Phi(x,y)\,ds(y)$$

tends to zero as $R \to \infty$. To do this we first prove that $\int_{|y|=R}|E|^2 ds$ is bounded w.r.t. R. The binomial theorem yields

$$\begin{aligned}\int_{|y|=R}|\sqrt{\varepsilon_0}\,E - \sqrt{\mu_0}\,H\times\nu|^2 ds \;&=\; \varepsilon_0\int_{|y|=R}|E|^2 ds \;+\; \mu_0\int_{|y|=R}|H\times\nu|^2 ds \\ &\quad -2\sqrt{\varepsilon_0\mu_0}\,\operatorname{Re}\int_{|y|=R} E\cdot(\overline{H}\times\nu)\,ds\,.\end{aligned}$$

We have by the divergence theorem

$$\begin{aligned}\int_{|y|=R} E\cdot(\overline{H}\times\nu)\,ds \;&=\; \int_{\partial D} E\cdot(\overline{H}\times\nu)\,ds \;+\; \iint_{D_R}\operatorname{div}(E\times\overline{H})\,dx \\ &=\; \int_{\partial D} E\cdot(\overline{H}\times\nu)\,ds \;+\; \iint_{D_R}\big[\overline{H}\big)\cdot\operatorname{curl} E - E\cdot\operatorname{curl}\overline{H}\big]\,dx \\ &=\; \int_{\partial D} E\cdot(\overline{H}\times\nu)\,ds \;+\; \iint_{D_R}\big[i\omega\mu_0|H|^2 - i\omega\varepsilon_0|E|^2\big]\,dx\,.\end{aligned}$$

This term is purely imaginary, thus

$$\int_{|y|=R} |\sqrt{\varepsilon_0}\, E - \sqrt{\mu_0}\, H \times \nu|^2 ds \;=\; \varepsilon_0 \int_{|y|=R} |E|^2 ds \;+\; \mu_0 \int_{|y|=R} |H \times \nu|^2 ds \\ -\, 2\sqrt{\varepsilon_0 \mu_0}\, \mathrm{Re} \int_{\partial D} E \cdot (\overline{H} \times \nu)\, ds\,.$$

From this the boundedness of $\int_{|y|=R} |E|^2 ds$ follows since the left hand side tends to zero by the radiation condition (3.38b).
Now we write I_R in the form

$$I_R \;=\; \mathrm{curl} \int_{|y|=R} \left\{ \big[\nu(y) \times E(y)\big]\, \Phi(x,y) \;+\; \frac{1}{ik}\, E(y) \times \nabla_y \Phi(x,y) \right\} ds(y) \\ -\, \frac{1}{i\omega\varepsilon_0}\, \mathrm{curl} \int_{|y|=R} \left(\big[\nu(y) \times H(y)\big] + \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E(y) \right) \times \nabla_y \Phi(x,y)\, ds(y)\,.$$

Let us first consider the second term. The bracket $(\cdots)$ tends to zero as $1/R^2$ by the radiation condition (3.38a). Taking the curl of the integral results in second order differentiations of Φ. Since Φ and all derivatives decay as $1/R$ the total integrand decays as $1/R^3$. Therefore, this second term tends to zero because the surface area is only $4\pi R^2$.
For the first term we observe that

$$\big[\nu(y) \times E(y)\big]\, \Phi(x,y) \;+\; \frac{1}{ik}\, E(y) \times \nabla_y \Phi(x,y) \\ =\; E(y) \times \left[-\frac{y}{R}\, \Phi(x,y) \;+\; \frac{1}{ik}\, \frac{y-x}{|y-x|}\, \Phi(x,y) \left(ik - \frac{1}{|y-x|} \right) \right].$$

For fixed x and arbitrary y with $|y| = R$ the bracket $[\cdots]$ tends to zero of order $1/R^2$. The same holds true for all of the partial derivatives w.r.t. x. Therefore, the first term can be estimated by the inequality of Cauchy-Schwartz

$$\frac{c}{R^2} \int_{|y|=R} \big|E(y)\big|\, ds \;\le\; \frac{c}{R^2} \sqrt{\int_{|y|=R} 1^2\, ds}\, \sqrt{\int_{|y|=R} \big|E(y)\big|^2\, ds} \;=\; \frac{c\sqrt{4\pi}}{R} \sqrt{\int_{|y|=R} \big|E(y)\big|^2\, ds}$$

and this tends also to zero.
This proves the representation of E by using the first radiation condition (3.38a). The representation of $H(x)$ follows again by computing $H = \frac{1}{i\omega\mu_0}\, \mathrm{curl}\, E$. If the second radiation condition (3.38b) is assumned one can argue as before and derive the representation of H first. □

We draw the following conclusions from this result.

Remark 3.29 *(a) If E, H are solutions of Maxwell's equations in vacuum $\mathbb{R}^3 \setminus \overline{D}$ then each of the radiation conditions implies the other one.*

(b) The Silver-Müller radiation condition for solutions E, H of the Maxwell system is equivalent to the Sommerfeld radiation condition (3.2) for every component of E and H. This follows from the fact that the fundamental solution Φ and every derivative of Φ satisfies the Sommerfeld radiation condition.

(c) The asymptotic behaviour of Φ yields

$$E(x) \;=\; \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{and} \quad H(x) \;=\; \mathcal{O}\left(\frac{1}{|x|}\right)$$

for $|x| \to \infty$ uniformly w.r.t. all directions $x/|x|$.

Sometimes it is conveniant to eliminate one of the fields E or H from the Maxwell system and work with only one of them. If we eliminate H then E solves the second order equation

$$\operatorname{curl} E \;-\; k^2 E \;=\; 0 \tag{3.39}$$

where again $k = \omega\sqrt{\mu_0\varepsilon_0}$ denotes the wave number. If, on the other hand, E satisfies (3.39) then E and $H = \frac{1}{i\omega\mu_0} E$ solve the Maxwell system. Indeed, the first Maxwell equation is satisfied by the definition of H. Also the second Maxwell equation is satisfied because

$$\operatorname{curl} H \;=\; \frac{1}{i\omega\mu_0}\operatorname{curl}^2 E \;=\; \frac{1}{i\omega\mu_0}\big[\nabla \underbrace{\operatorname{div} E}_{=0} \;-\; \underbrace{\Delta E}_{=-k^2 E}\,\big] \;=\; \frac{k^2}{i\omega\mu_0} E \;=\; -i\omega\varepsilon_0 E\,.$$

The Silver-Müller radiation condition (3.36a) or (3.36b) turn into

$$\operatorname{curl} E \times \hat{x} \;-\; ik\,E \;=\; \mathcal{O}\big(|x|^{-2}\big)\,, \quad |x| \to \infty\,, \tag{3.40}$$

uniformly with respect to $\hat{x} = x/|x|$.

3.2.3 Vector Potentials and Boundary Integral Operators

In Subsection 3.2.4 we will prove existence of solutions of the scattering problem by a boundary integral equation method. Analogously to the scalar case we have to introduce **vector potentials**. Motivated by the Stratton-Chu formulas we have to consider the curl and the double-curl of the single layer potential

$$v(x) \;=\; \int_{\partial D} a(y)\,\Phi(x,y)\,ds(y)\,, \quad x \in \mathbb{R}^3\,, \tag{3.41}$$

where $a \in C^\alpha(\partial D)^3$ is a tangential field, i.e. $a(y)\cdot\nu(y) = 0$ for all $y \in \partial D$.

Lemma 3.30 *Let v be defined by (3.41). Then $E = \operatorname{curl} v$ satisfies (3.39) in all of $\mathbb{R}^3 \setminus \partial D$ and also the radiation condition (3.40).*

The **proof** follows immediately from Lemma 3.27. □

In the next theorem we study the behaviour of E at the boundary.

Theorem 3.31 *The curl of the potential v from (3.41) with Hölder continuous tangential field $a \in C^\alpha(\partial D)^3$ can be continuously extended from D to $\overline{D}$ and from $\mathbb{R}^3 \setminus \overline{D}$ to $\mathbb{R}^3 \setminus D$. The limiting values of the tangential components are*

$$\nu(x) \times \operatorname{curl} v(x)\big|_\pm \;=\; \pm\frac{1}{2}\,a(x) \;+\; \nu(x) \times \int_{\partial D} \operatorname{curl}_x\big[a(y)\Phi(x,y)\big]\,ds(y)\,, \quad x \in \partial D\,. \quad (3.42)$$

The integral exists as an improper integral.

If, in addition, the surface divergence $\operatorname{Div} a$ *(see end of Chapter 1) is continuous, then* $\operatorname{div} v$ *is continuous in all of* $\mathbb{R}^3$.

If, furthermore, $\operatorname{Div} a \in C^\alpha(\partial D)$ *then* $\operatorname{curl}\operatorname{curl} v$ *can be continuously extended from D to $\overline{D}$ and from $\mathbb{R}^3 \setminus \overline{D}$ to $\mathbb{R}^3 \setminus D$.*

The limiting values are

$$\begin{aligned}
\operatorname{div} v(x)\big|_\pm &= \int_{\partial D} \Phi(x,y)\,\operatorname{Div} a(y)\,ds(y)\,, \quad x \in \partial D\,,\\
\nu \times \operatorname{curl}\operatorname{curl} v\big|_+ &= \nu \times \operatorname{curl}\operatorname{curl} v\big|_- \quad \text{on } \partial D\,, \qquad (3.43)\\
\nu(x)\cdot \operatorname{curl}\operatorname{curl} v(x)\big|_\pm &= \mp\frac{1}{2}\,\operatorname{Div} a(x)\\
&\quad + \int_{\partial D}\left[\operatorname{Div} a(y)\,\frac{\partial\Phi}{\partial\nu(x)}(x,y) + k^2\nu(x)\cdot a(y)\,\Phi(x,y)\right] ds(y)\,,\; x \in \partial D\,.
\end{aligned}$$

Proof: First we write the second term on the right hand side of (3.42) in the form

$$\begin{aligned}
\nu(x) \times \int_{\partial D} \operatorname{curl}_x\big[a(y)\Phi(x,y)\big]\,ds(y) \;&=\; \int_{\partial D} \nu(x) \times \big[\nabla_x\Phi(x,y) \times a(y)\big]\,ds(y)\\
= \int_{\partial D}\Big\{\big[\nu(x)-\nu(y)\big] \times \big[\nabla_x\Phi(x,y)\times a(y)\big] \;&+\; \frac{\partial\Phi}{\partial\nu(y)}(x,y)\,a(y)\Big\}ds(y)
\end{aligned}$$

and the integrand is weakly singular.
The components of $\operatorname{curl} v$ are combinations of partial derivatives of the single layer potential. Therefore, by Theorem 3.15 the field $\operatorname{curl} v$ has continuous extensions to ∂D from both sides. It remains to show the representation of the tangential components of these extensions on the boundary. We recall the special neighborhoods H_ρ of ∂D from Subsection 1.5 and write $x \in H_{\rho_0}$ in the form $x = z + t\nu(z)$, $z \in \partial D$, $0 < |t| < \rho_0$. Then we have

$$\begin{aligned}
\nu(z) \times \operatorname{curl} v(x) &= \int_{\partial D} \nu(z) \times \big[\nabla_x\Phi(x,y)\times a(y)\big]\,ds(y) \qquad (3.44)\\
&= \int_{\partial D}\big[\nu(z)-\nu(y)\big] \times \big[\nabla_x\Phi(x,y)\times a(y)\big]\,ds(y) \;+\; \int_{\partial D} a(y)\,\frac{\partial\Phi}{\partial\nu(y)}(x,y)\,ds(y)\,.
\end{aligned}$$

The first term is continuous in all of $\mathbb{R}^3$ by Lemma 3.12, the second in $\mathbb{R}^3 \setminus \partial D$ because it is a double layer potential. The limiting values are

$$\begin{aligned}
\nu(x) \times \operatorname{curl} v(x)\big|_\pm &= \int_{\partial D} \big[\nu(x) - \nu(y)\big] \times \big[\nabla_x \Phi(x,y) \times a(y)\big]\, ds(y) \\
&\quad \pm \frac{1}{2}\, a(x) \;+\; \int_{\partial D} a(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y)\, ds(y) \\
&= \pm \frac{1}{2}\, a(x) \;+\; \int_{\partial D} \nu(x) \times \big[\nabla_x \Phi(x,y) \times a(y)\big]\, ds(y)
\end{aligned}$$

which has the desired form. For the divergence we write

$$\begin{aligned}
\operatorname{div} v(x) &= \int_{\partial D} a(y) \cdot \nabla_x \Phi(x,y)\, ds(y) \;=\; -\int_{\partial D} a(y) \cdot \nabla_y \Phi(x,y)\, ds(y) \\
&= -\int_{\partial D} a(y) \cdot \operatorname{Grad}_y \Phi(x,y)\, ds(y) \;=\; \int_{\partial D} \Phi(x,y)\, \operatorname{Div} a(y)\, ds(y)
\end{aligned}$$

and this is a single layer potential and thus continuous. Finally, since $\operatorname{curl}\operatorname{curl} = \nabla \operatorname{div} - \Delta$ and $\Delta_x \Phi(x,y) = -k^2 \Phi(x,y)$, we conclude that

$$\begin{aligned}
\operatorname{curl}\operatorname{curl} v(x) &= \nabla \operatorname{div} \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y) \;+\; k^2 \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y) \\
&= \nabla \int_{\partial D} \Phi(x,y)\, \operatorname{Div} a(y)\, ds(y) \;+\; k^2 \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y)
\end{aligned}$$

from which the assertion follows by Theorem 3.15. □

The continuity properties of the derivatives of v give rise to corresponding boundary integral operators. It is convenient to not only define the spaces $C_t(\partial D)$ and $C_t^\alpha(\partial D)$ of continuous and Hölder continuous tangential fields, respectively, but also of Hölder continuous tangential fields such that the surface divergence is also Hölder continuous. Therefore, we define:

$$\begin{aligned}
C_t(\partial D) &= \big\{a \in C(\partial D)^3 : a(y) \cdot \nu(y) = 0 \text{ on } \partial D\big\}\,, \\
C_t^\alpha(\partial D) &= C_t(\partial D) \cap C^\alpha(\partial D)^3\,, \\
C_{Div}^\alpha(\partial D) &= \big\{a \in C_t^\alpha(\partial D) : \operatorname{Div} a \in C^\alpha(\partial D)\big\}\,.
\end{aligned}$$

We equip $C_t(\partial D)$ and $C_t^\alpha(\partial D)$ with the ordinary norms of $C(\partial D)^3$ and $C^\alpha(\partial D)^3$, respectively, and $C_{Div}^\alpha(\partial D)$ with the norm $\|a\|_{C_{Div}^\alpha} = \|a\|_{C^\alpha} + \|\operatorname{Div} a\|_{C^\alpha}$. Then we can prove:

Theorem 3.32 *(a) The boundary operator $\mathcal{M} : C_t(\partial D) \to C_t^\alpha(\partial D)$, defined by*

$$(\mathcal{M}a)(x) \;=\; \nu(x) \times \int_{\partial D} \operatorname{curl}_x \big[a(y)\, \Phi(x,y)\big]\, ds(y)\,, \quad x \in \partial D\,, \tag{3.45}$$

is well defined and bounded.

(b) $\mathcal{M}$ is well defined and compact from $C^\alpha_{Div}(\partial D)$ into itself.

(c) The boundary operator $\mathcal{N} : C^\alpha_{Div}(\partial D) \to C^\alpha_{Div}(\partial D)$, defined by

$$(\mathcal{N}a)(x) \;=\; \nu(x) \times \operatorname{curl}\operatorname{curl} \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y)\,, \quad x \in \partial D\,, \tag{3.46}$$

is well defined and bounded. Here, the right hand side is the trace of $\operatorname{curl}^2 v$ *with* v *from (3.41), see (3.43).*

Proof: (a) We see from (3.44) that the kernel of this integral operator has the form

$$G(x,y) \;=\; \nabla_x \Phi(x,y) \big[\nu(x) - \nu(y)\big]^\top \;-\; \frac{\partial \Phi}{\partial \nu(x)}(x,y)\, I\,.$$

The component $g_{j,\ell}$ of the first term is given by

$$g_{j,\ell}(x,y) \;=\; \frac{\partial \Phi}{\partial x_j}(x,y) \big[\nu_\ell(x) - \nu_\ell(y)\big]\,,$$

and this satisfies certainly the first assumption of part (a) of Theorem 3.16 for $\alpha = 1$. For the second assumption we write

$$g_{j,\ell}(x_1,y) - g_{j,\ell}(x_2,y) \;=\; \big[\nu_\ell(x_1) - \nu_\ell(x_2)\big] \frac{\partial \Phi}{\partial x_j}(x_1,y) + \big[\nu_\ell(x_2) - \nu_\ell(y)\big] \left[\frac{\partial \Phi}{\partial x_j}(x_1,y) - \frac{\partial \Phi}{\partial x_j}(x_2,y)\right]$$

and thus by the same arguments as in the proof Lemma 3.12

$$\big|g_{j,\ell}(x_1,y) - g_{j,\ell}(x_2,y)\big| \;\le\; c\,\frac{|x_1 - x_2|}{|x_1 - y|^2} \;+\; c|x_2 - y|\,\frac{|x_1 - x_2|}{|x_1 - y|^3} \;\le\; c'\,\frac{|x_1 - x_2|}{|x_1 - y|^2}\,.$$

This settles the first term of G. For the second term we observe that

$$\frac{\partial \Phi}{\partial \nu(x)}(x,y) \;=\; -\frac{\partial \Phi}{\partial \nu(y)}(x,y) \;+\; \big[\nu(x) - \nu(y)\big] \cdot \nabla_x \Phi(x,y)\,.$$

The first term is just the kernel of the double layer operator treated in the previous theorem. For the second we can apply the first part again because it is just $\sum_{\ell=1}^3 g_{\ell,\ell}(x,y)$.

(b) We note that the space $C^\alpha_{\mathrm{Div}}(\partial D)$ is a subspace of $C^\alpha_t(\partial D)$ with bounded imbedding and, furthermore, the space $C^\alpha_t(\partial D)$ is compactly imbedded in $C_t(\partial D)$ by Lemma 3.17. Therefore, $\mathcal{M}$ is compact from $C^\alpha_{\mathrm{Div}}(\partial D)$ into $C^\alpha_t(\partial D)$. It remains to show that $\operatorname{Div}\mathcal{M}$ is compact $C^\alpha_{\mathrm{Div}}(\partial D)$ into $C^\alpha(\partial D)$.
For $a \in C^\alpha_{\mathrm{Div}}(\partial D)$ we define the potential v by

$$v(x) \;=\; \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y)\,, \quad x \in D\,.$$

Then $\mathcal{M}a = \nu \times \operatorname{curl} v|_- + \frac{1}{2}a$ by Theorem 3.31. Furthermore, by Theorem 3.31 again we conclude that for $x \notin \partial D$

$$\begin{aligned}
\operatorname{curl}\operatorname{curl} v(x) \;&=\; (\nabla \operatorname{div} - \Delta) \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y) \\
&=\; \nabla \int_{\partial D} \operatorname{Div} a(y)\, \Phi(x,y)\, ds(y) \;+\; k^2 \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y)
\end{aligned}$$

and thus by Corollary 1.16 and the jump condition of the derivative of the single layer (Theorem 3.15)

$$\begin{aligned}
\operatorname{Div}(\mathcal{M}a)(x) &= -\nu(x)\cdot \operatorname{curl}\operatorname{curl} v(x)|_- + \frac{1}{2}\operatorname{Div} a(x) \\
&= -\int_{\partial D}\left[\operatorname{Div} a(y)\,\frac{\partial\Phi}{\partial\nu(x)}(x,y) + k^2\nu(x)\cdot a(y)\,\Phi(x,y)\right] ds(y) \\
&= -\mathcal{D}'(\operatorname{Div} a) \;-\; k^2\,\nu\cdot\mathcal{S}a\,.
\end{aligned}$$

The assertion follows since $\mathcal{D}'\circ\operatorname{Div}$ and $\nu\cdot\mathcal{S}$ are both compact from $C^\alpha_{\mathrm{Div}}(\partial D)$ into $C^\alpha(\partial D)$.

(c) Define

$$w(x) \;=\; \operatorname{curl}\operatorname{curl}\int_{\partial D} a(y)\,\Phi(x,y)\,ds(y)\,, \quad x\notin\partial D\,.$$

Writing again $\operatorname{curl}^2 = \nabla\operatorname{div} - \Delta$ yields for $x = z + t\nu(z)\in H_\rho\setminus\partial D$:

$$\begin{aligned}
w(x) &= \nabla\operatorname{div}\int_{\partial D}\Phi(x,y)\,a(y)\,ds(y) \;+\; k^2\int_{\partial D} a(y)\,\Phi(x,y)\,ds(y) \\
&= \int_{\partial D}\nabla_x\Phi(x,y)\;\operatorname{Div} a(y)\,ds(y) \;+\; k^2\int_{\partial D} a(y)\,\Phi(x,y)\,ds(y) \\
&= \int_{\partial D}\nabla_x\Phi(x,y)\,\big[\operatorname{Div} a(y) - \operatorname{Div} a(z)\big]\,ds(y) \;+\; \operatorname{Div} a(z)\int_{\partial D}\nabla_x\Phi(x,y)\,ds(y) \\
&\quad + k^2\int_{\partial D} a(y)\,\Phi(x,y)\,ds(y)
\end{aligned}$$

We use Lemma 3.14 and arrive at

$$\begin{aligned}
w(x) &= \int_{\partial D}\nabla_x\Phi(x,y)\,\big[\operatorname{Div} a(y) - \operatorname{Div} a(z)\big]\,ds(y) \;+\; k^2\int_{\partial D} a(y)\,\Phi(x,y)\,ds(y) \\
&\quad + \operatorname{Div} a(z)\int_{\partial D} H(y)\,\Phi(x,y)\,ds(y) \;-\; \operatorname{Div} a(z)\int_{\partial D}\nu(y)\,\frac{\partial\Phi}{\partial\nu(y)}(x,y)\,ds(y)\,.
\end{aligned}$$

Therefore, by Lemma 3.12, Theorems 3.11 and 3.13 the tangential component of w is continuous, thus $\mathcal{N}a$ is given by[9]

$$\begin{aligned}
(\mathcal{N}a)(x) &= \nu(x)\times w(x) \;=\; \nu(x)\times\int_{\partial D}\nabla_x\Phi(x,y)\,\big[\operatorname{Div} a(y) - \operatorname{Div} a(x)\big]\,ds(y) \\
&\quad + \operatorname{Div} a(x)\,\nu(x)\times\int_{\partial D} H(y)\,\Phi(x,y)\,ds(y) \\
&\quad - \operatorname{Div} a(x)\,\nu(x)\times\int_{\partial D}\nu(y)\frac{\partial\Phi}{\partial\nu(y)}(x,y)\,ds(y) \;+\; k^2\,\nu(x)\times\int_{\partial D} a(y)\,\Phi(x,y)\,ds(y)\,.
\end{aligned}$$

[9] This can also serve as a definition!

The boundedness of the last three terms as operators from $C^\alpha_{Div}(\partial D)$ into $C^\alpha_t(\partial D)$ follow from the boundedness of the single and double layer boundary operators $\mathcal{S}$ and $\mathcal{D}$. For the boundedness of the first term we apply part (b) of Theorem 3.16. The assumptions

$$\big|\nu(x) \times \nabla_x \Phi(x,y)\big| \;\le\; \frac{c}{|x-y|^2} \text{ for all } x,y \in \partial D\,,\ x \neq y\,, \quad \text{and}$$

$$\big|\nu(x_1) \times \nabla_x \Phi(x_1,y) - \nu(x_2) \times \nabla_x \Phi(x_2,y)\big| \;\le\; c\,\frac{|x_1-x_2|}{|x_1-y|^3}$$

for all $x_1, x_2, y \in \partial D$ with $|x_1 - y| \ge 3|x_1 - x_2|$ are simple to prove (cf. proof of Lemma 3.12). For the third assumption, namely

$$\left|\int_{\partial D \setminus K(x,r)} \nu(x) \times \nabla_x \Phi(x,y)\, ds(y)\right| \;\le\; \left|\int_{\partial D \setminus K(x,r)} \nabla_x \Phi(x,y)\, ds(y)\right| \;\le\; c \tag{3.47}$$

for all $x \in \partial D$ and $r > 0$ we refer to Lemma 3.14 below. This proves boundedness of $\mathcal{N}$ from $C^\alpha_{Div}(\partial D)$ into $C^\alpha_t(\partial D)$. We consider now the surface divergence $\operatorname{Div}\mathcal{N}$. By Corollary 1.16 and the form of w we conclude that $\operatorname{Div}\mathcal{N}a = \operatorname{Div}(\nu \times w) = -\nu \cdot \operatorname{curl} w$. For $x = z + t\nu(z) \in H_\rho \setminus \partial D$ we compute

$$\begin{aligned}
\operatorname{curl} w(x) &= \operatorname{curl}^3 \int_{\partial D} a(y)\,\Phi(x,y)\, ds(y) \;=\; k^2 \int_{\partial D} \nabla_x \Phi(x,y) \times a(y)\, ds(y) \\
&= -k^2 \int_{\partial D} \nabla_y \Phi(x,y) \times a(y)\, ds(y) \\
&= k^2 \int_{\partial D} \nabla_y \Phi(x,y) \times \big[a(z) - a(y)\big]\, ds(y) \;-\; k^2 \int_{\partial D} \nabla_y \Phi(x,y)\, ds(y) \;\times\; a(z) \\
&= k^2 \int_{\partial D} \nabla_y \Phi(x,y) \times \big[a(z) - a(y)\big]\, ds(y) \;-\; k^2 \int_{\partial D} \operatorname{Grad}_y \Phi(x,y)\, ds(y) \;\times\; a(z) \\
&\quad - k^2 \int_{\partial D} \nu(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y)\, ds(y) \;\times\; a(z) \\
&= k^2 \int_{\partial D} \nabla_y \Phi(x,y) \times \big[a(z) - a(y)\big]\, ds(y) \;+\; k^2 \int_{\partial D} H(y)\,\Phi(x,y)\, ds(y) \;\times\; a(z) \\
&\quad - k^2 \int_{\partial D} \nu(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y)\, ds(y) \;\times\; a(z)\,.
\end{aligned}$$

The normal component is bounded by the same arguments as above. □

3.2.4 Uniqueness and Existence

First we want to prove that there exists at most one solution of the exterior boundary value problem (3.37a), (3.37b),(3.38a), (3.38b) and therefore also to the scattering problem (3.35a), (3.35b),(3.38a), (3.38b).

Theorem 3.33 *For every tangential field $c \in C^\alpha_{Div}(\partial D)^3$ there exists at most one radiating solution $E^s, H^s \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ of the exterior boundary value problem (3.37a), (3.37b), (3.38a), (3.38b).*

Proof: Let E^s_j, H^s_j for $j = 1, 2$ be two solutions of the boundary value problem. Then the difference $E^s = E^s_1 - E^s_2$ and $H^s = H^s_1 - H^s_2$ solve the exterior boundary value problem for boundary data $c = 0$.
In the proof of the Stratton-Chu formula (Theorem 3.28) we have derived the following formula:

$$\int_{|y|=R} |\sqrt{\varepsilon_0}\, E^s - \sqrt{\mu_0}\, H^s \times \nu|^2 ds \;=\; \varepsilon_0 \int_{|y|=R} |E^s|^2 ds \;+\; \mu_0 \int_{|y|=R} |H^s \times \nu|^2 ds$$

$$- \,2\sqrt{\varepsilon_0 \mu_0}\,\mathrm{Re} \int_{\partial D} E^s \cdot (\overline{H^s} \times \nu)\, ds\,.$$

The integral $\int_{\partial D} E^s \cdot (\overline{H^s} \times \nu)\, ds = \int_{\partial D} \overline{H^s} \cdot (\nu \times E^s)\, ds$ vanishes by the boundary condition. Since the left hand side tends to zero by the radiation condition we conclude that $\int_{|y|=R} |E^s|^2 ds$ tends to zero as $R \to \infty$. Furthermore, by the Stratton-Chu formula (Theorem 3.28) in the exterior of $\overline{D}$ we can represent $E^s(x)$ in the form

$$E^s(x) \;=\; \mathrm{curl} \int_{\partial D} \big[\nu(y) \times E^s(y)\big]\, \Phi(x,y)\, ds(y) - \frac{1}{i\omega\varepsilon_0}\, \mathrm{curl}\,\mathrm{curl} \int_{\partial D} \big[\nu(y) \times H^s(y)\big]\, \Phi(x,y)\, ds(y)$$

for $x \notin \overline{D}$. From these two facts we observe that every component $u = E^s_j$ satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in the exterior of $\overline{D}$ and $\lim_{R\to 0} \int_{|x|=R} |u| ds = 0$. Furthermore, we recall that $\Phi(x, y)$ satisfies the Sommerfeld radiation condition (3.2) and therefore also $u = E^s_j$. The triangle inequality in the form $|z| \le |z-w| + |w|$, thus $|z|^2 \le 2|z-w|^2 + 2|w|^2$ yields

$$\int_{|x|=R} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 ds(x) \;\le\; 2 \int_{|x|=R} \left| \frac{x}{|x|} \cdot \nabla u(x) - iku(x) \right|^2 ds(x) + 2k^2 \int_{|x|=R} \big|u(x)\big|^2 ds(x)\,,$$

and this tends to zero as R tends to zero. We are now in the position to apply Rellich's Lemma 3.20 (or Lemma 3.21). This yields $u = 0$ in the exterior of $\overline{D}$ and ends the proof. □

Now we turn to the question of **existence** of the scattering problem (3.35a), (3.35b), (3.38a), (3.38b) or, more generally, the exterior boundary value problem (3.37a), (3.37b), (3.38a), (3.38b). First we prove an existence result which is not optimal but rather serves as a preliminary result to motivate a more complicated approach..

Theorem 3.34 *Assume that $w = 0$ is the only solution of the following interior boundary value problem:*

$$\mathrm{curl}\,\mathrm{curl}\, w - k^2 w \;=\; 0 \;\text{in } D\,, \quad \nu \times \mathrm{curl}\, w \;=\; 0 \;\text{on } \partial D\,. \tag{3.48}$$

Then, for every $c \in C^\alpha_{\mathrm{Div}}(\partial D)$, there exists a unique solution $E^s, H^s \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ of the exterior boundary value problem (3.37a), (3.37b), (3.38a), (3.38b). In particular, *under this assumption, the scattering problem (3.35a), (3.35b), (3.38a), (3.38b) has a unique solution for every incident field. The solution has the form of*

$$E^s(x) \;=\; \operatorname{curl} \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y)\,, \quad H^s(x) \;=\; \frac{1}{i\omega\mu_0} \operatorname{curl} E^s(x)\,, \quad x \notin D\,, \tag{3.49}$$

for $a \in C^\alpha_{\mathrm{Div}}(\partial D)$ which is the unique solution of the boundary integral equation

$$\frac{1}{2}\, a(x) \;+\; \nu(x) \times \operatorname{curl} \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y) \;=\; c(x)\,, \quad x \in \partial D\,. \tag{3.50}$$

Proof: Since $\mathcal{M}$ is compact we can apply the Fredholm theory[10] to equation (3.50): Existence is assured if uniqueness holds. Therefore, let $a \in C^\alpha_{\mathrm{Div}}(\partial D)$ be a solution of the homogeneous boundary integral equation (3.32), i.e. with the operator $\mathcal{M}$

$$\frac{1}{2}\, a \;+\; \mathcal{M} a \;=\; 0 \quad \text{on } \partial D\,.$$

Define E and H as in (3.49); that is,

$$E(x) \;=\; \operatorname{curl} \int_{\partial D} a(y)\, \Phi(x,y)\, ds(y)\,, \quad H \;=\; \frac{1}{i\omega\mu_0} \operatorname{curl} E\,, \quad x \notin \partial D\,.$$

Then E, H satisfy the Maxwell system and, by the jump condition of Theorem 3.31 and the homogeneous integral equation, $\nu \times E(x)\big|_+ = \frac{1}{2}\, a + \mathcal{M} a = 0$ on ∂D. The uniqueness result of Theorem 3.33 yields $E = 0$ in $\mathbb{R}^3 \setminus D$. Application of Theorem 3.31 again yields that $\nu \times \operatorname{curl} E$ is continuous on ∂D, thus $\nu \times \operatorname{curl} E\big|_- = 0$ on ∂D. From our assumption we conclude that E vanishes also inside of D. Now we apply Theorem 3.31 a third time and have that $a = \nu \times E\big|_- - \nu \times E\big|_+ = 0$. Therefore, the homogeneous integral equation admits only $a = 0$ as a solution and, therefore, there exists a unique solution of the inhomogeneous equation for every right hand side $c \in C^\alpha_{\mathrm{Div}}(\partial D)$. □

In general, there exist eigenvalues of the problem (3.48). First we show:

Lemma 3.35 *Let u solve the scalar Helmholtz equation $\Delta u + k^2 u = 0$ in the unit ball $B(0,1)$. Then $v(x) = \operatorname{curl}\big[u(x)x\big]$ solves $\operatorname{curl}\operatorname{curl} v - k^2 v = 0$ in $B(0,1)$ and $\nu \times v = \operatorname{Grad} u$ on $S^2 = \partial B(0,1)$ where $\operatorname{Grad} u$ is again the surface gradient of u.*

Proof: From $\operatorname{curl}\operatorname{curl} = \nabla \operatorname{div} - \Delta$ we conclude

$$\operatorname{curl}\operatorname{curl} v(x) \;=\; \operatorname{curl}\big[\nabla \operatorname{div}\big(u(x)x\big) - \Delta\big(u(x)x\big)\big] \;=\; -\operatorname{curl} \Delta\big(u(x)x\big)\,.$$

For any $j \in \{1,2,3\}$ we compute

$$\Delta\big(u(x)x_j\big) \;=\; \big(\Delta u(x)\big)x_j + 2\nabla u(x) \cdot \nabla x_j \;=\; -k^2 u(x) x_j + 2\frac{\partial u}{\partial x_j}(x)$$

and thus $\operatorname{curl}\operatorname{curl} v(x) = -\operatorname{curl} \Delta\big(u(x)x\big) = k^2 v(x)$. Furthermore, $\nu(x) \times v(x) = x \times \big[\nabla u(x) \times x\big] = \operatorname{Grad} u(x)$ on S^2. □

[10] This is actually a result by Riesz.

Example 3.36 *We claim that the field*

$$w(r,\theta,\phi) \;=\; \operatorname{curl}\operatorname{curl}\left[\sin\theta\left(k\,\cos(kr) - \frac{\sin(kr)}{r}\right)\hat{r}\right]$$

satisfies $\operatorname{curl}\operatorname{curl} w - k^2 w = 0$ *in* $B(0,1)$ *and* $\nu(x)\times\operatorname{curl} w(x) = k^2\cos\theta\big[k\,\cos k - \sin k\big]\,\hat{\theta}$ *on* $S^2 = \partial B(0,1)$.

Indeed, we can write $w = \operatorname{curl}\operatorname{curl}\big(u(x)x\big)$ *with*

$$u(r,\theta,\phi) \;=\; \sin\theta\left(k\,\frac{\cos(kr)}{r} - \frac{\sin(kr)}{r^2}\right) \;=\; \sin\theta\,\frac{d}{dr}\frac{\sin(kr)}{r}\,.$$

By direct evaluation of the Laplace operator in spherical coordinates we see that u *satisfies the Helmholtz equation* $\Delta u + k^2 u = 0$ *in all of* $\mathbb{R}^3$. *(Note that* $\frac{\sin(kr)}{r}$ *is analytic in* $\mathbb{R}$*!) By the previous lemma we have that* $v(x) = \operatorname{curl}\big(u(x)x\big)$ *satisfies* $\operatorname{curl}\operatorname{curl} v - k^2 v = 0$*, and thus also* w *satisfies* $\operatorname{curl}\operatorname{curl} w - k^2 w = 0$. *We compute* $\operatorname{curl} w$ *as* $\operatorname{curl} w(x) = \operatorname{curl}\big[\nabla\operatorname{div} - \Delta\big]\big(u(x)x\big) = k^2\operatorname{curl}\big(u(x)x\big) = k^2 v(x)$ *and thus by the previous lemma* $\nu(x)\times\operatorname{curl} w(x) = k^2\operatorname{Grad} u(x) = k^2\cos\theta\big[k\,\cos k - \sin k\big]\,\hat{\theta}$ *on* S^2.
Therefore, if $k > 0$ *is any zero of* $\psi(k) = k\,\cos k - \sin k$ *then the corresponding field* w *satisfies (3.48).*

Instead of the insuffcent ansatz (3.49) we propose a modified one of the form

$$E^s(x) \;=\; \operatorname{curl}\int_{\partial D} a(y)\,\Phi(x,y)\,ds(y) \tag{3.51a}$$

$$+\;\eta\,\operatorname{curl}\operatorname{curl}\int_{\partial D}\big[\nu(y)\times(\hat{S}_i^2 a)(y)\big]\,\Phi(x,y)\,ds(y)\,, \tag{3.51b}$$

$$H^s \;=\; \frac{1}{i\omega\mu_0}\,\operatorname{curl} E^s \tag{3.51c}$$

for some constant $\eta\in\mathbb{C}$, some $a\in C^\alpha_{Div}(\partial D)$, and where the bounded operator $\hat{S}_i : C^\alpha(\partial D)^3 \to C^{1,\alpha}(\partial D)^3$ is the single layer boundary operator for the value $k = i$, considered componentwise; that is, $\hat{S}_i a = (S_i a_1, S_i a_2, S_i a_3)^\top$ for $a = (a_1,a_2,a_3)^\top \in C^\alpha(\partial D)^3$. We note that $\hat{S}_i$ is bounded from $C^\alpha(\partial D)^3$ into $C^{1,\alpha}(\partial D)^3$ by Theorem 3.19. Therefore, the operator $\mathcal{K} : a \mapsto \nu\times\hat{S}_i^2 a$ is compact from $C^\alpha_{Div}(\partial D)$ into itself. We need the following additional result for the single layer boundary operator S_i for wavenumber $k = i$.

Lemma 3.37 *The operator* S_i *is selfadjoint with respect to* $\langle\varphi,\psi\rangle_{\partial D} = \int_{\partial D}\varphi\,\psi\,ds$*; that is*

$$\langle S_i\varphi,\psi\rangle_{\partial D} \;=\; \langle\varphi, S_i\psi\rangle_{\partial D}\quad \textit{for all } \varphi,\psi\in C^\alpha(\partial D)$$

and one-to-one.

Proof: Let $\varphi, \psi \in C^\alpha(\partial D)$ and define $u = \tilde{S}_i\varphi$ and $v = \tilde{S}_i\psi$ as the single layer potentials with densities φ and ψ, respectively. Then u and v are solutions of the Helmholtz equation $\Delta u - u = 0$ and $\Delta v - v = 0$ in $\mathbb{R}^3 \setminus \partial D$. Furthermore, u and v and their derivatives decay exponentially as $|x|$ tends to infinity. u and v are continuous in all of $\mathbb{R}^3$ and $\partial v/\partial\nu|_- - \partial v/\partial\nu|_+ = \psi$. By Green's formula we have

$$\begin{aligned} \langle S_i\varphi, \psi\rangle_{\partial D} &= \int_{\partial D} (S_i\varphi)\, \psi \, ds \;=\; \int_{\partial D} u \left(\frac{\partial v}{\partial \nu}\bigg|_- - \frac{\partial v}{\partial \nu}\bigg|_+ \right) ds \\ &= \iint_D (\nabla u \cdot \nabla v + u\, v)\, dx + \iint_{B(0,R)\setminus \overline{D}} (\nabla u \cdot \nabla v + u\, v)\, dx - \int_{|x|=R} u\, \frac{\partial v}{\partial r} ds\,. \end{aligned}$$

The integral over the sphere $\{x : |x| = R\}$ tends to zero and thus

$$\langle S_i\varphi, \psi\rangle_{\partial D} \;=\; \iint_{\mathbb{R}^3} (\nabla u \cdot \nabla v + u\, v)\, dx\,.$$

This term is symmetric with respect to u and v. Furthermore, if $S_i\varphi = 0$ we conclude for $\psi = \overline{\varphi}$ that u vanishes in $\mathbb{R}^3$. The jump condition $\partial u/\partial\nu|_- - \partial u/\partial\nu|_+ = \varphi$ implies that φ vanishes which shows injectivity of S_i. □

Analogously to the beginning of the previous section we observe (with the help of Theorem 3.31) that the ansatz (3.51b), (3.51c) solves the exterior boundary value problem if $a \in C^\alpha_{Div}(\partial D)$ solves the equation

$$\frac{1}{2}\, a \;+\; \mathcal{M}a \;+\; \eta\mathcal{N}\mathcal{K}a \;=\; c \quad \text{on } \partial D \tag{3.52}$$

where again $\mathcal{K}a = \nu \times \hat{S}_i^2 a$. Finally, we can prove the general existence theorem.

Theorem 3.38 *For every $c \in C^\alpha_{\mathrm{Div}}(\partial D)$, there exists a unique radiating solution $E, H \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ of the exterior boundary value problem. In particular, under this assumption, the scattering problem has a unique solution for every incident field. The solution has the form of (3.51b), (3.51c) for any $\eta \in \mathbb{C} \setminus \mathbb{R}$ and some $a \in C^\alpha_{\mathrm{Div}}(\partial D)$ which solves the boundary equation (3.52).*

Proof: We make the ansatz (3.51b), (3.51c) and have to discuss the boundary equation (3.52). The compactness of $\mathcal{M}$ and $\mathcal{K}$ and the boundedness of $\mathcal{N}$ yields compactness of the composition $\mathcal{N}\mathcal{K}$. Therefore, the Fredholm alternative is applicable to (3.52).
To show uniqueness, let a be a solution of the homogeneous equation. Then, with the ansatz (3.51b), (3.51c), we conclude that $\nu \times E^s|_+ = 0$ and thus $E^s = 0$ in $\mathbb{R}^3 \setminus D$ by the uniqueness result. From the jump conditions of Theorem 3.31 we conclude that (note that $\mathrm{curl}^3 \int_{\partial D} a(y)\, \Phi(x, y)\, ds(y) = -\,\mathrm{curl}\, \Delta \int_{\partial D} a(y)\, \Phi(x, y)\, ds(y) = k^2\, \mathrm{curl} \int_{\partial D} a(y)\, \Phi(x, y)\, ds(y)$)

$$\begin{aligned} \nu \times E^s|_- &= \nu \times E^s|_- - \nu \times E^s|_+ \;=\; -a\,, \\ \nu \times \mathrm{curl}\, E^s|_- &= \nu \times \mathrm{curl}\, E^s|_- - \nu \times \mathrm{curl}\, E^s|_+ \;=\; -\eta\, k^2 \mathcal{K}a\,. \end{aligned}$$

and thus

$$\begin{aligned}
\int_{\partial D} (\nu \times \operatorname{curl} E^s|_-) \cdot \overline{E^s}\, ds &= \eta k^2 \int_{\partial D} (\mathcal{K}a) \cdot (\overline{a} \times \nu)\, ds \\
&= \eta k^2 \int_{\partial D} (\nu \times \hat{S}_i^2 a) \cdot (\overline{a} \times \nu)\, ds \\
&= -\eta k^2 \int_{\partial D} \hat{S}_i^2 a \cdot \overline{a}\, ds \\
&= -\eta k^2 \int_{\partial D} |\hat{S}_i a|^2\, ds\,.
\end{aligned}$$

The left hand side is real valued by Green's theorem applied in D. Also the integral on the right hand side is real. Because η is not real we conclude that both integrals vanish and thus also a by the injectivity of $\hat{S}_i$. □