Research Statement — Oleksandr Bondarenko, March 2016

In my PhD thesis I studied an inverse problem arising from electromagnetic scattering by a medium covered with a very thin and highly conducting layer. The objective of the thesis is to show that the Factorization Method (FM) (an inverse problem solution algorithm) can be applied to detect the position and shape of such objects from the measurements of the scattered waves at large distances. Such problems originate from applications such as landmine detection, radar or seismic imaging.

We consider two special cases of the problem which are derived as the TM- (transverse magnetic) and TE- (transverse electric) modes from the full Maxwell system. The studies of both modes are divided into the following parts:

(1) Instead of considering the full model with a thin highly conductive layer of given thickness δ, we derive an approximate one. With the scaled asymptotic expansions technique [11], one can show that for the layers of thickness δ and conductivity proportional to δ^{-1}, the model with the well-known conductive transmission conditions [21], [22] can be used as the first order approximation for the full model.

(2) As it is always the case with inverse problems, prior to the investigation of the inverse problem, the corresponding forward problem needs to be understood completely. The well-posedness for both modes is established by a variational approach involving the Dirichlet-to-Neumann map on an auxiliary interface. This approach is also used to solve the direct problems numerically.

(3) In [1] and [2] we showed that the FM can be applied to solve the inverse problem. In particular, we showed that for the TM-mode [2] the FM works for partially coated obstacles. For the TE-mode one has to make the assumption that the obstacle is fully coated. More details on the FM are provided in Section 2.

(4) To test the factorization method numerically, we developed solvers for the direct problems and computed the forward data. We used two approaches: the combined integral equation and finite element method as suggested in [19], [20] (implemented in MATLAB); the finite element method (using the FreeFem++ solver [14]). Section 3 provides numerical examples and the details of the implementation of the FM.

(5) For the FM one has to make the assumption that the wave number of the incident waves is not an eigenvalue of a related *interior eigenvalue problem*. Such an interior eigenvalue problem appeared in our case too while proving the FM. It is easy to see, that for both modes, the interior eigenvalues form at most a discrete set. This implies that in practice the FM would always work (the probability of 'hitting' exactly the interior eigenvalue is zero). In recent years the study of interior transmission eigenvalues itself became an important area in the inverse scattering research (see [6]). It has been shown that with the knowledge of the transmission eigenvalues it is possible to get information about the material properties of the scatterer [4], [7], [9], [13]. Moreover, the transmission eigenvalues can be determined from the far field data [5], [17]. In a collaboration with I.Harris (Texas A&M) and A.Kleefel (Jülich Supercomputing Center) we showed that for the TE mode for real valued boundary parameter such wave numbers (interior eigenvalues) exist. We also

established monotonicity results which suggest that it is possible to retrieve information about the boundary parameter (if the refractive index is fixed) or about the refractive index (if the boundary parameter is fixed) from the knowledge of the interior eigenvalues.

To study the problems techniques from the following areas were used:

- Inverse problems
- PDE
- Perturbation Theory
- Integral Equations
- Functional analysis
- Numerical analysis, Finite Element Method.

As a future research it is reasonable to consider

- the full Maxwell system, and see to which extent the methods applied in the scalar case transfer;
- the numerical techniques, such as boundary element method, to solve the forward problem in 3D;
- other methods to reconstruct a scatterer from a single far field measurement (as done e.g. in [10]);
- the problem a far field design:
  how to choose the boundary material of the scatterer to achieve a certain far field?

In the following we briefly define the scattering problems and describe the idea of the FM (Section 2). In the Section 3 some numerical examples for TE- and TM-modes, as well as for the full model with a thin layer, are presented.

1 Definitions of the Scattering Problems

In both problems an incident plane wave

$$u^i(x;d) = e^{ikd\cdot x},\ x \in \mathbb{R}^2,\ k > 0,$$

with the direction of incidence $d \in S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$, is scattered by the medium. This results in the appearance of a scattered wave u^s.

We call the sum of the incident and the scattered field $u = u^i + u^s$ the total field. In the problem corresponding to the TM-mode u satisfies the Helmholtz equation

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^2 \backslash \partial D, \tag{1}$$

with conductive transmissions of the form

$$u_+ - u_- = 0 \ \text{ on } \partial D \quad \text{and} \quad \frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} + i\eta u = 0 \ \text{ on } \partial D. \tag{2}$$

Here, n is a complex valued index of refraction which characterizes the scattering medium, η is a (real-valued) surface conductivity, and $D \subset \mathbb{R}^2$ is a bounded simply connected domain with C^2 smooth boundary. For smooth solutions, we can think of $u_\pm$ and $\partial u_\pm/\partial\nu$ as the limits of u

and $\partial u/\partial\nu$ from the exterior (+) and the interior (−), respectively.

From the physical considerations, u^s satisfies the Sommerfeld radiation condition,

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u^s}{\partial\nu}-iku\right)=0,\ r=|x|, \tag{3}$$

uniformly for all directions $x/|x|$. The radiation condition implies [8] that at large distances the scattered wave behaves as an outgoing spherical wave with the scattering amplitude, also called far field, u^∞:

$$u^s(x)=\frac{e^{ik|x|}}{\sqrt{x}}\left\{u^\infty(\hat{x})+O\left(\frac{1}{|x|}\right)\right\},\ |x|\to\infty,$$

uniformly with respect to $\hat{x}=x/|x|\in S^1$. In the following, to indicate that the far field corresponds the scattered field due to the incident plane field with direction $d\in S^1$, we will write $u^\infty=u^\infty(\cdot,d)$.

In the TE-mode the total field satisfies the generalized Helmholtz equation

$$\nabla\cdot\left(\frac{1}{n}\nabla u\right)+k^2u=0\ \text{ in }\mathbb{R}^2\backslash\partial D \tag{4}$$

with conductive transmission conditions

$$\frac{\partial u_+}{\partial\nu}-\frac{1}{n_-}\frac{\partial u}{\partial\nu}=0\ \text{ on }\partial D\quad\text{ and }\quad u_+-u_--i\eta\frac{\partial u_+}{\partial\nu}=0\ \text{ on }\partial D. \tag{5}$$

We distinguish between the *direct* (or, also called *forward*) and *inverse problem*. In the *forward problem* we assume that the information about the scattering medium is given, and study the behavior of the scattered field. I showed, in particular, that the forward problem is well-posed, that is, for any incident wave u^i the solution u^s exists, is unique, and depends continuously on the incident field in an appropriate norm.

The *inverse problem* I considered in my thesis is to determine the location and the shape of the domain D from the knowledge of the far field patterns $u^\infty(\hat{x},d)$ for all $\hat{x},d\in S^1$. I showed, that in both cases, the inverse problems corresponding to the TE- and TM-modes, can be solved by the factorization method.

2 The Factorization Method

The FM belongs to the family of the sampling methods. The idea of the sampling methods is to construct a criteria on the known data to decide whether a given point z is inside or outside of the unknown domain D. For the FM, the criteria has the following form:

$$z\in D\quad\text{if and only if}\quad\widetilde{F}g(\hat{x})=e^{-ikz\cdot\hat{x}},\quad\hat{x}\in S^1,$$

is solvable in $L^2(S^1)$. Here, $\widetilde{F}$ is an operator which is computed from the well-known far field operator $F:L^2(S^1)\to L^2(S^1)$ defined as:

$$(Fg)(\hat{x})=\int_{S^1}u^\infty(\hat{x},d)g(d)\,\mathrm{d}s(d)\qquad\text{for }\hat{x}\in S^1. \tag{6}$$

As we can see the FM provides a necessary and sufficient criterion on z. Therefore, its important by-product is an *explicit proof of uniqueness of the inverse scattering problem.*

The FM has been introduced by Kirsch in 1998 for scattering by impenetrable sound soft or sound hard obstacles [15]. Since then it has been applied to a variety of problems from acoustic and electromagnetic scattering and electrical impedance tomography. The monograph [16] captures just a few of them. The FM is a non-iterative method, and does not require solving a sequence of forward problems. Another advantage of the FM is, that it works without prior knowledge on material properties or the number of components of the medium. Computationally, the method is fast and easy to implement.

The justification of the FM, regardless if we consider inverse acoustic or electromagnetic scattering, or electrical impedance tomography, follows always the same scenario. We

(1) derive a factorization of the far field operator F in the form $F = H^*TH$;

(2) characterize the scattering domain D by test functions;

(3) link the test functions and the data operator F.

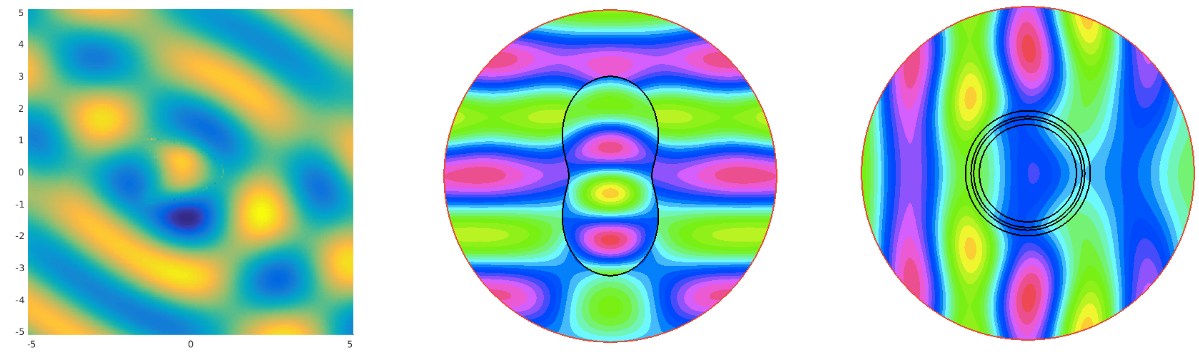
Although the three steps are always the same, the analysis of the FM for each particular case can be very different. In a collaboration with A. Kirsch and X. Liu we studied the FM for the following cases:

- In [3] we have shown the FM works for sound soft obstacles in a layered media.
- In [2] X.Liu and I studied the FM for a generalized version of the problem corresponding to the TM-mode with conductive transmission conditions.
- Prior to considering the FM for the TE-mode I studied the work of A.Kirsch and X.Liu [18], where there is no thin layer, i.e, the conductivity is equal zero. It turned out that with another choice of the factorization of the operator F, the (rather technical) proof in [18] of the FM can be simplified. Those ideas were used to prove the FM for the TE-mode [1].

3 Implementation and Numerical Tests

The forward problem corresponding to the TM-mode is computed by the coupled finite element and integral equation method analysed by A.Kirsch and P.Monk [19] and [20]. Figure 1 (a) shows the real part of total field for a kite-shaped obstacle, corresponding to the plane $u^i(x) = e^{ikx\cdot d}, x \in \mathbb{R}^2$ with $k = 2$, $d = [\cos(\pi/3)\ \ \sin(\pi/3)]^\top$, $n(x) = 1 + 10i|\sin(x)| + (x^2 + y^2)$, for $x \in D$, and $\eta(x) = |x| + y^2$ for $x \in \partial D$. For the numerical treatment of the integral equation, I used the Nystrom method 128 quadrature points, and for the finite element method, the MATLAB PDE toolbox with piecewise linear finite elements.

The direct problem corresponding to the TE-mode is solved with the help of FreeFem++ package [14]. Figure 1 (b) represents the real part of the total field corresponding to the incident field with incident direction $d = [\cos(1.5\pi)\ \sin(1.5\pi)]^\top$. The scatterer is a peanut-shaped



(a) (b) (c)

Figure 1: *Real part of the total field of (a) a kite-shaped obstacle corresponding to the incidence direction* $d = [\cos(\pi/3), \sin(\pi/3)]^\top$*; (b) a peanut-shaped object corresponding to* $d = [\cos(1.5\pi), \sin(1.5\pi)]^\top$*; (c) the unit disk with highly conductive layer of thickness* $\delta = 0.05$ *with incidence direction* $d = [1, 0]^\top$*.*

object. Further, $k = 3, \eta = 0.5$ and $n = 0.5$. The solution is obtained with P^1 finite elements discretization.

As mentioned in the overview, the problem with conductive transmission conditions is an approximation of the model involving a thin layer. Figure 1 (c) represents real part of the total field field corresponding to the incidence direction $d = [1\ 0]^\top$. The scatterer is a unit disk with a highly conductive layer of thickness $\delta = 0.05$. The problem is computed with the following parameters: $k = 3$, the wave number inside the layer is $k_l = \sqrt{3 + \frac{i}{2\delta}}$ and $n(x) = 1+0.5(x^2+y^2)$ for $x \in D$. The total field is computed again using the FreeFem++ package with P^1 finite elements.

In each case the data set for the inverse problem is represented by a matrix $F \in \mathbb{C}^{64\times 64}$, where $F_{jl} = u^\infty(\theta_j, \theta_l), j, l \in \{1, \ldots 64\}$, and $u^\infty(\theta_j, \theta_l), j, l \in \{1, \ldots 64\}$ are the far fields corresponding to the incident direction of the plane wave $\theta_j = 2\pi j/64$ and the observation point $\theta_l = 2\pi l/64$.

The implementation of the factorization method is done as follows. We compute the matrix $F_\sharp \in \mathbb{C}^{64\times 64}$ given by

$$F_\sharp = \text{Re } |F| + \text{Im } |F|,$$

with

$$\text{Re } F = \frac{F + F^*}{2} \qquad \text{and} \qquad \text{Im } F = \frac{F - F^*}{2i},$$

The absolute value of a matrix $A \in C^{N\times N}$ with a singular value decomposition $A = U\Lambda V^*$ is defined by

$$|A| = U|\Lambda|V^*,$$

with $|\Lambda|$ =diag$|\lambda_j|$, $j = 1, \ldots, N$.

For the reconstructions we use a grid $\mathcal{G}$ of 200×200 equally spaced sampling points on

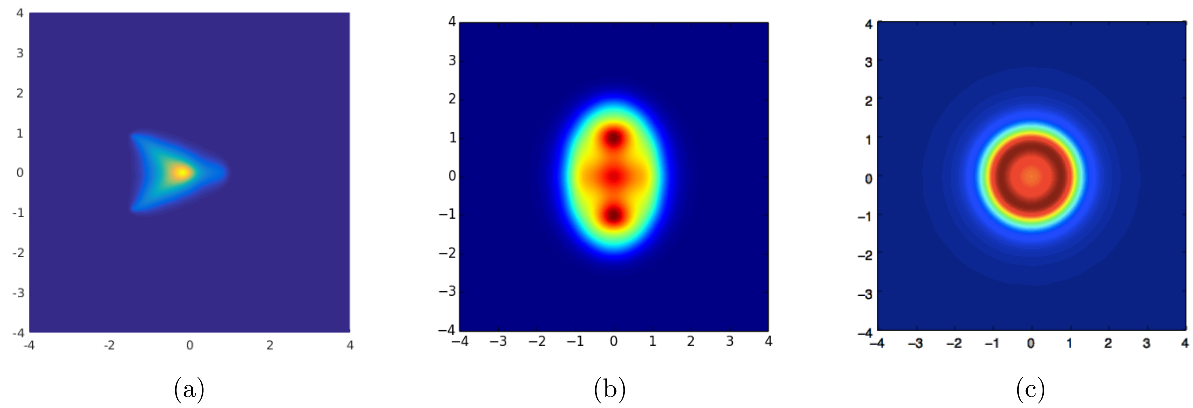


Figure 2: *Reconstructions by the FM for the obstacles presented in Figure 1.*

the rectangle $[-4,4]\times[-4,4]$. Let $\{(\sigma_n,\psi_n) : n = 1,...,64\}$ represent the eigensystem of the matrix $F_\sharp$. Finally, we compute the 'indicator function'

$$W(z) = \left[\sum_{j=1}^{64} \frac{|\phi_z^* \psi_n|^2}{|\sigma_n|}\right]^{-1}, \; z \in \mathcal{G},$$

where $\phi_z = (e^{-ik\theta_1\cdot z}, e^{-ik\theta_2\cdot z}, \ldots, e^{-ik\theta_{64}\cdot z})^\top \in \mathbb{C}^{64}$. (As we can see, the regularization method chosen here is the truncated SVD). We expect the value of $W(z)$ to be much larger for the points belonging to D then for those lying outside of the domain.

Figure 2 represents the reconstructions of the scatterers by the factorization method corresponding to the three scattering problems described above.

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