Chapter 3

Scattering From a Perfect Conductor

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In Section 2.6 of the previous chapter we studied the scattering of a plane wave by balls. In this chapter we investigate the same problem for arbitrary shapes. Also, as in the previous chapter, we consider first the simpler scattering problem for the scalar Helmhotz equation in Section 3.1 before we turn to Maxwell's equations in Section 3.2.

3.1 A Scattering Problem for the Helmholtz Equation

3.1.1 Formulation of the Problem

Throughout this chapter we make the following assumptions on the data:

Assumption: Let the wave number be given by $k = \omega\sqrt{\varepsilon_0\mu_0} > 0$ with constants $\varepsilon_0, \mu_0 > 0$. Let $D \subset \mathbb{R}^3$ be a finite union of bounded domains D_j such that $\overline{D}_j \cap \overline{D}_\ell = \emptyset$ for $j \neq \ell$. Furthermore, we assume that the boundary ∂D is C^2-smooth and that the complement $\mathbb{R}^3 \setminus \overline{D}$ is connected.

Scattering Problem: Given an incident field u^i; that is, a solution u^i of the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in all of $\mathbb{R}^3$, find the total field $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ such that

$$\Delta u \;+\; k^2 u \;=\; 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}\,, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D\,, \tag{3.1}$$

and such that the scattered field $u^s = u - u^i$ satisfies the Sommerfeld radiation condition (2.34); that is,

$$\frac{\partial u^s(r\hat{x})}{\partial r} \;-\; ik\, u^s(r\hat{x}) \;=\; \mathcal{O}\left(\frac{1}{r^2}\right) \quad \text{for } r \to \infty\,, \tag{3.2}$$

uniformly with respect to $\hat{x} \in S^2$. For smooth fields $u \in C^1(\mathbb{R}^3 \setminus D)$ and smooth boundaries the normal derivative is given by $\partial u/\partial \nu = \nabla u \cdot \nu$ where $\nu = \nu(x)$ denotes the exterior unit normal vector at $x \in \partial D$.

Before we investigate uniqueness and existence of solution of this scattering problem we study general properties of solutions of the Helmholtz equation $\Delta u + k^2 u = 0$ in bounded and unbounded domains.

3.1.2 Representation Theorems

We begin with the (really!) fundamental solution of the Helmholtz equation, compare (2.39).

Lemma 3.1 *For $k \in \mathbb{C}$ the function $\Phi_k : \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \neq y\} \to \mathbb{C}$, defined by*

$$\Phi_k(x,y) \;=\; \frac{e^{ik|x-y|}}{4\pi|x-y|}\,, \quad x \neq y\,,$$

is called the ***fundamental solution*** *of the Helmholtz equation, i.e. it holds that*

$$\Delta_x \Phi_k(x,y) \;+\; k^2 \Phi_k(x,y) \;=\; 0 \quad \text{for } x \neq y\,.$$

Proof: This is easy to check. □

We often suppress the index k, i.e. write Φ for Φ_k.

We continue with the existence of certain special improper integrals.

Lemma 3.2 *(a) Let $K : \{(x,y) \in \mathbb{R}^3 \times \overline{D} : x \neq y\} \to \mathbb{C}$ be continuous. Assume that there exists $c > 0$ and $\beta \in (0,1]$ such that*

$$\big|K(x,y)\big| \;\leq\; \frac{c}{|x-y|^{3-\beta}} \quad \text{for all } x \in \mathbb{R}^3 \text{ and } y \in D \text{ with } x \neq y\,.$$

Then $\iint_D K(x,y)\,dy$ exists as an improper integral and there exists $\hat{c}_\beta > 0$ with

$$\iint_{D\setminus B(x,\tau)} \big|K(x,y)\big|\,dy \;\leq\; \hat{c}_\beta \quad \text{for all } x \in \mathbb{R}^3 \text{ and all } \tau > 0\,, \tag{3.3a}$$

$$\iint_{D\cap B(x,\tau)} \big|K(x,y)\big|\,dy \;\leq\; \hat{c}_\beta\,\tau^\beta \quad \text{for all } x \in \mathbb{R}^3 \text{ and all } \tau > 0\,. \tag{3.3b}$$

(b) Let $K : \{(x,y) \in \partial D \times \partial D : x \neq y\} \to \mathbb{C}$ be continuous. Assume that there exists $c > 0$ and $\beta \in (0,1]$ such that

$$\big|K(x,y)\big| \;\leq\; \frac{c}{|x-y|^{2-\beta}} \quad \text{for all } x, y \in \partial D \text{ with } x \neq y\,.$$

Then $\int_{\partial D} K(x,y)\,ds(y)$ exists as an improper integral and there exists $\hat{c}_\beta > 0$ with

$$\int_{\partial D\setminus B(x,\tau)} \big|K(x,y)\big|\,ds(y) \;\leq\; \hat{c}_\beta \quad \text{for all } x \in \partial D \text{ and } \tau > 0\,, \tag{3.3c}$$

$$\int_{\partial D\cap B(x,\tau)} \big|K(x,y)\big|\,ds(y) \;\leq\; \hat{c}_\beta\,\tau^\beta \quad \text{for all } x \in \partial D \text{ and } \tau > 0\,. \tag{3.3d}$$

(c) Let $K : \{(x,y) \in \partial D \times \partial D : x \neq y\} \to \mathbb{C}$ *be continuous. Assume that there exists* $c > 0$ *and* $\beta \in (0,1)$ *such that*

$$\big|K(x,y)\big| \;\leq\; \frac{c}{|x-y|^{3-\beta}} \quad \text{for all } x, y \in \partial D \text{ with } x \neq y\,.$$

Then there exists $\hat{c}_\beta > 0$ *with*

$$\int_{\partial D \setminus B(x,\tau)} \big|K(x,y)\big|\, ds(y) \;\leq\; \hat{c}_\beta\, \tau^{\beta-1} \quad \text{for all } x \in \partial D \text{ and } \tau > 0\,. \tag{3.3e}$$

Proof: (a) Fix $x \in \mathbb{R}^3$ and choose $R > 0$ such that $\overline{D} \subset B(0,R)$.
First case: $|x| \leq 2R$. Then $D \subset B(x, 3R)$ and thus, using spherical polar coordinates w.r.t. x,

$$\iint_{D \setminus B(x,\tau)} \frac{1}{|x-y|^{3-\beta}}\, dy \;\leq\; \iint_{\tau < |y-x| < 3R} \frac{1}{|x-y|^{3-\beta}}\, dy \;=\; 4\pi \int_\tau^{3R} \frac{1}{r^{3-\beta}}\, r^2\, dr$$

$$= \; \frac{4\pi}{\beta}\left[(3R)^\beta - \tau^\beta\right] \;\leq\; \frac{4\pi}{\beta}\,(3R)^\beta\,.$$

Second case: $|x| > 2R$. Then $|x-y| \geq |x| - |y| \geq R$ for $y \in D$ and thus

$$\iint_{D \setminus B(x,\tau)} \frac{1}{|x-y|^{3-\beta}}\, dy \;\leq\; \frac{1}{R^{3-\beta}} \iint_{B(0,R)} dy \;=\; \frac{1}{R^{3-\beta}}\, \frac{4\pi}{3}\, R^3\,.$$

This proves (3.3a). For (3.3b) we compute

$$\iint_{D \cap B(x,\tau)} \big|K(x,y)\big|\, dy \;\leq\; c \iint_{|x-y|<\tau} \frac{1}{|x-y|^{3-\beta}}\, dy \;=\; 4\pi c \int_0^\tau \frac{1}{r^{3-\beta}}\, r^2\, dr \;=\; \frac{4\pi c}{\beta}\, r^\beta\,.$$

(b) We cover ∂D with finitely many open sets U_j; that is $\partial D = \bigcup_{j=1}^m U_j$ and choose a *partition of unity* corresponding to this covering; that is, a family of functions ϕ_j, $j = 1, \ldots, m$, with

- $\phi_j \in C^\infty(\mathbb{R}^3)$ and $0 \leq \phi_j \leq 1$ in $\mathbb{R}^3$,
- the support $S_j := \mathrm{supp}(\phi_j)$ is contained in U_j, and
- $\sum_{j=1}^m \phi_j(x) = 1$ for all $x \in \partial D$.

Then, obviously, $\partial D \supset \bigcup_{j=1}^m S_j$. Furthermore, there exists $\delta > 0$ such that $|x - y| \geq \delta$ for all $(x,y) \in S_j \times \bigcup_{\ell \neq j} (U_\ell \setminus \overline{U_j})$ for every j. Indeed, otherwise there would exist some j and a a sequence $(x_k, y_k) \in S_j \times \bigcup_{\ell \neq j}(U_\ell \setminus \overline{U_j})$ with $|x_k - y_k| \to 0$. There exist convergent subsequences $x_k \to z$ and $y_k \to z$. Then $z \in S_j$ and $z \in \bigcup_{\ell \neq j}(\overline{U_\ell} \setminus U_j)$ which is a contradiction

because $S_j \subset U_j$.
By Definition 1.4 we can use local coordinates $y = \Psi(v)$, $v \in B_2(0,1) \subset \mathbb{R}^2$ and thus

$$\int\limits_{\substack{\partial D\\ |x-y|>\tau}} \frac{ds(y)}{|x-y|^{2-\beta}} = \sum_{\ell=1}^{m} \int\limits_{\substack{\partial D\cap U_\ell\\ |x-y|>\tau}} \frac{\phi_j(y)}{|x-y|^{2-\beta}}\, ds(y)\,.$$

Let $x \in U_j$ for some j. Then $x = \Psi_j(u)$. Since Ψ_j is an isomorphism and Ψ_j' is regular we have an estimate of the form

$$c_1|x-y| \;\le\; |u-v| \;\le c_2|x-y| \quad \text{for all } x, y \in U_j\,.$$

For $y \notin U_j$ we have that $|x-y| \ge \delta$. Therefore, the integrals over $\partial D \cap U_\ell$ for $\ell \not= j$ are easily estimated. For the integral over $\partial D \cap U_j$ we use polar coordinates w.r.t. u; that is,

$$\begin{aligned}
\int\limits_{\substack{\partial D\\ |x-y|>\tau}} \frac{ds(y)}{|x-y|^{2-\beta}} &\le c_2^{2-\beta} \int\limits_{B_2(0,1)\setminus B_2(u,c_1\tau)} \frac{1}{|u-v|^{2-\beta}} \left| \frac{\partial\Psi}{\partial v_1}(v) \times \frac{\partial\Psi}{\partial v_2}(v) \right| dv \\
&\le \|\Psi'\|_\infty^2\, c_2^{2-\beta} \int\limits_{B_2(u,2)\setminus B_2(u,c_1\tau)} \frac{dv}{|u-v|^{2-\beta}} \\
&= \|\Psi'\|_\infty^2\, c_2^{2-\beta}\, 4\pi \int_{c_1\tau}^{2} \frac{1}{r^{2-\beta}}\, r\, dr \;=\; \|\Psi'\|_\infty^2\, \frac{4\pi\, c_2^{2-\beta}}{\beta} \left[2^\beta - (c_1\tau)^\beta\right] \\
&\le \|\Psi'\|_\infty^2\, \frac{4\pi\, c_2^{2-\beta}}{\beta}\, 2^\beta\,.
\end{aligned}$$

The proofs of (3.3d) and part (c) follow the same lines. □

The following representation theorem implies that any solution of the Helmholtz equation is already determined by its Dirichlet- and Neumann data on the boundary. This theorem is totally equivalent to Cauchy's integral representation formula for holomorphic functions.

Theorem 3.3 *(Green's representation theorem in the interior of D)*

For any $k \in \mathbb{C}$ and $u \in C^2(D) \cap C^1(\overline{D})$ we have the representation

$$\iint_D \Phi(x,y)\big[\Delta u(y) + k^2 u(y)\big]\, dy \;+\; \int_{\partial D} \left\{ u(y)\, \frac{\partial\Phi}{\partial\nu(y)}(x,y) - \Phi(x,y)\, \frac{\partial u}{\partial\nu}(y) \right\} ds(y) \;=$$

$$= \begin{cases} -u(x)\,, & x \in D\,, \\ -\frac{1}{2}\, u(x)\,, & x \in \partial D\,, \\ 0\,, & x \notin \overline{D}\,. \end{cases}$$

The domain integral as well as the surface integral (for $x \in \partial D$) exist as improper integrals.

Remarks:

- This theorem tells us that, for $x \in D$, any function u can be expressed as a sum of three potentials:

$$(\tilde{S}\varphi)(x) = \int_{\partial D} \varphi(y)\, \Phi(x,y)\, ds(y)\,, \quad x \notin \partial D\,, \tag{3.4a}$$

$$(\tilde{D}\varphi)(x) = \int_{\partial D} \varphi(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y)\, ds(y)\,, \quad x \notin \partial D\,, \tag{3.4b}$$

$$(V\varphi)(x) = \iint_D \varphi(y)\, \Phi(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,, \tag{3.4c}$$

  which are called **single layer potential**, **double layer potential**, and **volume potential**, respectively, with density φ. We will investigate these potential in detail in Subsection 3.1.3 below.

- The one-dimensional analogon is (for $x \in D = (a,b) \subset \mathbb{R}$)

$$u(x) = \frac{1}{2ik} \int_a^b e^{ik|x-y|} \big[u''(y) + k^2 u(y)\big]\, dy \; + \; \frac{1}{2ik} \left[u(y) \, \frac{d}{dy} e^{ik|x-y|} - u'(y)\, e^{ik|x-y|} \right]_a^b .$$

  Therefore, the one-dimensional fundamental solution is $\Phi(x,y) = -\exp(ik|x-y|)/(2ik)$. One should try to prove this representation in the one-dimensional case!

Proof of Theorem 3.3: First we fix $x \in D$ and a small closed ball $B[x,r] \subset D$ centered at x with radius $r > 0$. For $y \in \partial B(x,r)$ the normal vector $\nu(y) = \frac{x-y}{|y-x|} = (x-y)/r$ is directed into the interior of $B(x,r)$. We apply Green's second identity to u und $v(y) := \Phi(x,y)$ in the domain $D_r := D \setminus B[x,r]$. Then

$$\int_{\partial D} \left\{ u(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y) - \Phi(x,y)\, \frac{\partial u}{\partial \nu}(y) \right\} ds(y) \tag{3.5a}$$

$$+ \int_{\partial B(x,r)} \left\{ u(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y) - \Phi(x,y)\, \frac{\partial u}{\partial \nu}(y) \right\} ds(y) \; = \tag{3.5b}$$

$$= \iint_{D_r} \big\{ u(y)\, \Delta_y \Phi(x,y) - \Phi(x,y)\, \Delta u(y) \big\}\, dy \; = \; - \iint_{D_r} \Phi(x,y) \big[\Delta u(y) + k^2 u(y)\big]\, dy\,,$$

if one uses the Helmholtz equation for Φ. We compute the integral (3.5b). We observe that

$$\nabla_y \Phi(x,y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|} \left(ik - \frac{1}{|x-y|} \right) \frac{y-x}{|x-y|}$$

and thus for $|y-x| = r$:

$$\Phi(x,y) = \frac{\exp(ikr)}{4\pi r}\,,$$

$$\frac{\partial \Phi}{\partial \nu(y)}(x,y) = \frac{x-y}{r} \cdot \nabla_y \Phi(x,y) = -\frac{\exp(ikr)}{4\pi r} \left(ik - \frac{1}{r} \right).$$

Therefore, we compute the integral (3.5b) as

$$\begin{aligned} I_r(x) &:= \int_{\partial B(x,r)} \left\{ u(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y) - \Phi(x,y)\, \frac{\partial u}{\partial \nu}(y) \right\} ds(y) \\ &= \frac{\exp(ikr)}{4\pi r} \int_{|y-x|=r} \left\{ u(y) \left(\frac{1}{r} - ik \right) - \frac{\partial u}{\partial \nu}(y) \right\} ds \\ &= \frac{\exp(ikr)}{4\pi r^2} \int_{|y-x|=r} u(y)\, ds \;-\; \frac{\exp(ikr)}{4\pi r} \int_{|y-x|=r} \left\{ ik\, u(y) + \frac{\partial u}{\partial \nu}(y) \right\} ds\,. \end{aligned}$$

For $r \to 0$ the first term tends to $u(x)$, the second term to zero since the surface area of $\partial B(x,r)$ is just $4\pi r^2$. Therefore, also the limit of the volume integral exists as $r \to 0$ and yields the desired formula for $x \in D$.

Let now $x \in \partial D$. Then we procceed in the same way. The domains of integration in (3.5a) and (3.5a) have to be replaced by $\partial D \setminus B(x,r)$ and $\partial B(x,r) \cap D$, respectively. In the computation the region $\{y \in \mathbb{R}^3 : |y-x| = r\}$ has to be replaced by $\{y \in D : |y-x| = r\}$. By Lemma 1.6 its surface area is $2\pi r^2 + \mathcal{O}(r^3)$ which gives the factor $1/2$ of $u(x)$.

For $x \notin \overline{D}$ the functions u and $v = \Phi(x,\cdot)$ are both solutions of the Helmholtz equation in all of D. Application of Green's second identity in D yields the assertion. □

We note that the volume integral vanishes if u is a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ in D. In this case the function u can be expressed solely as a combination of a single and a double layer surface potential.

From the proof we observe that our assumptions on the smoothness of the boundary ∂D are too strong. The domain D has to satisfy exactly the assumptions which are needed for Green's theorems to hold.

As a corollary we have:

Conclusion 3.4 *Let $u \in C^2(D)$ be a (real- or complex valued) solution of the Helmholtz equation in D. Then u is analytic, i.e. one can locally expand u into a power series of the form*

$$u(x) \;=\; \sum_{n \in \mathbb{N}^3} a_n x_1^{n_1} x_2^{n_2} x_3^{n_3}$$

where we use the notation $\mathbb{N} = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

Proof: From the previous representation of $u(x)$ as a difference of a single and a double layer and the smoothness of the kernels $x \mapsto \Phi(x,y)$ and $x \mapsto \partial\Phi(x,y)/\partial\nu(y)$ for $x \neq y$ it follows immediately that $u \in C^\infty(D)$. The proof of analyticity is technically not easy if one avoids methods from complex analysis.[1] If one uses these methods then one can argue as

[1] We refer to *[?] E. Martensen: Potentialtheorie* for a proof.

follows: Fix $\hat{x} \in D$ and choose $r > 0$ such that $B[\hat{x}, r] \subset D$. Define the region $R \subset \mathbb{C}^3$ and the function $v : R \to \mathbb{C}$ by

$$R = \left\{ z \in \mathbb{C}^3 : \left|\operatorname{Re} z - \hat{x}\right| < r/2, \ \left|\operatorname{Im} z\right| < r/2 \right\},$$

$$v(z) = \int_{\partial D} \left[\frac{\exp\left[ik\sqrt{\sum_{j=1}^3 (z_j - y_j)^2}\right]}{4\pi\sqrt{\sum_{j=1}^3 (z_j - y_j)^2}} \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial}{\partial \nu(y)} \frac{\exp\left[ik\sqrt{\sum_{j=1}^3 (z_j - y_j)^2}\right]}{4\pi\sqrt{\sum_{j=1}^3 (z_j - y_j)^2}} \right] ds(y)$$

for $z \in R$. Taking the square root (principal value, cut along the negative real axis) of the complex number $\sum_{j=1}^3 (z_j - y_j)^2$ is not a problem since $\operatorname{Re} \sum_{j=1}^3 (z_j - y_j)^2 = \sum_{j=1}^3 (\operatorname{Re} z_j - y_j)^2 - (\operatorname{Im} z_j)^2 = |\operatorname{Re} z - y|^2 - |\operatorname{Im} z|^2 > 0$ because of $|\operatorname{Re} z - y| \ge |y - \hat{x}| - |\hat{x} - \operatorname{Re} z| > r - r/2 = r/2$ and $|\operatorname{Im} z| < r/2$. Obviously, the function v is holomorphic in R and thus (complex) analytic. □

As a second corollary we can easily prove the following version of **Holmgren's uniqueness theorem**.

Theorem 3.5 *Let D be a domain with C^2-boundary and $u \in C^1(\overline{D}) \cap C^2(D)$ be a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ in D. Let, furthermore, U be an open set such that $U \cap \partial D \neq \emptyset$ and $u = 0$ and $\partial u/\partial \nu = 0$ on $U \cap \partial D$. Then u vanishes in all of D.*

Proof: Let $z \in U \cap \partial D$ and $B \subset U$ a ball centered at z. Set $\Gamma = D \cap \partial B$. Then $\partial(B \cap D) = \Gamma \cup (B \cap \partial D)$. (The reader should sketch the situation.) We define v by

$$v(x) = \int_\Gamma \left\{ \Phi(x,y) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \Phi}{\partial \nu(y)}(x,y) \right\} ds(y), \quad x \in B.$$

Then v satisfies the Helmholtz equation in B. Application of Green's representation formula of Theorem 3.3 to u in $B \cap D$ yields[2]

$$u(x) = \int_{\partial(B\cap D)} \left\{ \Phi(x,y) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \Phi}{\partial \nu(y)}(x,y) \right\} ds(y) = v(x), \quad x \in B \cap D,$$

since the integral vanishes on $\partial D \cap B$. By the same theorem we conclude that $v(x) = 0$ for $x \in B \setminus \overline{D}$. Since v is analytic by the previous corollary we conclude that v vanishes in all of B. In particular, u vanishes in $B \cap D$. Again, u is analytic in D and D is connected, thus also u vanishes in all of D. □

For radiating solutions of the Helmholtz equation we have the following version of Green's representation theorem.

Theorem 3.6 *(Green's representation theorem in the exterior of D)*

[2] In this case the region $B \cap D$ does not meet the smoothness assumptions of the beginning of this section. The representation theorem still holds by the remark following Theorem 3.3.

Let $k \in \mathbb{R}_{>0}$ and $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ be a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. Furthermore, let u satisfy the Sommerfeld radiation condition (3.2). Then we have the representation

$$\int_{\partial D} \left\{ u(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y) - \Phi(x,y)\, \frac{\partial u}{\partial \nu}(y) \right\} ds(y) \;=\; \begin{cases} u(x)\,, & x \notin \overline{D}\,, \\ \frac{1}{2}\, u(x)\,, & x \in \partial D\,, \\ 0\,, & x \in D\,. \end{cases}$$

The domain integral as well as the surface integral (for $x \in \partial D$) exists as improper integrals.

Proof: Let first $x \notin \overline{D}$. We choose $R > |x|$ such that $\overline{D} \subset B(0,R)$ and apply Green's representation Theorem 3.3 in the annular region $B(0,R) \setminus \overline{D}$. Noting that $\Delta u + k^2 u = 0$ and that $\nu(y)$ for $y \in \partial D$ is directed into the interior of $B(0,R) \setminus \overline{D}$ yields

$$\begin{aligned} u(x) \;&=\; \int_{\partial D} \left\{ u(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y) - \Phi(x,y)\, \frac{\partial u}{\partial \nu}(y) \right\} ds(y) \\ &\quad - \int_{|y|=R} \left\{ u(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y) - \Phi(x,y)\, \frac{\partial u}{\partial \nu}(y) \right\} ds(y)\,. \end{aligned}$$

We show that the surface integral over $\partial B(0,R)$ tends to zero as R tends to infinity. We write this surface integral (for fixed x) as

$$I_R \;=\; \int_{|y|=R} u(y) \left[\frac{\partial \Phi}{\partial \nu(y)}(x,y) - ik\, \Phi(x,y) \right] ds(y) \;-\; \int_{|y|=R} \Phi(x,y) \left[\frac{\partial u}{\partial \nu}(y) - ik\, u(y) \right] ds(y)$$

and use the Cauchy-Schwarz inequality:

$$\begin{aligned} |I_R|^2 \;&\leq\; \int_{|y|=R} |u|^2 ds \int_{|y|=R} \left| \frac{\partial \Phi}{\partial \nu(y)}(x,y) - ik\, \Phi(x,y) \right|^2 ds(y) \\ &\quad + \int_{|y|=R} \big|\Phi(x,y)\big|^2 ds(y) \int_{|y|=R} \left| \frac{\partial u}{\partial \nu} - ik\, u \right|^2 ds(y) \end{aligned}$$

From the radiations of $\Phi(x,\cdot)$ (now for fixed x and with respect to y) and u we conclude that the integrands of the second and forth integral behave as $\mathcal{O}(1/R^4)$ as $R \to \infty$. Since the surface area of $\partial B(0,R)$ is equal to $4\pi R^2$ we conclude that second and forth integral tend to zero as R tends to infinity. Furthermore, the integrand of the third integral behaves as $\mathcal{O}(1/R^2)$ as $R \to \infty$. Therefore, the third integral is bounded. It remains to show that also $\int_{|y|=R} |u|^2 ds$ is bounded. This follows again from the radiation condition. Indeed, from the radiation condition we conclude that

$$\int\limits_{|x|=R} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds \;=\; \int\limits_{|x|=R} \left\{ \left| \frac{\partial u}{\partial r} \right|^2 + k^2 |u|^2 \right\} ds \;+\; 2k\, \mathrm{Im} \int\limits_{|x|=R} u\, \frac{\partial \overline{u}}{\partial r}\, ds\,.$$

Green's theorem, applied in $B(0,R) \setminus D$ to the function u yields that

$$\int\limits_{|x|=R} u\, \frac{\partial \overline{u}}{\partial r}\, ds \;=\; \int\limits_{\partial D} u\, \frac{\partial \overline{u}}{\partial r}\, ds \;+\; \iint\limits_{B(0,R)\setminus D} \left[|\nabla u|^2 - k^2 |u|^2 \right] dx\,.$$

The volume integral is real valued. Therefore, its imaginary part vanishes and we conclude that

$$\int\limits_{|x|=R} \left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2\, ds \;=\; -2k \,\mathrm{Im} \int\limits_{\partial D} u\, \frac{\partial \overline{u}}{\partial r}\, ds \;+\; \mathcal{O}\left(\frac{1}{R^2}\right).$$

This implies, in partular, that $\int_{|y|=R} |u|^2 ds$ is bounded. Altogether, we have shown that I_R tends to zero as R tends to infinity. □

3.1.3 Volume and Surface Potentials

We have seen in the preceding section that any function can be represented by a combination of volume and surface potentials. The integral equation method for solving Maxwell's equations rely heavily on the smoothness properties of these potentials. This subsection is concerned with the investigation of these potentials. The analysis is quite technical and uses the tools from differential geometry from Subsection 1.5 of Chapter 1.

We recall the fundamental solution (in this subsection only for $k \in \mathbb{R}$, $k \geq 0$)

$$\Phi(x,y) \;=\; \frac{e^{ik|x-y|}}{4\pi|x-y|}\,, \quad x \neq y\,, \tag{3.6}$$

and begin with the **volume potential**

$$w(x) \;=\; \iint_D \varphi(y)\, \Phi(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,. \tag{3.7}$$

Lemma 3.7 *Let $\varphi : \overline{D} \to \mathbb{C}$ be piecewise continuous.*[3] *Then $w \in C^1(\mathbb{R}^3)$ and*

$$\frac{\partial w}{\partial x_j}(x) \;=\; \iint_D \varphi(y)\, \frac{\partial \Phi}{\partial x_j}(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,,\; j = 1,2,3\,. \tag{3.8}$$

Proof: We fix $j \in \{1,2,3\}$ and a real valued function $\eta \in C^1(\mathbb{R})$ with $0 \leq \eta(t) \leq 1$ and $\eta(t) = 0$ for $t \leq 1$ and $\eta(t) = 1$ for $t \geq 2$. We set

$$v(x) \;=\; \iint_D \varphi(y)\, \frac{\partial \Phi}{\partial x_j}(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,,$$

and note that the integal exists as an improper integral by Lemma 3.2 since $\left|\frac{\partial \Phi}{\partial x_j}(x,y)\right| \leq c_0\big(|x-y|^{-2}\big)$ for some $c_0 > 0$. Furthermore, set

$$w_\varepsilon(x) \;=\; \iint_D \varphi(y)\, \Phi(x,y)\, \eta\big(|x-y|/\varepsilon\big)\, dy\,, \quad x \in \mathbb{R}^3\,.$$

Then $w_\varepsilon \in C^1(\mathbb{R}^3)$ and

$$v(x) - \frac{\partial w_\varepsilon}{\partial x_j}(x) \;=\; \iint_{|y-x|\leq 2\varepsilon} \varphi(y)\, \frac{\partial}{\partial x_j}\big\{\Phi(x,y)\, \big[1 - \eta\big(|x-y|/\varepsilon\big)\big]\big\}\, dy\,,$$

[3]This means: There exist finitely many domains D_j with $\overline{D} = \cup_j \overline{D_j}$ such that $\varphi|_{D_j}$ has a continuous extension to $\overline{D_j}$ for every j.

and thus

$$\begin{aligned} \left|v(x) - \frac{\partial w_\varepsilon}{\partial x_j}(x)\right| &\le \|\varphi\|_\infty \iint_{|y-x|\le 2\varepsilon} \left\{\left|\frac{\partial \Phi}{\partial x_j}(x,y)\right| + \frac{\|\eta'\|_\infty}{\varepsilon}\,\big|\Phi(x,y)\big|\right\} dy \\ &\le c_1 \iint_{|y-x|\le 2\varepsilon} \left[\frac{1}{|x-y|^2} + \frac{1}{\varepsilon\,|x-y|}\right] dy \\ &= c_1 \int_0^{2\pi}\int_0^{\pi}\int_0^{2\varepsilon} \left[\frac{1}{r^2} + \frac{1}{\varepsilon\, r}\right] r^2 \sin\theta \, dr\, d\theta\, d\varphi \;=\; 16\pi\, c_1\, \varepsilon\,. \end{aligned}$$

Therefore, $\partial w_\varepsilon / \partial x_j \to v$ uniformly in $\mathbb{R}^3$. Thus $w \in C^1(\mathbb{R}^3)$ and $\partial w/\partial x_j = v$. □

For any set T and $\alpha \in (0,1]$ we define the space $C^\alpha(T)$ of bounded Hölder-continuous functions $v : T \to \mathbb{C}$ by

$$C^\alpha(T) \;:=\; \left\{ v \in C(T) : v \text{ bounded and } \sup_{x,y\in T,\; x\neq y} \frac{\big|v(x)-v(y)\big|}{|x-y|^\alpha} \;<\; \infty \right\}$$

Note that any Hölder-continuous function is uniformly continuous and, therefore, has a continuous extension to $\overline{T}$. The space $C^\alpha(T)$ is a normed space with norm

$$\|v\|_{C^\alpha(T)} \;:=\; \underbrace{\sup_{x\in T}\big|v(x)\big|}_{=\,\|v\|_\infty} \;+\; \sup_{x,y\in T,\; x\neq y} \frac{\big|v(x)-v(y)\big|}{|x-y|^\alpha}\,. \tag{3.9}$$

Theorem 3.8 *Let $\varphi \in C^\alpha(D)$ and let w be the volume potential. Then $w \in C^2(D) \cap C^\infty(\mathbb{R}^3 \setminus \overline{D})$ and*

$$\Delta w + k^2 w \;=\; \begin{cases} -\varphi\,, & \text{in } D\,,\\ 0\,, & \text{in } \mathbb{R}^3 \setminus \overline{D}\,. \end{cases}$$

Furthermore, if D_0 is any C^2-smooth domain with $D \subset D_0$ then

$$\frac{\partial^2 w}{\partial x_i \partial x_j}(x) \;=\; \iint_{D_0} \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,y)\,\big[\varphi(y) - \varphi(x)\big]\,dy \;-\; \varphi(x) \int_{\partial D_0} \frac{\partial \Phi}{\partial x_i}(x,y)\,\nu_j(y)\,ds(y) \tag{3.10}$$

for $x \in D$ where we have extended φ by zero in $D_0 \setminus D$.

Proof: First we note that the volume integral in the last formula exists. Indeed, we fix $x \in D$ and split the region of integration into $D_0 = D \cup (D_0 \setminus D)$. The integral over $D_0 \setminus D$ exists because the integrand is smooth. The integral over D exists again by Lemma 3.2 since $\big|\varphi(y) - \varphi(x)\big|\big|\partial^2\Phi/(\partial x_i \partial x_j)(x,y)\big| \le c\,|x-y|^{\alpha-3}$ for $y \in D$. The surface integral is no problem since $x \notin \partial D_0$.
We fix $i, j \in \{1,2,3\}$ and the same funtion $\eta \in C^1(\mathbb{R})$ as in the previous lemma. We define $v := \partial w/\partial x_i$ and

$$\begin{aligned} u(x) &:= \iint_{D_0} \frac{\partial^2 \Phi}{x_i x_j}(x,y)\,\big[\varphi(y) - \varphi(x)\big]\,dy \;-\; \varphi(x) \int_{\partial D_0} \frac{\partial \Phi}{\partial x_i}(x,y)\,\nu_j(y)\,ds(y) \\ v_\varepsilon(x) &:= \iint_D \varphi(y)\,\eta\big(|x-y|/\varepsilon\big)\,\frac{\partial \Phi}{\partial x_i}(x,y)\,dy\,, \quad x \in \mathbb{R}^3\,. \end{aligned}$$

Then $v_\varepsilon \in C^1(D)$ and for $x \in D$

$$\begin{aligned}
\frac{\partial v_\varepsilon}{\partial x_j}(x) &= \iint_D \varphi(y) \frac{\partial}{\partial x_j} \left\{ \eta\big(|x-y|/\varepsilon\big) \frac{\partial \Phi}{\partial x_i}(x,y) \right\} dy \\
&= \iint_{D_0} [\varphi(y) - \varphi(x)] \frac{\partial}{\partial x_j} \left\{ \eta\big(|x-y|/\varepsilon\big) \frac{\partial \Phi}{\partial x_i}(x,y) \right\} dy \\
&\quad + \varphi(x) \iint_{D_0} \frac{\partial}{\partial x_j} \left\{ \eta\big(|x-y|/\varepsilon\big) \frac{\partial \Phi}{\partial x_i}(x,y) \right\} dy \\
&= \iint_{D_0} [\varphi(y) - \varphi(x)] \frac{\partial}{\partial x_j} \left\{ \eta\big(|x-y|/\varepsilon\big) \frac{\partial \Phi}{\partial x_i}(x,y) \right\} dy \\
&\quad - \varphi(x) \int_{\partial D_0} \frac{\partial \Phi}{\partial x_i}(x,y)\, \nu_j(y)\, ds(y)
\end{aligned}$$

provided $2\varepsilon \le d(x, \partial D_0)$. In the last step we used the Divergence Theorem! Therefore,

$$\begin{aligned}
\left| u(x) - \frac{\partial v_\varepsilon}{\partial x_j}(x) \right| &\le \iint_{|y-x|\le 2\varepsilon} |\varphi(y) - \varphi(x)| \left| \frac{\partial}{\partial x_j} \left\{ \big(1 - \eta\big(|x-y|/\varepsilon\big)\big) \frac{\partial \Phi}{\partial x_i}(x,y) \right\} \right| dy \\
&\le c \iint_{|y-x|\le 2\varepsilon} \left(\frac{1}{|y-x|^3} + \frac{\|\eta'\|_\infty}{\varepsilon\, |y-x|^2} \right) |y-x|^\alpha\, dy \\
&= \int_0^{2\varepsilon} \left(\frac{1}{r^{1-\alpha}} + \frac{\|\eta'\|_\infty}{\varepsilon}\, r^\alpha \right) dr \\
&\le 4\pi\, c \left[\frac{(2\varepsilon)^\alpha}{\alpha} + \frac{(2\varepsilon)^{1+\alpha}}{(1+\alpha)\,\varepsilon} \right] \;\le\; \tilde{c}\, \varepsilon\,,
\end{aligned}$$

provided $2\varepsilon \le \operatorname{dist}(x, \partial D)$. Therefore, $\partial v_\varepsilon / \partial x_j \to u$ uniformly on compact subsets of D. Also, $v_\varepsilon \to v$ uniformly on compact subsets of D and thus $w \in C^2(D)$ and $u = \partial^2 w / (\partial x_i \partial x_j)$. This proves (3.10). Finally, setting $D_0 = B(x, R)$:

$$\begin{aligned}
\Delta w(x) &= -k^2 \iint_{B(x,R)} \Phi(x,y) \big[\varphi(y) - \varphi(x)\big]\, dy \\
&\quad - \varphi(x) \sum_{j=1}^{3} \frac{y_j - x_j}{R} \left[\frac{\exp(ikR)}{4\pi\, R} \big(ik - 1/R\big) \frac{x_j - y_j}{R} \right] 4\pi\, R^2 \\
&= -k^2 w(x) \;+\; k^2 \varphi(x) \iint_{B(x,R)} \frac{\exp\big(ik|x-y|\big)}{4\pi\, |x-y|}\, dy \;-\; \varphi(x)\, e^{ikR} (1 - ikR)\,,
\end{aligned}$$

i.e.

$$\Delta w(x) + k^2 w(x) \;=\; -\varphi(x) \underbrace{\left[e^{ikR}(1 - ikR) - k^2 \iint_{B(x,R)} \frac{\exp\big(ik|x-y|\big)}{4\pi\, |x-y|}\, dy \right]}_{=\,1}$$

since

$$k^2 \iint_{B(x,R)} \frac{\exp\big(ik|x-y|\big)}{4\pi\,|x-y|}\,dy \;=\; 4\pi\,k^2 \int_0^R \frac{\exp(ikr)}{4\pi\,r}\,r^2 dr \;=\; k^2 \int_0^R r\,e^{ikr} dr$$
$$= \;\cdots\; = \; -ikR\,e^{ikR} \;+\; e^{ikR} \;-1\,.$$

□

Corollary 3.9 *Let D be C^2-smooth and $A \subset \mathbb{R}^3$ be a closed set with $A \subset D$ or $A \subset \mathbb{R}^3 \setminus \overline{D}$. Furthermore, let w be the volume integral with density $\varphi \in C^\alpha(\partial D)$. Then there exists $c > 0$ with*

$$\|w\|_{C^1(\mathbb{R}^3)} \;\le\; c\,\|\varphi\|_\infty \quad \textit{and} \quad \|w\|_{C^2(A)} \;\le\; c\,\|\varphi\|_{C^\alpha(\partial D)} \tag{3.11}$$

for all $\varphi \in C^\alpha(\partial D)$.

Proof: We estimate

$$\big|\Phi(x,y)\big| \;=\; \frac{1}{4\pi|x-y|}\,,$$
$$\left|\frac{\partial\Phi}{\partial x_j}(x,y)\right| \;\le\; c_1\left[\frac{1}{|x-y|^2} \;+\; \frac{1}{|x-y|}\right],$$
$$\left|\frac{\partial^2\Phi}{\partial x_i \partial x_j}(x,y)\right| \;\le\; c_2\left[\frac{1}{|x-y|^3} \;+\; \frac{1}{|x-y|^2} \;+\; \frac{1}{|x-y|}\right].$$

For $x \in \mathbb{R}^3$ we estimate, using (3.8) and Lemma 3.2 above,

$$\big|w(x)\big| \;\le\; \|\varphi\|_\infty \iint_D \frac{1}{4\pi|x-y|}\,dy \;\le\; c\,\|\varphi\|_\infty\,,$$
$$\left|\frac{\partial w}{\partial x_j}(x)\right| \;\le\; \|\varphi\|_\infty \iint_D \left|\frac{\partial\Phi}{\partial x_j}(x,y)\right| dy \;\le\; c\,\|\varphi\|_\infty\,,$$

where the constant is independent of x and φ. This proves already the first estimate of (3.11). Let now $x \in A$. If $A \subset D$ then there exists $\delta > 0$ with $|x-y| \ge \delta$ for all $x \in A$ and $y \in \partial D$. By (3.10) for $D = D_0$ we have

$$\left|\frac{\partial^2 w}{\partial x_i \partial x_j}(x)\right| \;\le\; \|\varphi\|_{C^\alpha(\partial D)} \iint_D \left|\frac{\partial^2\Phi}{\partial x_i\partial x_j}(x,y)\right| |x-y|^\alpha\,dy \;+\; \|\varphi\|_\infty \int_{\partial D} \left|\frac{\partial\Phi}{\partial x_i}(x,y)\right| ds(y)$$
$$\le\; c_3\,\|\varphi\|_{C^\alpha(\partial D)} \iint_D \frac{dy}{|x-y|^{3-\alpha}} \;+\; c_4\,\|\varphi\|_\infty \int_{\partial D} \frac{ds(y)}{|x-y|^2}$$
$$\le\; c_5\,\|\varphi\|_{C^\alpha(\partial D)} \;+\; \frac{c_4}{\delta^2}\|\varphi\|_\infty \int_{\partial D} ds$$
$$\le\; \tilde{c}\|\varphi\|_{C^\alpha(\partial D)}\,.$$

If $A \subset \mathbb{R}^3 \setminus \overline{D}$ then there exists $\delta > 0$ with $|x-y| \geq \delta$ for all $x \in A$ and $y \in D$. Therefore, we can estimate

$$\left|\frac{\partial^2 w}{\partial x_i \partial x_j}(x)\right| \;\leq\; \|\varphi\|_\infty \iint_D \left|\frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x,y)\right| dy \;\leq\; \|\varphi\|_\infty \frac{c}{\delta^3} \iint_D dy\,.$$

□

We continue with the **single layer surface potential**

$$v(x) \;=\; \int_{\partial D} \varphi(y)\, \Phi(x,y)\, ds(y)\,, \quad x \in \mathbb{R}^3\,. \tag{3.12}$$

The investigation of this potential requires some elementary facts from differential geometry which we have collected in Subsection 1.5 of Chapter 1.

First we note that for continuous densities φ the integral exists (as an improper integral) even for $x \in \partial D$ since Φ has a singularity of the form $\big|\Phi(x,y)\big| = 1/(4\pi|x-y|)$. This follows from Lemma 3.2 above. Before we prove continuity of v we make the following general remark.

Remark 3.10 *A function $v : \mathbb{R}^3 \to \mathbb{C}$ is Hölder continuous if*

(a) v is bounded,

(b) v is Hölder continuous in some H_ρ,

(c) v is Lipschitz continuous in $\mathbb{R}^3 \setminus U_\delta$ for every $\delta > 0$.

Again, H_ρ is the strip around ∂D of thickness ρ and U_δ is the neighborhood of ∂D with thickness δ (see Lemma 1.6).

Proof: Choose $\delta > 0$ such that $U_{3\delta} \subset H_\rho$.
Ist case: $|x_1 - x_2| < \delta$ and $x_1 \in U_{2\delta}$. Then $x_1, x_2 \in U_{3\delta} \subset H_\rho$ and Hölder continuity follows from (b).
2nd case: $|x_1 - x_2| < \delta$ and $x_1 \notin U_{2\delta}$. Then $x_1, x_2 \notin U_\delta$ and thus from (c):

$$\big|v(x_1) - v(x_2)\big| \;\leq\; c|x_1 - x_2| \;\leq\; c\delta^{1-\alpha}|x_1 - x_2|^\alpha\,.$$

3rd case: $|x_1 - x_2| \geq \delta$. Then, by (a),

$$\big|v(x_1) - v(x_2)\big| \;\leq\; 2\|v\|_\infty \;\leq\; \frac{2\|v\|_\infty}{\delta^\alpha}\,|x_1 - x_2|^\alpha\,.$$

Theorem 3.11 *The single-layer potential v from (3.12) with continuous density φ is uniformly Hölder continuous in all of $\mathbb{R}^3$, and for every $\alpha \in (0,1)$ there exists $c > 0$ (independent of φ) with*

$$\|v\|_{C^\alpha(\mathbb{R}^3)} \;\leq\; c\|\varphi\|_\infty\,. \tag{3.13}$$

Proof: We check the conditions (a), (b), (c) of Remark 3.10. Boundedness follows from Lemma 3.2 since $\big|\Phi(x,y)\big| = 1/(4\pi|x-y|)$. For (b) and (c) we write

$$\big|v(x_1) - v(x_2)\big| \;\le\; \|\varphi\|_\infty \int\limits_{\partial D} \big|\Phi(x_1,y) - \Phi(x_2,y)\big|\, ds(y) \tag{3.14}$$

and estimate

$$\begin{aligned}\big|\Phi(x_1,y) - \Phi(x_2,y)\big| \;&\le\; \frac{1}{4\pi}\left|\frac{1}{|x_1-y|} - \frac{1}{|x_2-y|}\right| \;+\; \frac{1}{4\pi|x_1-y|}\left|e^{ik|x_1-y|} - e^{ik|x_1-y|}\right| \\ &\le\; \frac{|x_1-x_2|}{4\pi\,|x_1-y||x_2-y|} \;+\; \frac{k\,|x_1-x_2|}{4\pi\,|x_1-y|} \end{aligned}\tag{3.15}$$

since $\big|\exp(it) - \exp(is)\big| \le |t-s|$ for all $t,s \in \mathbb{R}$. Now (c) follows since $|x_j - y| \ge \delta$ for $x_j \notin U_\delta$.

To show (b); that is, Hölder continuity in H_{ρ_0}, let $x_1, x_2 \in H_{\rho_0}$. We set $\Gamma_{z,r} = \big\{y \in \partial D : |y-z| < r\big\}$ and split the domain of integration into $\Gamma_{z_1,r}$ and $\partial D \setminus \Gamma_{z_1,r}$ where we set $r = 3|x_1 - x_2|$. The integral over $\Gamma_{z_1,r}$ is simply estimated by

$$\begin{aligned}\int\limits_{\Gamma_{z_1,r}} \big|\Phi(x_1,y) - \Phi(x_2,y)\big|\, ds(y) \;&\le\; \frac{1}{4\pi}\int\limits_{\Gamma_{z_1,r}} \frac{ds(y)}{|x_1-y|} \;+\; \frac{1}{4\pi}\int\limits_{\Gamma_{z_1,r}} \frac{ds(y)}{|x_2-y|} \\ &\le\; \frac{1}{2\pi}\int\limits_{\Gamma_{z_1,r}} \frac{ds(y)}{|z_1-y|} \;+\; \frac{1}{2\pi}\int\limits_{\Gamma_{z_2,2r}} \frac{ds(y)}{|z_2-y|}\end{aligned}$$

where we used the unique representation of x_j in the form $x_j = z_j + t_j\nu(z_j)$ with $z_j \in \partial D$ and $|t_j| < \rho_0$ (see Lemma 1.6). By this lemma we conclude that $|z_j - y| \le 2|x_j - y|$ for $j = 1,2$ and $\Gamma_{z_1,r} \subset \Gamma_{z_2,2r}$ because $|y - z_2| \le |y - z_1| + |z_1 - z_2| \le |y - z_1| + 2|x_1 - x_2| \le |y - z_1| + r$. The estimate

$$\int\limits_{\Gamma_{z,\rho}} \frac{ds(y)}{|z-y|} \;\le\; \hat{c}_1\,\rho$$

for some $\hat{c}_1$ independent of z and ρ has been proven in (3.3d). Therefore, we have shown that

$$\int\limits_{\Gamma_{z_1,r}} \big|\Phi(x_1,y) - \Phi(x_2,y)\big|\, ds(y) \;\le\; \frac{c}{2\pi}\,(r + 2r) \;=\; \frac{9c}{2\pi}\,|x_1-x_2| \;\le\; \tilde{c}\,|x_1-x_2|^\alpha$$

where $\tilde{c}$ is independent of x_j.

Now we continue with the integral over $\partial D \setminus \Gamma_{z_1,r}$. For $y \in \partial D \setminus \Gamma_{z_1,r}$ we have $3|x_1 - x_2| = r \le |y - z_1| \le 2|y - x_1|$, thus $|x_2 - y| \ge |x_1 - y| - |x_1 - x_2| \ge (1 - 2/3)\,|x_1 - y| = |x_1 - y|/3$ and therefore

$$\big|\Phi(x_1,y) - \Phi(x_2,y)\big| \;\le\; \frac{3|x_1-x_2|}{4\pi\,|x_1-y|^2} \;+\; \frac{k\,|x_1-x_2|}{4\pi\,|x_1-y|} \;\le\; \frac{3|x_1-x_2|}{\pi\,|z_1-y|^2} \;+\; \frac{k\,|x_1-x_2|}{2\pi\,|z_1-y|}$$

since $|z_1 - y| \le 2|x_1 - y|$. Then we estimate

$$\begin{aligned}
&\int\limits_{\partial D\setminus\Gamma_{z_1,r}} \big|\Phi(x_1,y) - \Phi(x_2,y)\big|\,ds(y) \\
&\le \frac{|x_1-x_2|}{\pi} \int\limits_{\partial D\setminus\Gamma_{z_1,r}} \left[\frac{3}{|z_1-y|^2} + \frac{k}{2|z_1-y|}\right] ds(y) \\
&= \frac{|x_1-x_2|^\alpha}{\pi} \int\limits_{\partial D\setminus\Gamma_{z_1,r}} (r/3)^{1-\alpha} \left[\frac{3}{|z_1-y|^2} + \frac{k}{2|z_1-y|}\right] ds(y) \\
&\le \frac{|x_1-x_2|^\alpha}{\pi\, 3^{1-\alpha}} \int\limits_{\partial D\setminus\Gamma_{z_1,r}} \left[\frac{3}{|z_1-y|^{2-(1-\alpha)}} + \frac{k}{2\,|z_1-y|^{1-(1-\alpha)}}\right] ds(y)
\end{aligned}$$

since $r^{1-\alpha} \le |y - z_1|^{1-\alpha}$ for $y \in \partial D \setminus \Gamma_{z_1,r}$. Therefore,

$$\int\limits_{\partial D\setminus\Gamma_{z_1,r}} \big|\Phi(x_1,y) - \Phi(x_2,y)\big|\,ds(y) \;\le\; \frac{1}{\pi\, 3^{1-\alpha}}\,|x_1-x_2|^\alpha \left[3\,\hat c_{1-\alpha} + \frac{k}{2}\,\hat c_{2-\alpha}\right]$$

with the constants $\hat c_\beta$ from Lemma 3.2, part (b). Altogether we have shown the existence of $c > 0$ with

$$\big|v(x_1) - v(x_2)\big| \;\le\; c\,\|\varphi\|_\infty\,|x_1-x_2|^\alpha \quad \text{for all } x_1, x_2 \in H_{\rho_0}$$

which ends the proof. □

Before we continue with the investigation of the double layer surface potential we prove an auxiliary result which we will use often in the following.

Lemma 3.12 *For $\varphi \in C^\alpha(\partial D)$ and $a \in C(\partial D)^3$ define*

$$w(x) \;=\; \int_{\partial D} \big[\varphi(y) - \varphi(z)\big]\, a(y)\cdot\nabla_y\Phi(x,y)\,ds(y)\,, \quad x \in H_{\rho_0}\,,$$

where $x = z + t\nu(z) \in H_{\rho_0}$ with $|t| < \rho_0$ and $z \in \partial D$. Then the integral exists for $x \in \partial D$ as an improper integral and w is Hölder continuous in H_{ρ_0} for any exponent $\beta < \alpha$. Furthermore, there exists $c > 0$ with $\big|w(x)\big| \le c\|\varphi\|_{C^\beta(\partial D)}$ for all $x \in H_{\rho_0}$, and the constant c does not depend on x and φ (but may depend on a).

Proof: For $x_\ell = z_\ell + t_\ell\nu(z_\ell) \in H_{\rho_0}$, $\ell = 1, 2$, we have to estimate

$$\begin{aligned}
&w(x_1) - w(x_2) \\
&= \int\limits_{\partial D} \big[\varphi(y) - \varphi(z_1)\big]\, a(y)\cdot\nabla_y\Phi(x_1,y) - \big[\varphi(y) - \varphi(z_2)\big]\, a(y)\cdot\nabla_y\Phi(x_2,y)\,ds(y) \qquad (3.16) \\
&= \big[\varphi(z_2) - \varphi(z_1)\big] \int_{\partial D} a(y)\cdot\nabla_y\Phi(x_1,y)\,ds(y) \;+ \\
&\quad + \int_{\partial D} \big[\varphi(y) - \varphi(z_2)\big]\, a(y)\cdot\big[\nabla_y\Phi(x_1,y) - \nabla_y\Phi(x_2,y)\big]\,ds(y)
\end{aligned}$$

and thus

$$\begin{aligned} \big|w(x_1) - w(x_2)\big| &\le \big|\varphi(z_2) - \varphi(z_1)\big| \left| \int_{\partial D} a(y) \cdot \nabla_y \Phi(x_1, y)\, ds(y) \right| + \\ &+ \|a\|_\infty \int_{\partial D} \big|\varphi(y) - \varphi(z_2)\big| \, \big|\nabla_y \Phi(x_1, y) - \nabla_y \Phi(x_2, y)\big| \, ds(y) \quad (3.17) \end{aligned}$$

We need the following estimates of $\nabla_y \Phi$: There exists $\hat{c} > 0$ with

$$\begin{aligned} \big|\nabla_y \Phi(x, y)\big| &\le \frac{\hat{c}}{|x - y|^2}, \quad x \in H_{\rho_0},\ y \in \partial D, \\ \big|\nabla_y \Phi(x_1, y) - \nabla_y \Phi(x_2, y)\big| &\le \hat{c}\, \frac{|x_1 - x_2|}{|x_1 - y|^3}, \quad y \in \partial D,\ x_\ell \in H_{\rho_0} \text{ with } |x_1 - y| \le 3|x_2 - y|. \end{aligned}$$

Proof of these estimates: The first one is obvious. For the second one we observe that $\Phi(x, y) = \phi(|x - y|)$ with $\phi(t) = \exp(ikt)/(4\pi t)$, thus $\phi'(t) = \phi(t)(ik - 1/t)$ and $\phi''(t) = \phi'(t)(ik - 1/t) + \phi(t)/t^2 = \phi(t)\big[(ik - 1/t)^2 + 1/t^2\big]$ and therefore $\big|\phi'(t)\big| \le c_1/t^2$ and $\big|\phi''(t)\big| \le c_2/t^3$ for $0 < t \le 1$. For $t \le 3s$ we have

$$\begin{aligned} \big|\phi'(t) - \phi'(s)\big| &= \left| \int_s^t \phi''(\tau)\, d\tau \right| \le c_2 \left| \int_s^t \frac{d\tau}{\tau^3} \right| = \frac{c_2}{2} \left|t^{-2} - s^{-2}\right| \\ &= \frac{c_2}{2} \frac{|t^2 - s^2|}{t^2 s^2} \le \frac{c_2}{2} |t - s| \left(\frac{1}{t s^2} + \frac{1}{t^2 s} \right) \\ &\le \frac{c_2}{2} |t - s| \left(\frac{9}{t^3} + \frac{3}{t^3} \right) = 6 c_2 \frac{|t - s|}{t^3}. \end{aligned}$$

Setting $t = |x_1 - y|$ and $s = |x_2 - y|$ and observing that $|t - s| \le |x_1 - x_2|$ yields the estimate

$$\begin{aligned} \big|\nabla_y \Phi(x_1, y) - \nabla_y \Phi(x_2, y)\big| &= \left| \phi'(|x_1 - y|) \frac{y - x_1}{|y - x_1|} - \phi'(|x_2 - y|) \frac{y - x_2}{|y - x_2|} \right| \\ &\le \big|\phi'(|x_1 - y|) - \phi'(|x_2 - y|)\big| \\ &\quad + \big|\phi'(|x_2 - y|)\big| \left| \frac{y - x_1}{|y - x_1|} - \frac{y - x_2}{|y - x_2|} \right| \\ &\le 6\, c_2 \frac{|x_2 - x_1|}{|y - x_1|^3} + 2 \big|\phi'(|x_2 - y|)\big| \frac{|x_2 - x_1|}{|y - x_1|} \\ &\le 6\, c_2 \frac{|x_2 - x_1|}{|y - x_1|^3} + 2\, c_1 \frac{|x_2 - x_1|}{|y - x_1||y - x_2|^2} \\ &\le (6 c_2 + 18 c_1) \frac{|x_2 - x_1|}{|y - x_1|^3}. \end{aligned}$$

This yields the second estimate.

Now we split the region of integration again into $\Gamma_{z_1,r}$ and $\partial D \setminus \Gamma_{z_1,r}$ with $r = 3|x_1 - x_2|$ where again $\Gamma_{z,r} = \{y \in \partial D : |y - z| < r\}$. The integral (in the form (3.16)) over $\Gamma_{z_1,r}$ is estimated by[4]

$$\begin{aligned}
&\int_{\Gamma_{z_1,r}} \big|\big[\varphi(y) - \varphi(z_1)\big]\, a(y) \cdot \nabla_y \Phi(x_1, y) - \big[\varphi(y) - \varphi(z_2)\big]\, a(y) \cdot \nabla_y \Phi(x_2, y)\big|\, ds(y) \quad (3.18)\\
&\quad\leq\; c \int_{\Gamma_{z_1,r}} \left[|y - z_1|^\alpha \frac{1}{|y - x_1|^2} + |y - z_2|^\alpha \frac{1}{|y - x_2|^2}\right] ds(y)\\
&\quad\leq\; c \int_{\Gamma_{z_1,r}} |y - z_1|^\alpha \frac{1}{|y - z_1|^2}\, ds(y) \;+\; c \int_{\Gamma_{z_2,2r}} |y - z_2|^\alpha \frac{1}{|y - z_2|^2}\, ds(y)
\end{aligned}$$

since $\Gamma_{z_1,r} \subset \Gamma_{z_2,2r}$ and $|x_\ell - y| \geq |z_\ell - y|/2$. Therefore, using (3.3d), this term behaves as $r^\alpha = 3^\alpha |x_1 - x_2|^\alpha \leq c|x_1 - x_2|^\beta$.
We finally consider the integral over $\partial D \setminus \Gamma_{z_1,r}$ und use the form (3.17):

$$\begin{aligned}
I \;:=\;& \big|\varphi(z_2) - \varphi(z_1)\big| \left|\int_{\partial D \setminus \Gamma_{z_1,r}} a(y) \cdot \nabla_y \Phi(x_1, y)\, ds(y)\right| +\\
&+ \|a\|_\infty \int_{\partial D \setminus \Gamma_{z_1,r}} \big|\varphi(y) - \varphi(z_2)\big|\, \big|\nabla_y \Phi(x_1, y) - \nabla_y \Phi(x_2, y)\big|\, ds(y)\\
\leq\;& c \int\limits_{\partial D \setminus \Gamma_{z_1,r}} \left[|z_2 - z_1|^\alpha \frac{1}{|x_1 - y|^2} + |y - z_2|^\alpha \frac{|x_1 - x_2|}{|x_1 - y|^3}\right] ds(y)
\end{aligned}$$

Since $y \in \partial D \setminus \Gamma_{z_1,r}$ we use the estimates

- $|x_1 - x_2| = \frac{r}{3} \leq \frac{1}{3}|y - z_1| \leq \frac{2}{3}|y - x_1| < |y - x_1|$ and
- $|y - z_2| \leq 2|y - x_2| \leq 2|y - x_1| + 2|x_1 - x_2| \leq 4|y - x_1|$,

thus

$$\begin{aligned}
I \;\leq\;& c \int\limits_{\partial D \setminus \Gamma_{z_1,r}} \left[|x_2 - x_1|^\alpha \frac{1}{|z_1 - y|^2} + \frac{|x_1 - x_2|}{|x_1 - y|^{3-\alpha}}\right] ds(y)\\
\leq\;& c' |x_1 - x_2|^\beta \int\limits_{\partial D \setminus \Gamma_{z_1,r}} \left[|x_2 - x_1|^{\alpha-\beta} \frac{1}{|z_1 - y|^2} + \frac{|x_1 - x_2|^{1-\beta}}{|z_1 - y|^{3-\alpha}}\right] ds(y)\\
\leq\;& c|x_1 - x_2|^\beta \int\limits_{\partial D \setminus \Gamma_{z_1,r}} \frac{1}{|z_1 - y|^{2-(\alpha-\beta)}}\, ds(y) \;\leq\; c\,\hat{c}_{\alpha-\beta} |x_1 - x_2|^\beta
\end{aligned}$$

with the constant $\hat{c}_{\alpha-\beta}$ from Lemma 3.2. This, together with (3.18) proves the Hölder-continuity of w. The proof of the estimate $\big|w(x)\big| \leq v\|\varphi\|_{C^\beta(\partial D)}$ for $x \in H_{\rho_0}$ is simpler and left to the reader. □

[4] The constant c is different from line to line.

Now we continue with the **double layer surface potential**

$$v(x) \;=\; \int_{\partial D} \varphi(y)\,\frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\,ds(y)\,,\quad x \in \mathbb{R}^3 \setminus \partial D\,, \tag{3.19}$$

for Hölder-continuous densities φ. Here we indicate the dependence on $k \geq 0$ by writing Φ_k.

Theorem 3.13 *The double layer potential v from (3.19) with Hölder-continuous density $\varphi \in C^\alpha(\partial D)$ can be continuously extended from D to $\overline{D}$ and from $\mathbb{R}^3 \setminus \overline{D}$ to $\mathbb{R}^3 \setminus D$ with limiting values*

$$\lim_{\substack{x\to x_0\\ x\in D}} v(x) \;=\; -\frac{1}{2}\,\varphi(x_0) \;+\; \int_{\partial D} \varphi(y)\,\frac{\partial \Phi_k}{\partial \nu(y)}(x_0,y)\,ds(y)\,,\quad x_0 \in \partial D\,, \tag{3.20a}$$

$$\lim_{\substack{x\to x_0\\ x\notin \overline{D}}} v(x) \;=\; +\frac{1}{2}\,\varphi(x_0) \;+\; \int_{\partial D} \varphi(y)\,\frac{\partial \Phi_k}{\partial \nu(y)}(x_0,y)\,ds(y)\,,\quad x_0 \in \partial D\,. \tag{3.20b}$$

v is Hölder-continuous in D and in $\mathbb{R}^3 \setminus \overline{D}$ with exponent β for every $\beta < \alpha$. The integrals exist as improper integrals.

Proof: First we note that the integrals exist since for $x_0, y \in \partial D$ we can estimate

$$\left|\frac{\partial \Phi_k}{\partial \nu(y)}(x_0,y)\right| \;=\; \left|\frac{\exp(ik|x_0-y|)}{4\pi|x_0-y|}\left(ik - \frac{1}{|x_0-y|}\right)\right| \frac{\bigl|\nu(y)\cdot(y-x_0)\bigr|}{|y-x_0|} \;\leq\; \frac{c}{|x_0-y|}\,.$$

by Lemma 1.6. Furthermore, v has a decomposition into $v = v_0 + v_1$ where

$$v_0(x) \;=\; \int_{\partial D} \varphi(y)\,\frac{\partial \Phi_0}{\partial \nu(y)}(x,y)\,ds(y)\,,\quad x \in \mathbb{R}^3 \setminus \partial D\,,$$

$$v_1(x) \;=\; \int_{\partial D} \varphi(y)\,\frac{\partial(\Phi_k - \Phi_0)}{\partial \nu(y)}(x,y)\,ds(y)\,,\quad x \in \mathbb{R}^3 \setminus \partial D\,.$$

It is easily seen that the kernel $K(x,y) = \varphi(y)\,\frac{\partial(\Phi_k-\Phi_0)}{\partial\nu(y)}(x,y)$ of the integral in the definition of v_1 is continuous in $\mathbb{R}^3 \times \mathbb{R}^3$ and continuously differentiable with respect to x and bounded on the set $\bigl\{(x,y) \in \mathbb{R}^3 \times \partial D : x \neq y\bigr\}$. From this it follows that v_1 is Hölder-continuous in all of $\mathbb{R}^3$. Also this is easy to prove.
We continue with the analysis of v_0 and note that we have again prove estimates of the form (a) and (b) of Remark 3.10.
First, for $x \in U_{\delta/4}$ we have $x = z + t\nu(z)$ with $|t| < \delta/4$ and $z \in \partial D$. We write $v_0(x)$ in the form

$$v_0(x) \;=\; \underbrace{\int_{\partial D} \bigl[\varphi(y) - \varphi(z)\bigr]\,\frac{\partial \Phi_0}{\partial \nu(y)}(x,y)\,ds(y)}_{=\,\tilde v_0(x)} \;+\varphi(z)\int_{\partial D} \frac{\partial \Phi_0}{\partial \nu(y)}(x,y)\,ds(y)\,.$$

The function $\tilde v_0$ is Hölder continuous in $U_{\delta/4}$ with exponent $\beta < \alpha$ by Lemma 3.12 (take $a(y) = \nu(y)$). This proves the estimate (a) and (b) of Remark 3.10 for $\tilde v_0$.

Now we go back to the decomposition

$$v_0(x) \;=\; \tilde{v}_0(x) \;+\; \varphi(z) \int_{\partial D} \frac{\partial \Phi_0}{\partial \nu(y)}(x,y)\, ds(y)$$

By Green's representation (Theorem 3.3) for $k=0$ and $u=1$ we observe that

$$\int_{\partial D} \frac{\partial \Phi_0}{\partial \nu(y)}(x,y)\, ds(y) \;=\; \begin{cases} -1\,, & x \in D\,, \\ -1/2\,, & x \in \partial D\,, \\ 0\,, & x \notin \overline{D}\,. \end{cases}$$

Therefore,

$$\lim_{x \to x_0,\; x \in D} v_0(x) \;=\; \tilde{v}_0(x_0) - \varphi(x_0) \;=\; v_0(x_0) + \frac{1}{2}\,\varphi(x_0) - \varphi(x_0) \;=\; -\frac{1}{2}\,\varphi(x_0)\,,$$

$$\lim_{x \to x_0,\; x \notin \overline{D}} v_0(x) \;=\; \tilde{v}_0(x_0) \;=\; v_0(x_0) + \frac{1}{2}\,\varphi(x_0)\,.$$

This ends the proof. □

In the following we write

$$v(x_0)\big|_- \;=\; \lim_{\substack{x \to x_0 \\ x \in D}} v(x) \quad\text{and}\quad v(x_0)\big|_+ \;=\; \lim_{\substack{x \to x_0 \\ x \notin \overline{D}}} v(x) \quad\text{and}$$

$$\frac{\partial v}{\partial \nu}(x_0)\bigg|_- \;=\; \lim_{\substack{x \to x_0 \\ x \in D}} \nu(x_0)\cdot\nabla v(x) \quad\text{and}\quad \frac{\partial v}{\partial \nu}(x_0)\bigg|_+ \;=\; \lim_{\substack{x \to x_0 \\ x \notin \overline{D}}} \nu(x_0)\cdot\nabla v(x)\,.$$

We continue with the **derivative of the single layer potential** (3.12); that is,

$$v(x) \;=\; \int_{\partial D} \varphi(y)\, \Phi(x,y)\, ds(y)\,, \quad x \in \mathbb{R}^3\,.$$

First we show the following auxiliary result.

Lemma 3.14 *(a) There exists $c > 0$ with*

$$\left| \int_{\partial D \setminus K(x,\tau)} \nabla_x \Phi(x,y)\, ds(y) \right| \;\le c \quad \textit{for all } x \in \mathbb{R}^3\,,\ \tau > 0\,,$$

(b) $$\lim_{\tau \to 0} \int\limits_{\partial D \setminus K(x,\tau)} \nabla_x \Phi(x,y)\, ds(y) = \int\limits_{\partial D} H(y)\, \Phi(x,y)\, ds(y) - \int\limits_{\partial D} \nu(y)\, \frac{\partial \Phi}{\partial \nu(y)}(x,y)\, ds(y)$$

for $x \in \mathbb{R}^3$ where $H(y) = \big(\operatorname{Div} \hat{e}_t^1(y), \operatorname{Div} \hat{e}_t^2(y), \operatorname{Div} \hat{e}_t^3(y)\big)^\top \in \mathbb{R}^3$ and $\hat{e}_t^j(y) = \nu(y) \times \big(\hat{e}^j \times \nu(y)\big)$, $j = 1,2,3$, the tangential components of the unit vectors $\hat{e}^j$.

Proof: For any $x \in \mathbb{R}^3$ we have

$$\begin{aligned}
\int\limits_{\partial D \setminus K(x,\tau)} \nabla_x \Phi(x,y)\, ds(y) &= - \int\limits_{\partial D \setminus K(x,\tau)} \nabla_y \Phi(x,y)\, ds(y) \\
&= - \int\limits_{\partial D \setminus K(x,\tau)} \operatorname{Grad}_y \Phi(x,y)\, ds(y) - \int\limits_{\partial D \setminus K(x,\tau)} \frac{\partial \Phi}{\partial \nu(y)}(x,y)\, \nu(y)\, ds(y)
\end{aligned}$$

and thus for any fixed vector $a \in \mathbb{C}^3$ by the previous theorem (here again $\Gamma(x,\tau) = \partial D \cap K(x,\tau)$ and $a_t(y) = \nu(y) \times \big(a \times \nu(y)\big)$)

$$\begin{aligned}
a \cdot \int\limits_{\partial D \setminus K(x,\tau)} \nabla_x \Phi(x,y)\, ds(y) &= \int\limits_{\partial D \setminus K(x,\tau)} \operatorname{Div} a_t(y)\, \Phi(x,y)\, ds(y) \\
&\quad + \int\limits_{\partial \Gamma(x,\tau)} a_t(y) \cdot \nu_0(y)\, \Phi(x,y)\, ds(y) \\
&\quad + \int\limits_{\partial D \setminus K(x,\tau)} \frac{\partial \Phi}{\partial \nu(y)}(x,y)\, a \cdot \nu(y)\, ds(y)\,.
\end{aligned}$$

The first and third integrals converge uniformly with respect to $x \in \partial D$ when τ tends to zero because the integrands are weakly singular. For the second integral we note that

$$\begin{aligned}
\left| \int\limits_{\partial \Gamma(x,\tau)} a_t(y) \cdot \nu_0(y)\, \Phi(x,y)\, ds(y) \right| &= \frac{1}{4\pi\tau} \left| \int\limits_{\partial \Gamma(x,\tau)} a_t(y) \cdot \nu_0(y)\, ds(y) \right| \\
&= \frac{1}{4\pi\tau} \left| \int\limits_{\Gamma(x,\tau)} \operatorname{Div} a_t(y)\, ds(y) \right|
\end{aligned}$$

and this tends to zero uniformly with respect to $x \in \partial D$ when τ tends to zero. The conclusion follows if we take for a the unit coordinate vectors $\hat{e}^{(j)}$. $\square$

Theorem 3.15 *The derivative of the single layer potential v from (3.12) with Hölder-continuous density $\varphi \in C^\alpha(\partial D)$ can be continuously extended from D to $\overline{D}$ and from $\mathbb{R}^3 \setminus \overline{D}$ to $\mathbb{R}^3 \setminus D$. The tangential component is continuous, i.e.* $\operatorname{Grad} v|_- = \operatorname{Grad} v|_+$, *and the limiting values of the normal derivatives are*

$$\left.\frac{\partial v}{\partial \nu}(x)\right|_\pm = \mp \frac{1}{2}\, \varphi(x) + \int_{\partial D} \varphi(y)\, \frac{\partial \Phi}{\partial \nu(x)}(x,y)\, ds(y)\,, \quad x \in \partial D\,. \tag{3.21}$$

The integral exists as an improper integral.

Proof: We note that the integral exists (see proof of Theorem 3.13). First we consider the density 1, i.e. we set

$$v_1(x) \;=\; \int_{\partial D} \Phi(x,y)\,ds(y)\,, \quad x \in \mathbb{R}^3\,.$$

By the previous lemma we have that

$$\nabla v_1(x) \;=\; -\int_{\partial D} \nu(y)\,\frac{\partial \Phi}{\partial \nu(y)}(x,y)\,ds(y) \;+\; \int_{\partial D} H(y)\,\Phi(x,y)\,ds(y)\,, \quad x \notin \partial D\,, \qquad (3.22)$$

where again $H(y) = \big(\operatorname{Div}\hat{e}_t^1(y), \operatorname{Div}\hat{e}_t^2(y), \operatorname{Div}\hat{e}_t^3(y)\big)^\top \in \mathbb{R}^3$.

The right hand side is the sum of a double and a single layer potential. By Theorems 3.11 and 3.13 it has a continuous extension to the boundary from the inside and the outside with limiting values

$$\begin{aligned}\nabla v_1(x)\big|_\pm \;&=\; \mp\frac{1}{2}\nu(x) \;-\; \int_{\partial D} \nu(y)\,\frac{\partial \Phi}{\partial \nu(y)}(x,y)\,ds(y) \;+\; \int_{\partial D} H(y)\,\Phi(x,y)\,ds(y) \\ &=\; \mp\frac{1}{2}\nu(x) \;+\; \int_{\partial D} \nabla_x \Phi(x,y)\,ds(y)\end{aligned}$$

for $x \in \partial D$. The last integral has to be interpreted as a Cauchy principal value as in part (b) of the previous Lemma3.14. In particular, the tangential component is continuous and the normal derivatives jumps, and we have

$$\frac{\partial v_1}{\partial \nu}(x)\bigg|_\pm \;=\; \mp\frac{1}{2} \;+\; \int_{\partial D} \frac{\partial \Phi}{\partial \nu(x)}(x,y)\,ds(y)\,, \quad x \in \partial D\,.$$

for $x \in \partial D$. Now we consider v and have for $x = z + t\nu(z) \in H_{\rho_0} \setminus \partial D$ (that is, $t \neq 0$)

$$\nabla v(x) \;=\; \int_{\partial D} \varphi(y)\,\nabla_x \Phi(x,y)\,ds(y) \;=\; \underbrace{\int_{\partial D} \nabla_x \Phi(x,y)\,\big[\varphi(y) - \varphi(z)\big]\,ds(y)}_{=\;\tilde{v}(x)} \;+\; \varphi(z)\,\nabla v_1(x)\,.$$

Application of Lemma 3.12 yields that $\tilde{v}$ is Hölder continuous in all of H_{ρ_0} with limiting value

$$\tilde{v}(x_0) \;=\; \int_{\partial D} \nabla_x \Phi(x_0,y)\,\big[\varphi(y) - \varphi(x_0)\big]\,ds(y) \quad \text{for } x_0 \in \partial D\,.$$

This proves that the gradient has continuous extensions from the inside and outside of D and

$$\begin{aligned}\frac{\partial v}{\partial \nu}(x)\bigg|_\pm \;&=\; \mp\frac{1}{2}\varphi(x) + \varphi(x)\int_{\partial D} \frac{\partial \Phi}{\partial \nu(x)}(x,y)\,ds(y) + \int_{\partial D} \frac{\partial \Phi}{\partial \nu(x)}(x,y)\,\big[\varphi(y) - \varphi(x)\big]\,ds(y) \\ &=\; \mp\frac{1}{2}\varphi(x) \;+\; \int_{\partial D} \varphi(y)\,\frac{\partial \Phi}{\partial \nu(x)}(x,y)\,ds(y)\,,\end{aligned}$$

$$\operatorname{Grad} v(x)\big|_\pm \;=\; \int_{\partial D} \operatorname{Grad}_x \Phi(x,y)\,\big[\varphi(y) - \varphi(z)\big]\,ds(y) \;+\; \varphi(z)\,\operatorname{Grad} v_1(x) \qquad (3.23)$$

for $x \in \partial D$. □

3.1.4 Boundary Integral Operators

It is the aim of this subsection to investigate the mapping properties of the traces of the single and double layer potentials on the boundary ∂D. We start with a general theorem on boundary integral operators with singular kernels.

Theorem 3.16 *Let* $\Lambda = \big\{(x,y) \in \partial D \times \partial D : x \neq y\big\}$ *and* $G \in C(\Lambda)$.

(a) Let there exist $c > 0$ *and* $\alpha \in (0,1)$ *such that*

$$\big|G(x,y)\big| \;\leq\; \frac{c}{|x-y|^{2-\alpha}} \quad \text{for all } (x,y) \in \Lambda\,, \tag{3.24a}$$

$$\big|G(x_1,y) - G(x_2,y)\big| \;\leq\; c\,\frac{|x_1-x_2|}{|x_1-y|^{3-\alpha}} \quad \text{for all } (x_1,y),(x_2,y) \in \Lambda \tag{3.24b}$$

$$\text{with } |x_1-y| \geq 3|x_1-x_2|\,.$$

Then the operator $K_1 : C(\partial D) \to C^\alpha(\partial D)$, *defined by*

$$(K_1\varphi)(x) \;=\; \int_{\partial D} G(x,y)\,\varphi(y)\,ds(y)\,, \quad x \in \partial D\,,$$

is well defined and bounded.

(b) Let there exist $c > 0$ *such that:*

$$\big|G(x,y)\big| \;\leq\; \frac{c}{|x-y|^{2}} \quad \text{for all } (x,y) \in \Lambda\,, \tag{3.24c}$$

$$\big|G(x_1,y) - G(x_2,y)\big| \;\leq\; c\,\frac{|x_1-x_2|}{|x_1-y|^{3}} \quad \text{for all } (x_1,y),(x_2,y) \in \Lambda \tag{3.24d}$$

$$\text{with } |x_1-y| \geq 3|x_1-x_2|\,,$$

$$\left|\int_{\partial D\setminus K(x,r)} G(x,y)\,ds(y)\right| \;\leq\; c \quad \text{for all } x \in \partial D \text{ and } r > 0\,. \tag{3.24e}$$

Then the operator $K_2 : C^\alpha(\partial D) \to C^\alpha(\partial D)$, *defined by*

$$(K_2\varphi)(x) \;=\; \int_{\partial D} G(x,y)\,\big[\varphi(y) - \varphi(x)\big]\,ds(y)\,, \quad x \in \partial D\,,$$

is well defined and bounded.

Proof: (a) We follow the idea of the proof of Theorem 3.11 and write

$$\big|(K_1\varphi)(x_1) - (K_1\varphi(x_2)\big| \;\leq\; \|\varphi\|_\infty \int_{\partial D} \big|G(x_1,y) - G(x_2,y)\big|\,ds(y)\,.$$

We split the region of integration again into $\Gamma_{x_1,r}$ and $\partial D \setminus \Gamma_{x_1,r}$ where again $\Gamma_{x_1,r} = \{y \in \partial D : |y - x_1| < r\}$ and set $r = 3|x_1 - x_2|$. The integral over $\Gamma_{x_1,r}$ can be estimated with (3.3d) of Lemma 3.2 by

$$\begin{aligned}\int_{\Gamma_{x_1,r}} \big|G(x_1,y) - G(x_2,y)\big|\, ds(y) &\le c \int_{\Gamma_{x_1,r}} \frac{ds(y)}{|x_1 - y|^{2-\alpha}} + c \int_{\Gamma_{x_2,2r}} \frac{ds(y)}{|x_2 - y|^{2-\alpha}} \\ &\le c' r^\alpha = (c' 3^\alpha)\, |x_1 - x_2|^\alpha\end{aligned}$$

since $\Gamma_{x_1,r} \subset \Gamma_{x_2,2r}$. Here we used formula (3.3d) of Lemma 3.2.

For the integral over $\partial D \setminus \Gamma_{x_1,r}$ we note that $|y - x_1| \ge r = 3|x_1 - x_2|$ for $y \in \partial D \setminus \Gamma_{x_1,r}$, thus

$$\begin{aligned}\int_{\partial D \setminus \Gamma_{x_1,r}} \big|G(x_1,y) - G(x_2,y)\big|\, ds(y) &\le c\, |x_1 - x_2| \int_{\partial D \setminus \Gamma_{x_1,r}} \frac{ds(y)}{|x_1 - y|^{3-\alpha}} \\ &\le c' |x_1 - x_2|\, r^{\alpha-1} = c' 3^{\alpha-1}\, |x_1 - x_2|^\alpha\end{aligned}$$

where we used estimate (3.3e). The proof of $\big|(K_1\varphi)(x)\big| \le c\|\varphi\|_\infty$ is similar (and simpler) and is left to the reader.

For part (b) we follow the ideas of the proof of Lemma 3.12. We write

$$\begin{aligned}\big|(K_2\varphi)(x_1) - (K_2\varphi(x_2)\big| &\le \int_{\Gamma_{x_1,r}} \big|G(x_1,y)\big[\varphi(y) - \varphi(x_1)\big] - G(x_2,y)\big[\varphi(y) - \varphi(x_2)\big]\big|\, ds(y) \\ &\quad + \big|\varphi(x_2) - \varphi(x_1)\big| \left| \int_{\partial D \setminus \Gamma_{x_1,r}} G(x_1,y)\, ds(y) \right| \\ &\quad + \int_{\partial D \setminus \Gamma_{x_1,r}} \big|\varphi(y) - \varphi(x_2)\big|\, \big|G(x_1,y) - G(x_2,y)\big|\, ds(y) \\ &\le c\|\varphi\|_{C^\alpha(\partial D)} \left[\int_{\Gamma_{x_1,r}} \frac{ds(y)}{|y - x_1|^{2-\alpha}} + \int_{\Gamma_{x_2,2r}} \frac{ds(y)}{|y - x_2|^{2-\alpha}} \right] \\ &\quad + c\|\varphi\|_{C^\alpha(\partial D)} |x_1 - x_2|^\alpha \\ &\quad + c\|\varphi\|_{C^\alpha(\partial D)} \int_{\partial D \setminus \Gamma_{x_1,r}} |y - x_2|^\alpha \frac{|x_1 - x_2|}{|y - x_1|^3}\, ds(y) \\ &\le c\|\varphi\|_{C^\alpha(\partial D)} \left[r^\alpha + |x_1 - x_2|^\alpha + |x_1 - x_2| \int_{\partial D \setminus \Gamma_{x_1,r}} \frac{ds(y)}{|y - x_1|^{3-\alpha}} \right]\end{aligned}$$

since $|y - x_2| \le |y - x_1| + |x_1 - x_2| = |y - x_1| + r/3 \le 2|y - x_1|$. The last integral has been estimated by $r^{\alpha-1}$, see (3.3e). This proves that

$$\big|(K_2\varphi)(x_1) - (K_2\varphi(x_2)\big| \le c\|\varphi\|_{C^\alpha(\partial D)}\, |x_1 - x_2|^\alpha\,.$$

The proof of $\big|(K_2\varphi)(x)\big| \leq c\|\varphi\|_{C^\alpha(\partial D)}$ is again simpler and is left again to the reader. □

We need compactness properties of boundary operators in Hölderspaces. This follows from the previous theorem and the compact imbedding of $C^\alpha(\partial D)$ in $C(\partial D)$.

Lemma 3.17 *The imbedding $C^\alpha(\partial D) \to C(\partial D)$ is compact for every $\alpha \in (0,1)$.*

Proof: We have to prove that the unit ball $B = \big\{\varphi \in C^\alpha(\partial D) : \|\varphi\|_{C^\alpha(\partial D)} \leq 1\big\}$ is relatively compact[5] in $C(\partial D)$. This follows directly from the theorem of Arcela-Ascoli (see [?]). Indeed, B is equi-continuous since

$$\big|\varphi(x_1) - \varphi(x_2)\big| \ \leq \ \|\varphi\|_{C^\alpha(\partial D)}|x_1 - x_2|^\alpha \ \leq \ |x_1 - x_2|^\alpha$$

for all $x_1, x_2 \in \partial D$. Furthermore, B is bounded. □

Corollary 3.18 *Under the assumptions of Theorem 3.16 the operator K_1 is compact from $C^\alpha(\partial D)$ into itself for every $\alpha \in (0,1)$.*

Proof: This follows immediately from the boundedness of K_1 from $C(\partial D)$ into $C^\alpha(\partial D)$ and the compactness of the imbedding $C^\alpha(\partial D)$ into $C(\partial D)$. □

We apply this result to the **boundary integral operators** which appear in the traces of the single and double layer potentials of Theorems 3.11, 3.13, and 3.15.

Theorem 3.19 *The operators $S, D, D' : C^\alpha(\partial D) \to C^\alpha(\partial D)$, defined by*

$$(S\varphi)(x) \ = \ \int_{\partial D} \varphi(y)\, \Phi_k(x,y)\, ds(y)\,, \quad x \in \partial D\,, \tag{3.25a}$$

$$(D\varphi)(x) \ = \ \int_{\partial D} \varphi(y)\, \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\, ds(y)\,, \quad x \in \partial D\,, \tag{3.25b}$$

$$(D'\varphi)(x) \ = \ \int_{\partial D} \varphi(y)\, \frac{\partial \Phi_k}{\partial \nu(x)}(x,y)\, ds(y)\,, \quad x \in \partial D\,, \tag{3.25c}$$

are well defined and compact. The operator S is bounded from $C^\alpha(\partial D)$ into $C^{1,\alpha}(\partial D)$

Proof: We have to check the assumptions (3.24a) and (3.24b) of Theorem 3.16. For $x, y \in \partial D$ we have by the definition of the fundamental solution Φ and part (a) of Lemma 1.6 that

$$\begin{aligned}
\big|\Phi(x,y)\big| \ &= \ \frac{1}{4\pi|x-y|}\,, \\
\left|\frac{\partial}{\partial \nu(y)}\Phi(x,y)\right| \ &= \ \frac{1}{4\pi|x-y|}\left|ik - \frac{1}{|x-y|}\right| \frac{\big|(y-x)\cdot\nu(y)\big|}{|x-y|} \\
&\leq \ \frac{c}{4\pi|x-y|}\left|ik - \frac{1}{|x-y|}\right| |x-y| \\
&\leq \ \frac{c}{4\pi|x-y|}\big[k|x-y|+1\big] \ \leq \ \frac{c(kd+1)}{4\pi|x-y|}
\end{aligned}$$

[5] i.e. its closure is compact

where $d = \sup\{|x-y| : x, y \in \partial D\}$. The same stimate holds for $\partial\Phi(x,y)/\partial\nu(x)$. This proves (3.24a) with $\alpha = 1$. Furthermore, we will prove (3.24b) with $\alpha = 1$. Let $x_1, x_2, y \in \partial D$ such that $|x_1 - y| \geq 3|x_1 - x_2|$. Then, for any $t \in [0,1]$, we conclude that $|x_1 + t(x_2 - x_1) - y| \geq |x_1 - y| - |x_2 - x_1| \geq |x_1 - y| - |x_1 - y|/3 = 2|x_1 - y|/3$.
First we consider Φ and apply the mean value theorem:

$$\begin{aligned} \big|\Phi(x_1,y) - \Phi(x_2,y)\big| &\leq |x_1 - x_2| \sup_{0\leq t\leq 1} \big|\nabla_x \Phi\big(x_1 + t(x_2 - x_1), y\big)\big| \\ &\leq c \sup_{0\leq t\leq 1} \frac{|x_1 - x_2|}{|x_1 + t(x_2 - x_1) - y|^2} \leq c\,\frac{9}{4}\,\frac{|x_1 - x_2|}{|x_1 - y|^2}\,. \end{aligned}$$

To show the corresponding estimate for the normal derivative of the fundamental solution we can restrict ourselves to the case $k = 0$. Indeed, from the representation $\Phi_k(x,y) - \Phi_0(x,y) = A\big(|x-y|^2\big) + |x-y|\,B\big(|x-y|^2\big)$ with analytic functions A and B we observe that $\partial(\Phi_k - \Phi_0)/\partial\nu$ is continuous.
Let again $x_1, x_2, y \in \partial D$ such that $|x_1 - y| \geq 3|x_1 - x_2|$. Then

$$\begin{aligned} &\left|\frac{\partial}{\partial\nu(y)}\Phi_0(x_1,y) - \frac{\partial}{\partial\nu(y)}\Phi_0(x_2,y)\right| \\ &\leq \frac{1}{4\pi|x_1 - y|^3}\,\big|\underbrace{\nu(y)\cdot(y - x_1) - \nu(y)\cdot(y - x_2)}_{=\nu(y)\cdot(x_2 - x_1)}\big| \\ &\quad + \frac{1}{4\pi}\left|\frac{1}{|x_1 - y|^3} - \frac{1}{|x_2 - y|^3}\right|\big|\nu(y)\cdot(y - x_2)\big| \\ &\leq \frac{1}{4\pi|x_1 - y|^3}\Big[\big|(\nu(y) - \nu(x_1))\cdot(x_2 - x_1)\big| + \big|\nu(x_1)\cdot(x_2 - x_1)\big|\Big] \\ &\quad + \frac{1}{4\pi}\left|\frac{1}{|x_1 - y|^3} - \frac{1}{|x_2 - y|^3}\right|\big|\nu(y)\cdot(y - x_2)\big| \end{aligned}$$

Now we use estimates (a) and (b) of Lemma 1.6 for the first term and the mean value theorem for the second term. Using again $|x_1 + t(x_2 - x_1) - y| \geq 2|x_1 - y|/3$ we have

$$\left|\frac{\partial}{\partial\nu(y)}\Phi_0(x_1,y) - \frac{\partial}{\partial\nu(y)}\Phi_0(x_2,y)\right| \leq c\,\frac{|y - x_1||x_2 - x_1| + |x_1 - x_2|^2}{|x_1 - y|^3} + c\,\frac{|y - x_2|^2|x_1 - x_2|}{|x_1 - y|^4}$$

Estimate (3.24b) now follows from the estimates $|x_1 - x_2| \leq |x_1 - y|/3$ and $|y - x_2| \leq |y - x_1| + |x_1 - x_2| \leq 4|y - x_1|/3$. The proof for the normal derivative with respect to x follows the same arguments. Finally, we have to show that $\operatorname{Grad} S$ is bounded from $C^\alpha(\partial D)$ into $C^\alpha(\partial D)^3$. But this follows from the representation (3.23). □

3.1.5 Uniqueness and Existence

Now we come back to the scattering problem (3.1), (3.2) from the beginning of this section. First we study the question of uniqueness. The following lemma is fundamental for proving uniqueness and tells us, that a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for real and positive (!) k cannot decay faster than $1/|x|$ as x tends to infinity. We will give two proofs of this result. The first - and simpler - one uses the expansion arguments from the previous chapter. In particular, properties of the spherical Bessel- and Hankel functions are used. The second proof which goes back to the original work by Rellich (see [?]) avoids the use of these special functions but is far more technical and also needs a stronger assumption on the field. For completeness, we present both versions. We begin with the first form.

Lemma 3.20 *(Rellich's Lemma, first form) Let $u \in C^2(\mathbb{R}^3 \setminus B[0, R_0])$ be a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for $|x| > R_0$ and wave number $k \in \mathbb{R}_{>0}$ such that*

$$\lim_{R\to\infty} \int_{|x|=R} |u|^2 \, ds \;=\; 0\,.$$

Then u vanishes for $|x| > R_0$.

Proof: The general solution of the Helmholtz equation in the exterior of $B(0, R_0)$ is given by (2.37); that is,

$$u(r\hat{x}) \;=\; \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[a_n^m \, h_n^{(1)}(kr) + b_n^m \, j_n(kr)\right] Y_n^m(\hat{x})\,, \quad \hat{x} \in S^2\,,\; r > R\,,$$

for some $a_n^m, b_n^m \in \mathbb{C}$. The spherical harmonics $\left\{Y_n^m : |m| \le n,\; n \in \mathbb{N}_0\right\}$ form an orthogonal system. Therefore, Parceval's theorem yields

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m \, h_n^{(1)}(kr) + b_n^m \, j_n(kr)\right|^2 \;=\; \int_{S^2} \left|u(r\hat{x})\right|^2 ds(\hat{x})\,,$$

and from the assumption on u we note that $r^2 \int_{S^2} \left|u(r\hat{x})\right|^2 ds(\hat{x})$ tends to zero as r tends to infinity. Therefore, for every fixed $n \in \mathbb{N}_0$ and m with $|m| \le n$ we conclude that

$$r^2 \left|a_n^m \, h_n^{(1)}(kr) + b_n^m \, j_n(kr)\right|^2 \;\longrightarrow\; 0$$

as r tends to infinity. Defining $c_n^m = a_n^m + b_n^m$ we can write this as $(kr)\, i\, a_n^m \, y_n(kr) + (kr)\, c_n^m \, j_n(kr) \to 0$. Now we use the asymptotic behaviour of $j_n(kr)$ and $y_n(kr)$ as r tends to infinity. From Theorem 2.28 we conclude that

$$i\, a_n^m \, \mathrm{Im}\left[e^{ikr}(-i)^{n+1}\right] \;+\; c_n^m \, \mathrm{Re}\left[e^{ikr}(-i)^{n+1}\right] \;\longrightarrow\; 0\,.$$

The term $(-i)^{n+1}$ can take the values ± 1 and $\pm i$. Therefore, we have that (depending on n)

$$i\, a_n^m \sin(kr) \;+\; c_n^m \cos(kr) \;\longrightarrow\; 0 \quad \text{or} \quad i\, a_n^m \cos(kr) \;-\; c_n^m \sin(kr) \;\longrightarrow\; 0\,.$$

In any case, a_n^m and c_n^m have to vanish by taking particular sequences $r_j \to \infty$. This shows that also $b_n^m = 0$. Since this holds for all n and m we conclude that u vanishes. $\square$

The second proof avoids the use of the Bessel and Hankel functions but needs, however, a stronger assumption on u.

Lemma 3.21 *(Rellich's Lemma, second form) Let $u \in C^2(\mathbb{R}^3 \setminus B[0,R_0])$ be a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for $|x| > R_0$ with wave number $k \in \mathbb{R}_{>0}$ such that*

$$\lim_{r\to\infty} \int_{|x|=r} |u|^2\, ds \;=\; 0 \quad \textit{and} \quad \lim_{r\to\infty} \int_{|x|=r} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 ds \;=\; 0\,.$$

Then u vanishes for $|x| > R_0$.

Proof:[6] The proof is lengthy, and we will structure it. We assume that u is real valued (take real and imaginary parts separately).

1st step: Transforming the integral onto the unit sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ we conclude that

$$\int_{|\hat{x}|=1} \big|u(r\hat{x})|^2\, r^2\, ds(\hat{x}) = \int_{|x|=r} \big|u(x)\big|^2\, ds(x) \quad \text{and} \quad \int_{|\hat{x}|=1} \left| \frac{\partial u}{\partial r}(r\hat{x}) \right|^2 r^2\, ds(\hat{x}) \tag{3.26}$$

tend to zero as r tends to infinity. We transform the partial differential equation into an ordinary differential equation (not quite!) for the function $v(r,\hat{x}) = r\, u(r,\hat{x})$ w.r.t. r. We write $v(r)$ and $v'(r)$ and $v''(r)$ for $v(r,\cdot)$ and $\partial v(r,\cdot)/\partial r$ and $\partial^2 v(r,\cdot)/\partial r^2$, respectively. Then (3.26) yields that $\|v(r)\|_{L^2(S^2)} \to 0$ and $\|v'(r)\|_{L^2(S^2)} \to 0$ as $r \to \infty$. The latter follows from $\frac{\partial}{\partial r}\big(ru(r,\cdot,\cdot)\big) = \frac{1}{r}\big(ru(r,\cdot)\big) + r\frac{\partial u}{\partial r}(r,\cdot)$ and the triangle inequality.
We observe that $u = \frac{1}{r}v$, thus $r^2\frac{\partial u}{\partial r} = -v + r\frac{\partial v}{\partial r}$ and $\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big) = r\frac{\partial^2 v}{r^2}$, thus

$$\begin{aligned} 0 \;&=\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}(r,\theta,\phi) \right) \;+\; \frac{1}{r^2}\,\Delta_S u(r,\theta,\phi) \;+\; k^2 u(r,\theta,\phi) \\ &=\; \frac{1}{r}\left[\frac{\partial^2 v}{\partial r^2}(r,\theta,\phi) \;+\; k^2 v(r,\theta,\phi) \;+\; \frac{1}{r^2}\,\Delta_S v(r,\theta,\phi) \right], \end{aligned}$$

i.e.

$$v''(r) \;+\; k^2 v(r) \;+\; \frac{1}{r^2}\,\Delta_S v(r) \;=\; 0 \quad \text{for } r \ge R_0\,, \tag{3.27}$$

where again $\Delta_S = \operatorname{Div}\operatorname{Grad}$ denotes the Laplace-Beltrami operator; that is, in polar coordinates $\hat{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^\top$

$$(\Delta_S w)(\theta,\phi) \;=\; \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, \frac{\partial w}{\partial\theta}(\theta,\phi) \right) \;+\; \frac{1}{\sin^2\theta}\,\frac{\partial^2 w}{\partial\phi^2}(\theta,\phi)$$

for any $w \in C^2(S^2)$. It is easily seen either by direct integration or by application of Theorem **??** that Δ_S is selfadjoint and negative definite, i.e.

$$\big(\Delta_S v, w\big)_{L^2(S^2)} = \big(v, \Delta_S w\big)_{L^2(S^2)} \quad \text{and} \quad \big(\Delta_S v, v\big)_{L^2(S^2)} \le 0 \quad \text{for all } v, w \in C^2(S^2)\,.$$

[6] We took the proof from the monograph *Partielle Differentialgleichungen zweiter Ordnung* by R. Leis

2nd step: We introduce the functions E, v_m and F by

$$\begin{aligned} E(r) &:= \|v'(r)\|^2_{L^2(S^2)} + k^2\|v(r)\|^2_{L^2(S^2)} + \frac{1}{r^2}\big(\Delta_S v(r), v(r)\big)_{L^2(S^2)}, \quad r \geq R_0, \\ v_m(r) &:= r^m v(r), \quad r \geq R_0, \ m \in \mathbb{N}, \\ F(r,m,c) &:= \|v_m'(r)\|^2_{L^2(S^2)} + \left(k^2 + \frac{m(m+1)}{r^2} - \frac{2c}{r}\right)\|v_m(r)\|^2_{L^2(S^2)} \\ &\quad + \frac{1}{r^2}\big(\Delta_S v_m(r), v_m(r)\big)_{L^2(S^2)}, \end{aligned}$$

for $r \geq R_0$, $m \in \mathbb{N}$, $c \geq 0$. In the following we write $\|\cdot\|$ and $(\cdot,\cdot)$ for $\|\cdot\|_{L^2(S^2)}$ and $(\cdot,\cdot)_{L^2(S^2)}$, respectively. We show:

(a) E satisfies $E'(r) \geq 0$ for all $r \geq R_0$.

(b) The functions v_m solve the differential equation

$$v_m''(r) - \frac{2m}{r}v_m'(r) + \left(\frac{m(m+1)}{r^2} + k^2\right)v_m(r) + \frac{1}{r^2}\Delta_S v_m(r) = 0. \tag{3.28}$$

(c) For every $c > 0$ there exist $r_0 = r_0(c) \geq R_0$ and $m_0 = m_0(c) \in \mathbb{N}$ such that

$$\frac{\partial}{\partial r}\big[r^2 F(r,m,c)\big] \geq 0 \quad \text{for all } r \geq r_0, \ m \geq m_0.$$

(d) Expressed in terms of v the function F has the forms

$$\begin{aligned} F(r,m,c) &= r^{2m}\left\{\left\|v'(r) + \frac{m}{r}v(r)\right\|^2 + \left(k^2 + \frac{m(m+1)}{r^2} - \frac{2c}{r}\right)\|v(r)\|^2 \right. \\ &\quad \left. + \frac{1}{r^2}\big(\Delta_S v(r), v(r)\big)\right\} &(3.29a) \\ &= r^{2m}\left\{E(r) + \frac{2m}{r}\big(v(r), v'(r)\big) + \left(\frac{m(2m+1)}{r^2} - \frac{2c}{r}\right)\|v(r)\|^2\right\} &(3.29b) \end{aligned}$$

Proof of these statements:
(a) We just differentiate E and substitute the second derivative from (3.27). Note that $\frac{d}{dr}\|v(r)\|^2 = 2(v,v')$ and $\frac{d}{dr}\big(\Delta_S v, v\big) = 2\big(\Delta_S v, v'\big)$:

$$\begin{aligned} E'(r) &= 2\big(v'(r), v''(r)\big) + 2k^2\big(v(r), v'(r)\big) - \frac{1}{r^3}\big(\Delta_S v(r), v(r)\big) + \frac{2}{r^2}\big(\Delta_S v(r), v'(r)\big) \\ &= 2\left(v'(r), \left[v''(r) + k^2 v(r) + \frac{1}{r^2}\Delta_S v(r)\right]\right) - \frac{1}{r^3}\big(\Delta_S v(r), v(r)\big) \\ &= -\frac{1}{r^3}\big(\Delta_S v(r), v(r)\big) \geq 0. \end{aligned}$$

(b) We substitute $v(r) = r^{-m} v_m(r)$ into (3.27) and obtain directly (3.28). We omit the calculation.

(c) Again we differentiate $r^2 F(r,m,c)$ w.r.t. r, substitute the form of v''_m from (3.28) and obtain

$$\begin{aligned}\frac{\partial}{\partial r}\big[r^2 F(r,m,c)\big] &= 2r\|v'_m(r)\|^2 + 2r^2\big(v'_m(r), v''_m(r)\big) + 2(k^2 r - c)\|v_m(r)\|^2 \\ &\quad + 2r^2\left(k^2 + \frac{m(m+1)}{r^2} - \frac{2c}{r}\right)\big(v_m(r), v'_m(r)\big) + 2\big(\Delta_S v_m(r), v'_m(r)\big) \\ &= \cdots = 2r(1+2m)\|v'_m(r)\|^2 - 4cr\big(v'_m(r), v_m(r)\big) + 2(k^2 r - c)\|v_m(r)\|^2 \\ &= 2r\left[\left\|\sqrt{1+2m}\, v'_m(r) - \frac{c}{\sqrt{1+2m}}\, v_m(r)\right\|^2 + \left(k^2 - \frac{c}{r} - \frac{c^2}{1+2m}\right)\|v_m(r)\|^2\right].\end{aligned}$$

From this the assertion (c) follows if r_0 and m_0 are chosen such that the bracket $(\cdots)$ is positive.
(d) The first equation is easy to see by just inserting the form of v_m. For the second form one uses simply the binomial theorem for the first term and the definition of $E(r)$.

3rd step: We begin with the actual proof of the lemma and show first that there exists $R_1 \geq R_0$ such that $\|v(r)\| = 0$ for all $r \geq R_1$. Assume, on the contrary, that this is not the case. Then, for every $R \geq R_0$ there exists $\hat{r} \geq R$ such that $\|v(\hat{r})\| > 0$.
We choose the constants $\hat{c} > 0$, r_0, m_0, r_1, m_1 in the following order:

- Choose $\hat{c} > 0$ with $k^2 - \frac{2\hat{c}}{R_0} > 0$.
- Choose $r_0 = r_0(\hat{c}) \geq R_0$ and $m_0 = m_0(\hat{c}) \in \mathbb{N}$ according to property (c) above, i.e. such that $\frac{\partial}{\partial r}\big[r^2 F(r,m,\hat{c})\big] \geq 0$ for all $r \geq r_0$ and $m \geq m_0$.
- Choose $r_1 > r_0$ such that $\|v(r_1)\| > 0$.
- Choose $m_1 \geq m_0$ such that $m_1(m_1+1)\|v(r_1)\|^2 + \big(\Delta_S v(r_1), v(r_1)\big) > 0$.

Then, by (3.29a) and since $k^2 - \frac{2\hat{c}}{r_1} \geq k^2 - \frac{2\hat{c}}{R_0} > 0$, it follows that $F(r_1, m_1, \hat{c}) > 0$ and thus, by the monotonicity of $r \mapsto r^2 F(r, m_1, \hat{c})$ that also $F(r, m_1, \hat{c}) > 0$ for all $r \geq r_1$. Therefore, from (3.29b) we conclude that, for $r \geq r_1$,

$$\begin{aligned} 0 < r^{-2m_1} F(r, m_1, \hat{c}) &= E(r) + \frac{2m_1}{r}\big(v(r), v'(r)\big) + \left(\frac{m_1(2m_1+1)}{r^2} - \frac{2\hat{c}}{r}\right)\|v(r)\|^2 \\ &= E(r) + \frac{m_1}{r}\frac{d}{dr}\|v(r)\|^2 + \frac{1}{r}\left(\frac{m_1(2m_1+1)}{r} - 2\hat{c}\right)\|v(r)\|^2 .\end{aligned}$$

Choose now $r_2 \geq r_1$ such that $\frac{m_1(2m_1+1)}{r_2} - 2\hat{c} < 0$. Finally, choose $\hat{r} \geq r_2$ such that $\frac{d}{dr}\|v(\hat{r})\|^2 \leq 0$. (This is possible since $\|v(r)\|^2 \to 0$ as $r \to \infty$.) We finally have

$$0 < p := \hat{r}^{-2m_1} F(\hat{r}, m_1, \hat{c}) \leq E(\hat{r}) .$$

By the monotonicity of E we conclude that $E(r) \geq p$ for all $r \geq \hat{r}$. On the other hand, by the definition of $E(r)$ we have that $E(r) \leq \|v'(r)\|^2 + k^2\|v(r)\|^2$ and this tends to zero as r tends to infinity. This is a contradiction. Therefore, there exists $R_1 \geq R_0$ with $v(r) = 0$ for all $r \geq R_1$ and thus also Re $u = 0$ for $|x| \geq R_1$. The same holds for Im u and thus $u = 0$ for $|x| \geq R_1$. □

We can now prove uniqueness of the scattering problem.

Theorem 3.22 *For any incident field u^{inc} there exists at most one solution $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ of the scattering problem (3.1), (3.2).*

Proof: Let u be the difference of two solutions. Then u satisfies (3.1) and also the radiation condition (3.2). From the radiation condition we conclude that

$$\int\limits_{|x|=R} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds \;=\; \int\limits_{|x|=R} \left\{ \left| \frac{\partial u}{\partial r} \right|^2 + k^2|u|^2 \right\} ds \;+\; 2k \,\mathrm{Im} \int\limits_{|x|=R} u \, \frac{\partial \overline{u}}{\partial r} \, ds$$

tends to zero as R tends to infinity. Green's theorem, applied in $B_R \setminus D$ to the function u yields that

$$\int\limits_{|x|=R} u \, \frac{\partial \overline{u}}{\partial r} \, ds \;=\; \int\limits_{\partial D} u \, \frac{\partial \overline{u}}{\partial r} \, ds \;+\; \iint\limits_{B_R \setminus D} \left[|\nabla u|^2 - k^2|u|^2 \right] dx \;=\; \iint\limits_{B_R \setminus D} \left[|\nabla u|^2 - k^2|u|^2 \right] dx$$

since the surface integral over ∂D vanishes by the boundary condition. The volume integral is real valued. Therefore, its imaginary part vanishes and we conclude that

$$\int\limits_{|x|=R} \left| \frac{\partial u}{\partial r} \right|^2 + k^2|u|^2 ds$$

tends to zero as R tends to infinity. Rellich's lemma (in the form Lemma 3.20 or Lemma 3.21) implies that u vanishes outside of every ball which encloses ∂D. Finally, we note that u is an analytic function in the exterior of D. Since the exterior of D is connected we conclude that u vanishes in $\mathbb{R}^3 \setminus D$. □

We turn to the question of **existence** and choose the integral equation method for its treatment. We follow again the approach of [**?**] but prefer to work in the space $C^\alpha(\partial D)$ of Hölder continuous functions rather that in the space of merely continuous functions. This avoids the necessity to introduce the class of continuous functions for which the normal derivatives exist "in the uniform sense along the normal".

We recall the notion of the single layer potentials of (3.4a), see also (3.12), and make the ansatz for the scattered field in the form of a single layer potential. We remark already here that we will face some difficulties with this ansatz. Before we modify the ansatz below we try the single layer ansatz for the scattered field in the form

$$u^s(x) \;=\; \int_{\partial D} \varphi(y) \, \Phi(x,y) \, ds(y) \,, \quad x \in \mathbb{R}^3 \setminus \partial D \,, \tag{3.30}$$

where again $\Phi(x,y) = \exp(ik|x-y|)/(4\pi|x-y|)$ denotes the fundamental solution of the Helmholtz equation, and $\varphi \in C^\alpha(\partial D)$ is some density to be determined. First we note that u^s solves the Helmholtz equation in the exterior of $\overline{D}$ and also the radiation condition. This follows from the corresponding properties of the fundamental solution $\Phi(\cdot, y)$, uniformly with respect to y on the compact surface ∂D. Furthermore, by Theorems 3.11 and 3.15 the function u^s and its derivatives can be extended continuously (from the exterior) into $\mathbb{R}^3 \setminus D$ with limiting values

$$u^s(x)\Big|_+ \;=\; \int_{\partial D} \varphi(y)\,\Phi(x,y)\,ds(y) \;=\; (S\varphi)(x)\,, \quad x \in \partial D\,, \tag{3.31a}$$

$$\begin{aligned}\frac{\partial u^s}{\partial \nu}(x)\Big|_+ \;&=\; -\frac{1}{2}\,\varphi(x) \;+\; \int_{\partial D} \varphi(y)\,\frac{\partial}{\partial \nu(x)}\Phi(x,y)\,ds(y)\\ &=\; -\frac{1}{2}\,\varphi(x) \;+\; (D'\varphi)(x)\,, \quad x \in \partial D\,,\end{aligned} \tag{3.31b}$$

where we used the notations of the boundary integral operators from Theorem 3.19. Therefore, in order that $u = u^{inc} + u^s$ satisfies the boundary condition $\partial u/\partial \nu = 0$ on ∂D the density φ has to satisfy the boundary integral equation

$$-\frac{1}{2}\varphi \;+\; D'\varphi \;=\; -\frac{\partial u^{inc}}{\partial \nu} \quad \text{in } C^\alpha(\partial D)\,. \tag{3.32}$$

By Theorem 3.19 the operator D' is compact. Therefore, we can apply the Fredholm theory. In particular, existence follows from uniqueness. To prove uniqueness we assume that $\varphi \in C^\alpha(\partial D)$ satisfies the homogeneous equation $-\frac{1}{2}\varphi + D'\varphi = 0$. Define v to be the single layer potential with density φ just as in (3.30), but for arbitrary $x \notin \partial D$. Then, again from the jump contions of the normal derivative of the single layer, $\partial v/\partial \nu\big|_+ = -\frac{1}{2}\varphi + D'\varphi = 0$. Therefore, v is the solution of the exterior Neumann problem with vanishing boundary data. The uniqueness result of Theorem 3.22 yields that v vanishes in the exterior of D. Furthermore, v is continuous in $\mathbb{R}^3$, thus v is a solution of the Helmholtz equation in D with vanishing boundary data. This is the point where we wish to conclude that v vanishes also in D. However, this is not always the case. Indeed, this is not the case if, and only if, k^2 is an eigenvalue of $-\Delta$ in D with respect to Dirichlet boundary conditions. This is the reason why we have to modify the ansatz (3.30). There are several ways how to do it, see the discussion in [**?**]. We choose a modification which we have not found in the literature. It avoids the use of double layer potentials. We assume for simplicity that D is connected although this is not necessary as one observes from the following arguments.

We choose an open ball B with boundary Γ such that $\Gamma \subset D$ and such that k^2 is not an eigenvalue of $-\Delta$ inside B with respect to Dirichlet boundary conditions. By Theorem 2.30 from the previous chapter we observe that we have to choose the radius ρ of B such that $k\rho$ is not a zero of any of the Bessel functions j_n.

Now we make an ansatz for u^s as a sum of two single layer potentials in the form

$$u^s(x) \;=\; (\tilde{S}_{\partial D}\varphi)(x) + (\tilde{S}_\Gamma \psi)(x) \;=\; \int_{\partial D} \varphi(y)\,\Phi(x,y)\,ds(y) + \int_\Gamma \psi(y)\,\Phi(x,y)\,ds(y)\,, \quad x \notin \overline{D}\,, \tag{3.33}$$

where $\phi \in C^\alpha(\partial D)$ and $\psi \in C^\alpha(\Gamma)$ are two densities to be determined from the system of two boundary integral equations

$$-\frac{1}{2}\varphi \;+\; D'\varphi \;+\; \frac{\partial}{\partial\nu}\tilde{S}_\Gamma\psi \;=\; -\frac{\partial u^{inc}}{\partial\nu} \quad \text{on } \partial D\,, \tag{3.34a}$$

$$\left(\frac{\partial}{\partial\nu} + ik\right)\tilde{S}_{\partial D}\varphi \;-\; \frac{1}{2}\psi \;+\; D'_\Gamma\psi \;+\; ik\,S_\Gamma\psi \;=\; 0 \quad \text{on } \Gamma\,. \tag{3.34b}$$

The operators S_Γ and D'_Γ denote the boundary operators S and D', respectively, on the boundary Γ instead of ∂D. These two equations can be written in matrix form as

$$-\frac{1}{2}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \begin{pmatrix} D' & \partial\tilde{S}_\Gamma/\partial\nu \\ (\partial/\partial\nu + ik)\tilde{S}_{\partial D} & D'_\Gamma + ik\,S_\Gamma \end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \;=\; -\begin{pmatrix}\partial u^{inc}/\partial\nu \\ 0\end{pmatrix}$$

in $C^\alpha(\partial D)\times C^\alpha(\Gamma)$. The operators D', D'_Γ, $\partial\tilde{S}_\Gamma/\partial\nu$, and $(\partial/\partial\nu + ik)\tilde{S}_{\partial D}$ are all compact. Therefore, we can apply the Fredholm alternative to this system. Existence is assured if the homogeneous system admits only the trivial solution $\varphi = 0$ and $\psi = 0$. Therefore, let $(\varphi, \psi) \in C^\alpha(\partial D)\times C^\alpha(\Gamma)$ be a solution of the homogeneous system and define the v as the sum of the single layers with densities φ and ψ for all x in $\mathbb{R}^3 \setminus (\partial D \cup \Gamma)$. From the the jump condition for the normal derivative and the first (homogeneous) integral equation we conclude – just in the above case of only one single layer potential – that $\partial v/\partial\nu\big|_+ = -\frac{1}{2}\varphi + D'\varphi + \partial\tilde{S}_\Gamma\psi/\partial\nu = 0$. Again, v is a solution of the exterior Neumann problem with vanishing boundary data. Therefore, by the uniqueness theorem, v vanishes in the exterior of D. Furthermore, v is continuous in $\mathbb{R}^3$ and satisfies also the Helmholtz equation in $D \setminus \overline{B}$. From the jump conditions on the boundary Γ we conclude that

$$\frac{\partial v}{\partial\nu}\bigg|_+ + ikv \;=\; \left(\frac{\partial}{\partial\nu} + ik\right)\tilde{S}_{\partial D}\varphi \;-\; \frac{1}{2}\psi \;+\; D'_\Gamma\psi \;+\; ik\,S_\Gamma\psi \;=\; 0 \quad \text{on } \Gamma\,.$$

Therefore, $v = 0$ on ∂D and $\partial v/\partial\nu|_+ + ikv = 0$ on Γ. Application of Green's first theorem in $D \setminus \overline{B}$ yields

$$\iint\limits_{D\setminus\overline{B}} \left[|\nabla v|^2 - k^2|v|^2\right] dx \;=\; \int\limits_{\partial D} \overline{v}\,\frac{\partial v}{\partial\nu}\,ds - \int\limits_{\Gamma} \overline{v}\,\frac{\partial v}{\partial\nu}\bigg|_+ ds \;=\; ik\int\limits_{\Gamma} |v|^2\,ds\,.$$

Taking the imaginary part yields that v vanishes on Γ and therefore also $\partial v/\partial\nu|_+ = 0$ on Γ. Holmgren's uniqueness Theorem 3.5 implies that v vanishes in all of $D \setminus B$. The jump conditions on ∂D yield

$$0 \;=\; \frac{\partial v}{\partial\nu}\bigg|_- - \frac{\partial v}{\partial\nu}\bigg|_+ \;=\; \varphi \quad \text{on } \partial D\,.$$

Therefore, v is a single layer potential on Γ with density ψ and vanishes on Γ. The wave number k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B by the choice of the radius of B. Therefore, v vanishes also in B. The jump conditions on Γ yield

$$0 \;=\; \frac{\partial v}{\partial\nu}\bigg|_- - \frac{\partial v}{\partial\nu}\bigg|_+ \;=\; \psi \quad \text{on } \Gamma\,.$$

Therefore, $\varphi = 0$ on ∂D and $\psi = 0$ on Γ.

If D consists of several components $D = \bigcup_{m=1}^{M} D_m$ then one has to choose balls B_m in each of the domains D_m and make an ansatz as a sum of single layers on ∂D and ∂B_m for $m = 1, \ldots, M$.

Application of Fredholm's alternative yields the following result:

Theorem 3.23 *There exists a unique solution* $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ *of the scattering problem (3.1), (3.2).*

3.2 A Scattering Problem for the Maxwell System

3.2.1 Formulation of the Problem

For this chapter we make the following assumptions on the data:

Assumption: Let the wave number be given by $k = \omega\sqrt{\varepsilon_0 \mu_0} > 0$ with constants $\varepsilon_0, \mu_0 > 0$. Let $D \subset \mathbb{R}^3$ be bounded and C^2−smooth such that the complement $\mathbb{R}^3 \setminus \overline{D}$ is connected.

Scattering Problem: Given a solution (E^i, H^i) of the Maxwell system

$$\operatorname{curl} E^i - i\omega\mu_0 H^i = 0\,, \quad \operatorname{curl} H^i + i\omega\varepsilon_0 E^i = 0 \text{ in some neighborhood of } D\,,$$

determine the total fields $E, H \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ such that

$$\operatorname{curl} E - i\omega\mu_0 H = 0 \quad \text{and} \quad \operatorname{curl} H + i\omega\varepsilon_0 E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}\,, \tag{3.35a}$$

E satisfies the boundary condition

$$\nu \times E \;=\; 0 \quad \text{on } \partial D\,, \tag{3.35b}$$

and the radiating parts $E^s = E - E^i$ and $H^s = H - H^i$ satisfy the Silver Müller radiation conditions

$$\sqrt{\varepsilon_0}\, E^s(x) \;-\; \sqrt{\mu_0}\, H^s(x) \times \frac{x}{|x|} \;=\; \mathcal{O}\left(\frac{1}{|x|^2}\right)\,, \tag{3.36a}$$

and

$$\sqrt{\mu_0}\, H^s(x) \;+\; \sqrt{\varepsilon_0}\, E^s(x) \times \frac{x}{|x|} \;=\; \mathcal{O}\left(\frac{1}{|x|^2}\right)\,, \tag{3.36b}$$

uniformly w.r.t. $x/|x|$. Clearly, after renaming the unknown fields, this is a special case of the following problem:

Exterior Boundary Value Problem: Given a tangential field[7] $c \in C^\alpha(\partial D)^3$ such that $\operatorname{Div} c \in C^\alpha(\partial D)$ determine radiating[8] solutions $E^s, H^s \in C^1(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ of

$$\operatorname{curl} E^s - i\omega\mu_0 H^s = 0 \quad \text{and} \quad \operatorname{curl} H^s + i\omega\varepsilon_0 E^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}\,, \tag{3.37a}$$

[7] i.e. $\nu(x) \cdot c(x) = 0$ on ∂D

[8] i.e. E^s, H^s satisfy the radiating conditions (3.36a) and (3.36b)