

## MONOTONICITY IN INVERSE MEDIUM SCATTERING ON UNBOUNDED DOMAINS\*

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**Abstract.** We discuss a time-harmonic inverse scattering problem for the Helmholtz equation with compactly supported penetrable and possibly inhomogeneous scattering objects in an unbounded homogeneous background medium, and we develop a monotonicity relation for the far field operator that maps superpositions of incident plane waves to the far field patterns of the corresponding scattered waves. We utilize this monotonicity relation to establish novel characterizations of the support of the scattering objects in terms of the far field operator. These are related to and extend corresponding results known from factorization and linear sampling methods to determine the support of unknown scattering objects from far field observations of scattered fields. An attraction of the new characterizations is that they only require the refractive index of the scattering objects to be above or below the refractive index of the background medium locally and near the boundary of the scatterers. An important tool to prove these results are so-called localized wave functions that have arbitrarily large norm in some prescribed region while at the same time having arbitrarily small norm in some other prescribed region. We present numerical examples to illustrate our theoretical findings.

**Key words.** inverse scattering, Helmholtz equation, monotonicity, far field operator, inhomogeneous medium

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**1. Introduction.** Accurately recovering the location and the shape of unknown scattering objects from far field observations of scattered acoustic or electromagnetic waves is a basic but severely ill-posed inverse problem in remote sensing, and in the past twenty years efficient qualitative reconstruction methods for this purpose have received a lot of attention (see, e.g., [5, 7, 9, 36, 41] and the references therein). In this work we develop a new approach for this shape reconstruction problem that is based on a *monotonicity relation* for the far field operator that maps superpositions of incident plane waves, which are being scattered at the unknown scattering objects, to the far field patterns of the corresponding scattered waves. Throughout we assume that the scattering objects are penetrable, nonabsorbing, and possibly inhomogeneous.

The new monotonicity relation generalizes similar results for the Neumann-to-Dirichlet map for the Laplace equation on bounded domains that have been established in [26], where they have been utilized to justify and extend an earlier monotonicity based reconstruction scheme for electrical impedance tomography developed in [43], using so-called localized potentials introduced in [13]. This is also related to corresponding estimates for the Laplace equation developed in [29, 30]. The analysis from [26] has recently been extended for the Neumann-to-Dirichlet operator for the Helmholtz equation on bounded domains in [24], and the main contribution of the present work is the generalization of these results to the inverse medium scattering problem on unbounded domains with plane wave incident fields and far field observations of the scattered waves.

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The monotonicity relation for the far field operator essentially states that the real part of a suitable unitary transform of the difference of two far field operators corresponding to two different inhomogeneous media is positive or negative semidefinite up to a finite dimensional subspace if the difference of the corresponding refractive indices is either nonnegative or nonpositive pointwise almost everywhere. This can be translated into criteria and algorithms for shape reconstruction by comparing a given (or observed) far field operator to various virtual (or simulated) far field operators corresponding to a sufficiently small or large index of refraction on some probing domains to decide whether these probing domains are contained inside the support of the unknown scattering objects or whether the probing domains contain the unknown scattering objects. In fact the situation is even more favorable, since it turns out to be sufficient to compare the given far field operator to linearized versions of the probing far field operators, i.e., Born far field operators, which can be simulated numerically very efficiently. An advantage of these new characterizations is that they only require the refractive index of the scattering object to be above or below the refractive index of the background medium locally and near the boundary of the scatterers, i.e., they apply to a large class of so-called indefinite scatterers.

Besides the monotonicity relation, the second main ingredient of our analysis are so-called *localized wave functions*, which are special solutions to scattering problems corresponding to suitably chosen incident waves that have arbitrarily large norm on some prescribed region  $B \subseteq \mathbb{R}^d$ , while at the same time having arbitrarily small norm on a different prescribed region  $D \subseteq \mathbb{R}^d$ , assuming that  $\mathbb{R}^d \setminus \bar{D}$  is connected and  $B \not\subseteq D$ . This generalizes corresponding results on so-called localized potentials for the Laplace equation established in [13]. The arguments that we use to prove the existence of such localized wave functions are inspired by the analysis of the factorization method (see [4, 31, 32, 33] for the origins of the method and [16, 19, 36] for recent overviews) and of the linear sampling method for the inverse medium scattering problem (see, e.g., [5, 7, 8]).

It is interesting to note that the characterizations of the support of the scattering objects in terms of the far field operator developed in this work are independent of so-called transmission eigenvalues (see, e.g., [5, 6, 9, 39]). On the other hand, the monotonicity relation for the far field operator is somewhat related to well-known monotonicity principles for the phases of the eigenvalues of the so-called scattering operator, which have been discussed, e.g., in [37], where they have actually been utilized to characterize transmission eigenvalues. The latter have recently been extended to monotonicity relations for the difference of far field operators in [38] that are closely related to our results. Our work substantially extends the results in [38], using very different analytical tools.

For further recent contributions on monotonicity based reconstruction methods for various inverse problems for partial differential equations we refer to [2, 3, 10, 11, 12, 20, 21, 22, 23, 27, 40, 42, 44, 45, 46]. We further note that this approach has also been utilized to obtain theoretical uniqueness results for inverse problems (see, e.g., [1, 17, 18, 25, 28]).

The outline of this article is as follows. After briefly introducing the mathematical setting of the scattering problem in section 2, we develop the monotonicity relation for the far field operator in section 3. In section 4 we discuss the existence of localized wave functions for the Helmholtz equation in unbounded domains, and we use them to provide a converse of the monotonicity relation from section 3. In section 5 we establish rigorous characterizations of the support of scattering objects in terms of the far field operator. An efficient and suitably regularized numerical implementation

of these criteria is beyond the scope of this article, but we discuss a preliminary algorithm and two numerical examples for the sign definite case (i.e., when the refractive index of the scattering objects is either above or below the refractive index of the background medium) in section 6 to illustrate our theoretical findings. This preliminary algorithm cannot be considered competitive when compared against state-of-the-art implementations of linear sampling or factorization methods, but, as outlined in our final remarks, this may change in the future.

**2. Scattering by an inhomogeneous medium.** We use the Helmholtz equation as a simple model for the propagation of time-harmonic acoustic or electromagnetic waves in an isotropic nonabsorbing inhomogeneous medium in  $\mathbb{R}^d$ ,  $d = 2, 3$ . Assuming that the inhomogeneity is compactly supported, the *refractive index* can be written as  $n^2 = 1 + q$  with a real-valued *contrast function*  $q \in L^\infty_{0,+}(\mathbb{R}^d)$ , where  $L^\infty_{0,+}(\mathbb{R}^d)$  denotes the space of compactly supported  $L^\infty$ -functions satisfying  $q > -1$  a.e. on  $\mathbb{R}^d$ .

The wave motion caused by an *incident field*  $u^i$  satisfying

$$(2.1) \quad \Delta u^i + k^2 u^i = 0 \quad \text{in } \mathbb{R}^d,$$

with *wave number*  $k > 0$ , that is being scattered at the inhomogeneous medium is described by the *total field*  $u_q$ , which is a superposition

$$(2.2a) \quad u_q = u^i + u_q^s$$

of the incident field and the *scattered field*  $u_q^s$  such that the Helmholtz equation

$$(2.2b) \quad \Delta u_q + k^2 n^2 u_q = 0 \quad \text{in } \mathbb{R}^d$$

is satisfied together with the *Sommerfeld radiation condition*

$$(2.2c) \quad \lim_{r \rightarrow \infty} r^{\frac{d-1}{2}} \left( \frac{\partial u_q^s}{\partial r}(x) - i k u_q^s(x) \right) = 0, \quad r = |x|,$$

uniformly with respect to all directions  $x/|x| \in S^{d-1}$ .

*Remark 2.1.* Throughout this work, Helmholtz equations are always to be understood in distributional (or weak) sense. For instance,  $u_q \in H^1_{\text{loc}}(\mathbb{R}^d)$  is a solution to (2.2b) if and only if

$$\int_{\mathbb{R}^d} (\nabla u_q \cdot \nabla v - k^2 n^2 u_q v) \, dx = 0 \quad \text{for all } v \in C^\infty_0(\mathbb{R}^d).$$

Accordingly, standard regularity results yield smoothness of  $u_q$  and  $u_q^s$  in  $\mathbb{R}^d \setminus \overline{B_R(0)}$ , where  $B_R(0)$  is a ball containing the support of the contrast function  $\text{supp}(q)$ , and the entire solution  $u^i$  is smooth throughout  $\mathbb{R}^d$ . In particular the Sommerfeld radiation condition (2.2c) is well defined.<sup>1</sup>  $\square$

**LEMMA 2.2.** *Suppose that the incident field  $u^i \in H^1_{\text{loc}}(\mathbb{R}^d)$  satisfies (2.1); then the scattering problem (2.2) has a unique solution  $u_q \in H^1_{\text{loc}}(\mathbb{R}^d)$ . Furthermore, the scattered field  $u_q^s = u_q - u^i \in H^1_{\text{loc}}(\mathbb{R}^d)$  has the asymptotic behavior*

<sup>1</sup>As usual, we call a (weak) solution to a Helmholtz equation on an unbounded domain that satisfies the Sommerfeld radiation condition a *radiating solution*.

$$(2.3) \quad u_q^s(x) = C_d \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}} u_q^\infty(\hat{x}) + O(|x|^{-\frac{d+1}{2}}), \quad |x| \rightarrow \infty,$$

uniformly in all directions  $\hat{x} := x/|x| \in S^{d-1}$ , where

$$(2.4) \quad C_d = e^{i\pi/4}/\sqrt{8\pi k} \quad \text{if } n = 2 \quad \text{and} \quad C_d = 1/(4\pi) \quad \text{if } n = 3,$$

and the far field pattern  $u_q^\infty$  is given by

$$(2.5) \quad u_q^\infty(\hat{x}) = \int_{\partial B_R(0)} \left( u_q^s(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu_y} - e^{-ik\hat{x}\cdot y} \frac{\partial u_q^s}{\partial \nu}(y) \right) ds(y), \quad \hat{x} \in S^{d-1}.$$

*Proof.* The unique solvability follows, e.g., immediately from [9, Thm. 8.7] (see also [34, Thm. 6.9]), and the farfield asymptotics are, e.g., shown in [9, Thm. 2.6].  $\square$

For the special case of a *plane wave incident field*  $u^i(x; \theta) := e^{ik\theta \cdot x}$ , we explicitly indicate the dependence on the *incident direction*  $\theta \in S^{d-1}$  by a second argument, and accordingly we write  $u_q(\cdot; \theta)$ ,  $u_q^s(\cdot; \theta)$ , and  $u_q^\infty(\cdot; \theta)$  for the corresponding scattered field, total field, and far field pattern, respectively. As usual, we collect the far field patterns  $u_q^\infty(\hat{x}; \theta)$  for all possible observation and incident directions  $\hat{x}, \theta \in S^{d-1}$  in the *far field operator*

$$(2.6) \quad F_q : L^2(S^{d-1}) \rightarrow L^2(S^{d-1}), \quad (F_q g)(\hat{x}) := \int_{S^{d-1}} u_q^\infty(\hat{x}; \theta) g(\theta) ds(\theta),$$

which is compact and normal (see, e.g., [9, Thm. 3.24]). Moreover, the *scattering operator* is defined by

$$(2.7) \quad \mathcal{S}_q : L^2(S^{d-1}) \rightarrow L^2(S^{d-1}), \quad \mathcal{S}_q g := (I + 2ik|C_d|^2 F_q)g,$$

where  $C_d$  is again the constant from (2.4). The operator  $\mathcal{S}_q$  is unitary, and consequently the eigenvalues of  $F_q$  lie on the circle of radius  $1/(2k|C_d|^2)$  centered in  $i/(2k|C_d|^2)$  in the complex plane (cf., e.g., [9, pp. 285–286]).

By linearity, for any given function  $g \in L^2(S^{d-1})$ , the solution to the direct scattering problem (2.2) with incident field

$$(2.8a) \quad u_g^i(x) = \int_{S^{d-1}} e^{ikx \cdot \theta} g(\theta) ds(\theta), \quad x \in \mathbb{R}^d,$$

is given by

$$(2.8b) \quad u_{q,g}(x) = \int_{S^{d-1}} u_q(x; \theta) g(\theta) ds(\theta), \quad x \in \mathbb{R}^d,$$

and the corresponding scattered field

$$(2.8c) \quad u_{q,g}^s(x) = \int_{S^{d-1}} u_q^s(x; \theta) g(\theta) ds(\theta), \quad x \in \mathbb{R}^d,$$

has the far field pattern  $u_{q,g}^\infty = F_q g$  satisfying

$$(2.8d) \quad u_{q,g}^\infty(\hat{x}) = \int_{\partial B_R(0)} \left( u_{q,g}^s(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu_y} - e^{-ik\hat{x}\cdot y} \frac{\partial u_{q,g}^s}{\partial \nu}(y) \right) ds(y), \quad \hat{x} \in S^{d-1}.$$

Incident fields as in (2.8a) are usually called *Herglotz wave functions*.

**3. A monotonicity relation for the far field operator.** We will frequently be discussing relative orderings compact self-adjoint operators. The following extension of the Loewner order was introduced in [24]. Let  $A, B : X \rightarrow X$  be two compact self-adjoint linear operators on a Hilbert space  $X$ . We write

$$A \leq_r B \quad \text{for some } r \in \mathbb{N}$$

if  $B - A$  has at most  $r$  negative eigenvalues. Similarly, we write  $A \leq_{\text{fin}} B$  if  $A \leq_r B$  holds for some  $r \in \mathbb{N}$ , and the notations  $A \geq_r B$  and  $A \geq_{\text{fin}} B$  are defined accordingly.

The following result was shown in [24, Cor. 3.3].

LEMMA 3.1. *Let  $A, B : X \rightarrow X$  be two compact self-adjoint linear operators on a Hilbert space  $X$  with scalar product  $\langle \cdot, \cdot \rangle$ , and let  $r \in \mathbb{N}$ . Then the following statements are equivalent:*

- (a)  $A \leq_r B$ .
- (b) *There exists a finite dimensional subspace  $V \subseteq X$  with  $\dim(V) \leq r$  such that*

$$\langle (B - A)v, v \rangle \geq 0 \quad \text{for all } v \in V^\perp.$$

In particular this lemma shows that  $\leq_{\text{fin}}$  and  $\geq_{\text{fin}}$  are transitive relations (see [24, Lem. 3.4]) and thus preorders. We use this notation in the following *monotonicity relation* for the far field operator.

THEOREM 3.2. *Let  $q_1, q_2 \in L_{0,+}^\infty(\mathbb{R}^d)$ . Then there exists a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that*

$$(3.1) \quad \text{Re} \left( \int_{S^{d-1}} g \overline{\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})g} ds \right) \geq k^2 \int_{\mathbb{R}^d} (q_2 - q_1) |u_{q_1,g}|^2 dx \quad \text{for all } g \in V^\perp.$$

*In particular*

$$(3.2) \quad q_1 \leq q_2 \quad \text{implies that} \quad \text{Re}(\mathcal{S}_{q_1}^* F_{q_1}) \leq_{\text{fin}} \text{Re}(\mathcal{S}_{q_1}^* F_{q_2}),$$

where as usual the real part of a linear operator  $A : X \rightarrow X$  on a Hilbert space  $X$  is the self-adjoint operator given by  $\text{Re}(A) := \frac{1}{2}(A + A^*)$ .

Remark 3.3. Since the scattering operators  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are unitary, we find using (2.7) that

$$\begin{aligned} \mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1}) &= \frac{1}{2ik|C_d|^2} \mathcal{S}_{q_1}^*(\mathcal{S}_{q_2} - \mathcal{S}_{q_1}) = \frac{1}{2ik|C_d|^2} (\mathcal{S}_{q_1}^* \mathcal{S}_{q_2} - I) \\ &= \left( \frac{1}{2ik|C_d|^2} (I - \mathcal{S}_{q_2}^* \mathcal{S}_{q_1}) \right)^* = \left( \frac{1}{2ik|C_d|^2} \mathcal{S}_{q_2}^*(\mathcal{S}_{q_2} - \mathcal{S}_{q_1}) \right)^* = (\mathcal{S}_{q_2}^*(F_{q_2} - F_{q_1}))^*. \end{aligned}$$

Recalling that the eigenvalues of a compact linear operator and of its adjoint are complex conjugates of each other, we conclude that the spectra of  $\text{Re}(\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1}))$  and  $\text{Re}(\mathcal{S}_{q_2}^*(F_{q_2} - F_{q_1}))$  coincide. Consequently, the monotonicity relations (3.1)–(3.2) remain true if we replace  $\mathcal{S}_{q_1}^*$  by  $\mathcal{S}_{q_2}^*$  in these formulas.  $\square$

Interchanging the roles of  $q_1$  and  $q_2$ , except for  $\mathcal{S}_{q_1}^*$  (see Remark 3.3), we may restate Theorem 3.2 as follows.

**COROLLARY 3.4.** *Let  $q_1, q_2 \in L_{0,+}^\infty(\mathbb{R}^d)$ . Then there exists a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that*

$$(3.3) \quad \text{Re}\left(\int_{S^{d-1}} g \overline{\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})g} \, ds\right) \leq k^2 \int_{\mathbb{R}^d} (q_2 - q_1) |u_{q_2,g}|^2 \, dx \quad \text{for all } g \in V^\perp.$$

*Remark 3.5.* A well-known monotonicity principle for the phases of the eigenvalues of the far field operator, which has been discussed, e.g., in [37, Lem. 4.1], can be rephrased as  $\text{Re}(F_q) \geq_{\text{fin}} 0$  if  $q > 0$  and  $\text{Re}(F_q) \leq_{\text{fin}} 0$  if  $q < 0$  a.e. on the support of the contrast function  $\text{supp}(q)$ . This result can now also be obtained as a special case of (3.1) in Theorem 3.2 with  $q_1 = 0$  and  $q_2 = q$  if  $q > 0$  (or  $q_1 = q$  and  $q_2 = 0$  and  $\mathcal{S}_{q_1}^*$  replaced by  $\mathcal{S}_{q_2}^*$  (see Remark 3.3) if  $q < 0$ ).

The monotonicity relation (3.2), which is a consequence of the stronger result (3.1), has already been established in [38, Lem. 3], using rather different techniques.  $\square$

The proof of Theorem 3.2 is a simple corollary of the following lemmas. We begin by summarizing some useful identities for the solution of the scattering problem (2.2).

**LEMMA 3.6.** *Let  $q \in L_{0,+}^\infty(\mathbb{R}^d)$ ,  $n^2 = 1 + q$ , and let  $B_R(0)$  be a ball containing  $\text{supp}(q)$ . Then*

$$(3.4) \quad \int_{S^{d-1}} g \overline{F_{q,g}} \, ds = k^2 \int_{B_R(0)} q u_g^i \overline{u_{q,g}} \, dx \quad \text{for all } g \in L^2(S^{d-1}),$$

and, for any  $v \in H^1(B_R(0))$ ,

$$(3.5) \quad \int_{B_R(0)} (\nabla u_{q,g}^s \cdot \nabla v - k^2 n^2 u_{q,g}^s v) \, dx - \int_{\partial B_R(0)} v \frac{\partial u_{q,g}^s}{\partial \nu} \, ds = k^2 \int_{B_R(0)} q u_g^i v \, dx.$$

Furthermore, if  $q_1, q_2 \in L_{0,+}^\infty(\mathbb{R}^d)$  and  $B_R(0)$  is a ball containing  $\text{supp}(q_1) \cup \text{supp}(q_2)$ , then, for any  $j, l \in \{1, 2\}$ ,

$$(3.6) \quad \int_{\partial B_R(0)} \left( u_{q_j,g}^s \frac{\partial \overline{u_{q_l,g}^s}}{\partial \nu} - \overline{u_{q_l,g}^s} \frac{\partial u_{q_j,g}^s}{\partial \nu} \right) \, ds = -2ik |C_d|^2 \int_{S^{d-1}} F_{q_j,g} \overline{F_{q_l,g}} \, ds,$$

where  $C_d$  denotes the constant from (2.4).

*Proof.* Let  $g \in L^2(S^{d-1})$ ; then the scattered field  $u_{q,g}^s \in H_{\text{loc}}^1(\mathbb{R}^d)$  from (2.8c) solves

$$\Delta u_{q,g}^s + k^2 n^2 u_{q,g}^s = -\Delta u_{q,g}^i - k^2 n^2 u_{q,g}^i = -k^2 q u_{q,g}^i \quad \text{in } B_R(0),$$

and accordingly Green’s formula shows that, for any  $v \in H^1(B_R(0))$ ,

$$\int_{B_R(0)} \nabla u_{q,g}^s \cdot \nabla v \, dx = \int_{\partial B_R(0)} v \frac{\partial u_{q,g}^s}{\partial \nu} \, ds + k^2 \int_{B_R(0)} n^2 u_{q,g}^s v \, dx + k^2 \int_{B_R(0)} q u_{q,g}^i v \, dx,$$

which proves (3.5).

Likewise, we obtain from (2.8d) and Green’s formula that

$$\begin{aligned} u_{q,g}^\infty(\theta) &= \int_{\partial B_R(0)} \left( u_{q,g}^s(y) \frac{\partial e^{-ik\theta \cdot y}}{\partial \nu_y} - e^{-ik\theta \cdot y} \frac{\partial u_{q,g}^s}{\partial \nu}(y) \right) ds(y) \\ &= k^2 \int_{B_R(0)} q(y) u_{q,g}(y) e^{-ik\theta \cdot y} dy, \end{aligned}$$

and thus

$$\int_{S^{d-1}} g \overline{F_{qg}} ds = k^2 \int_{B_R(0)} q(y) \overline{u_{q,g}(y)} \int_{S^{d-1}} g(\theta) e^{ik\theta \cdot y} ds(\theta) dy.$$

Using (2.8a) this shows (3.4).

To see (3.6) let  $r > R$ . Then  $u_{q_j,g}^s, u_{q_l,g}^s \in H_{\text{loc}}^1(\mathbb{R}^d)$  solve (for  $q = q_j$  and  $q = q_l$ )

$$\Delta u_{q,g}^s + k^2 u_{q,g}^s = 0 \quad \text{in } B_r(0) \setminus \overline{B_R(0)},$$

and applying Green’s formula we obtain that

$$(3.7) \quad \int_{\partial B_r(0)} \left( u_{q_j,g}^s \frac{\overline{\partial u_{q_l,g}^s}}{\partial \nu} - \overline{u_{q_l,g}^s} \frac{\partial u_{q_j,g}^s}{\partial \nu} \right) ds = \int_{\partial B_r(0)} \left( u_{q_j,g}^s \frac{\overline{\partial u_{q_l,g}^s}}{\partial \nu} - \overline{u_{q_l,g}^s} \frac{\partial u_{q_j,g}^s}{\partial \nu} \right) ds.$$

Using the radiation condition (2.2c) and the far field expansion (2.3) (for  $q = q_j$  and  $q = q_l$ ) we find that, as  $r \rightarrow \infty$ ,

$$\begin{aligned} (3.8) \quad \int_{\partial B_r(0)} \left( u_{q_j,g}^s \frac{\overline{\partial u_{q_l,g}^s}}{\partial \nu} - \overline{u_{q_l,g}^s} \frac{\partial u_{q_j,g}^s}{\partial \nu} \right) ds &= -2ik \int_{\partial B_r(0)} u_{q_j,g}^s \overline{u_{q_l,g}^s} ds + o(1) \\ &= -2ik|C_d|^2 \int_{S^{d-1}} F_{q_j,g} \overline{F_{q_l,g}} ds + o(1). \end{aligned}$$

Substituting (3.8) into (3.7) and letting  $r \rightarrow \infty$  finally gives (3.6). □

The next tool we will use to prove the monotonicity relation for the far field operator in Theorem 3.2 is the following integral identity.

LEMMA 3.7. *Let  $q_1, q_2 \in L_{0,+}^\infty(\mathbb{R}^d)$ , and let  $B_R(0)$  be a ball containing  $\text{supp}(q_1) \cup \text{supp}(q_2)$ . Then, for any  $g \in L^2(S^{d-1})$ ,*

$$\begin{aligned} (3.9) \quad \int_{S^{d-1}} (g \overline{F_{q_2g}} - \overline{g} F_{q_1g}) ds + 2ik|C_d|^2 \int_{S^{d-1}} F_{q_1g} \overline{F_{q_2g}} ds + k^2 \int_{\mathbb{R}^d} (q_1 - q_2) |u_{q_1,g}|^2 dx \\ = \int_{B_R(0)} (|\nabla(u_{q_2,g}^s - u_{q_1,g}^s)|^2 - k^2 n_2^2 |u_{q_2,g}^s - u_{q_1,g}^s|^2) dx \\ - \int_{\partial B_R(0)} \frac{\overline{(u_{q_2,g}^s - u_{q_1,g}^s)} \partial(u_{q_2,g}^s - u_{q_1,g}^s)}{\partial \nu} ds. \end{aligned}$$

*Proof.* The identity (3.6) (with  $j = 1$  and  $l = 2$ ) immediately implies that

$$\begin{aligned}
 & 2 \operatorname{Re} \int_{\partial B_R(0)} \overline{u_{q_1, g}^s} \frac{\partial u_{q_2, g}^s}{\partial \nu} \, ds \\
 &= \int_{\partial B_R(0)} \left( \overline{u_{q_1, g}^s} \frac{\partial u_{q_2, g}^s}{\partial \nu} + \overline{u_{q_2, g}^s} \frac{\partial u_{q_1, g}^s}{\partial \nu} \right) \, ds - 2ik|C_d|^2 \int_{S^{d-1}} F_{q_1} g \overline{F_{q_2} g} \, ds.
 \end{aligned}$$

Using this and (3.5) we find that

$$\begin{aligned}
 & \int_{B_R(0)} (|\nabla u_{q_2, g}^s - \nabla u_{q_1, g}^s|^2 - k^2 n_2^2 |u_{q_2, g}^s - u_{q_1, g}^s|^2) \, dx \\
 & - \int_{\partial B_R(0)} \overline{(u_{q_2, g}^s - u_{q_1, g}^s)} \frac{\partial (u_{q_2, g}^s - u_{q_1, g}^s)}{\partial \nu} \, ds \\
 &= \int_{B_R(0)} (|\nabla u_{q_2, g}^s|^2 - k^2 n_2^2 |u_{q_2, g}^s|^2) \, dx + \int_{B_R(0)} (|\nabla u_{q_1, g}^s|^2 - k^2 n_2^2 |u_{q_1, g}^s|^2) \, dx \\
 & - 2 \operatorname{Re} \left( \int_{B_R(0)} (\nabla u_{q_2, g}^s \cdot \overline{\nabla u_{q_1, g}^s} - k^2 n_2^2 u_{q_2, g}^s \overline{u_{q_1, g}^s}) \, dx - \int_{\partial B_R(0)} \overline{u_{q_1, g}^s} \frac{\partial u_{q_2, g}^s}{\partial \nu} \, ds \right) \\
 & + 2ik|C_d|^2 \int_{S^{d-1}} F_{q_1} g \overline{F_{q_2} g} \, ds - \int_{\partial B_R(0)} \overline{u_{q_2, g}^s} \frac{\partial u_{q_2, g}^s}{\partial \nu} \, ds - \int_{\partial B_R(0)} \overline{u_{q_1, g}^s} \frac{\partial u_{q_1, g}^s}{\partial \nu} \, ds \\
 &= k^2 \int_{B_R(0)} q_2 u_g^i \overline{u_{q_2, g}^s} \, dx - 2 \operatorname{Re} \left( k^2 \int_{B_R(0)} q_2 u_g^i \overline{u_{q_1, g}^s} \, dx \right) + k^2 \int_{B_R(0)} q_1 u_g^i \overline{u_{q_1, g}^s} \, dx \\
 & + k^2 \int_{B_R(0)} (q_1 - q_2) |u_{q_1, g}^s|^2 \, dx + 2ik|C_d|^2 \int_{S^{d-1}} F_{q_1} g \overline{F_{q_2} g} \, ds.
 \end{aligned}$$

Further simple manipulations give

$$\begin{aligned}
 & \int_{B_R(0)} (|\nabla u_{q_2, g}^s - \nabla u_{q_1, g}^s|^2 - k^2 n_2^2 |u_{q_2, g}^s - u_{q_1, g}^s|^2) \, dx \\
 & - \int_{\partial B_R(0)} \overline{(u_{q_2, g}^s - u_{q_1, g}^s)} \frac{\partial (u_{q_2, g}^s - u_{q_1, g}^s)}{\partial \nu} \, ds \\
 &= k^2 \int_{B_R(0)} q_2 u_g^i \overline{u_{q_2, g}^s} \, dx - 2 \operatorname{Re} \left( k^2 \int_{B_R(0)} (q_2 - q_1) u_g^i \overline{u_{q_1, g}^s} \, dx \right) \\
 & - k^2 \int_{B_R(0)} q_1 \overline{u_g^i} u_{q_1, g}^s \, dx - k^2 \int_{B_R(0)} (q_2 - q_1) |u_{q_1, g}^s|^2 \, dx + 2ik|C_d|^2 \int_{S^{d-1}} F_{q_1} g \overline{F_{q_2} g} \, ds \\
 &= k^2 \int_{B_R(0)} q_2 u_g^i \overline{u_{q_2, g}^s} \, dx - k^2 \int_{B_R(0)} q_1 \overline{u_g^i} u_{q_1, g}^s \, dx - k^2 \int_{B_R(0)} (q_2 - q_1) |u_g^i|^2 \, dx \\
 & - 2 \operatorname{Re} \left( k^2 \int_{B_R(0)} (q_2 - q_1) \overline{u_g^i} u_{q_1, g}^s \, dx \right) - k^2 \int_{B_R(0)} (q_2 - q_1) |u_{q_1, g}^s|^2 \, dx \\
 & + 2ik|C_d|^2 \int_{S^{d-1}} F_{q_1} g \overline{F_{q_2} g} \, ds \\
 &= k^2 \int_{B_R(0)} q_2 u_g^i \overline{u_{q_2, g}^s} \, dx - k^2 \int_{B_R(0)} q_1 \overline{u_g^i} u_{q_1, g}^s \, dx - k^2 \int_{B_R(0)} (q_2 - q_1) |u_{q_1, g}^s|^2 \, dx \\
 & + 2ik|C_d|^2 \int_{S^{d-1}} F_{q_1} g \overline{F_{q_2} g} \, ds.
 \end{aligned}$$

Finally, applying (3.4) we obtain that

$$\begin{aligned} & \int_{B_R(0)} (|\nabla u_{q_2,g}^s - \nabla u_{q_1,g}^s|^2 - k^2 n_2^2 |u_{q_2,g}^s - u_{q_1,g}^s|^2) \, dx \\ & - \int_{\partial B_R(0)} \frac{\overline{(u_{q_2,g}^s - u_{q_1,g}^s)} \partial(u_{q_2,g}^s - u_{q_1,g}^s)}{\partial \nu} \, ds \\ & = \int_{S^{d-1}} (g \overline{F_{q_2} g} - \bar{g} F_{q_1} g) \, ds - k^2 \int_{B_R(0)} (q_2 - q_1) |u_{q_1,g}^s|^2 \, dx \\ & + 2ik|C_d|^2 \int_{S^{d-1}} F_{q_1} g \overline{F_{q_2} g} \, ds, \end{aligned}$$

which proves the assertion.  $\square$

*Remark 3.8.* Since the adjoint of the scattering operator  $\mathcal{S}_{q_1}$  from (2.7) is given by

$$\mathcal{S}_{q_1}^* = I - 2ik|C_d|^2 F_{q_1}^*,$$

we find that

$$\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1}) = F_{q_2} - F_{q_1} - 2ik|C_d|^2(F_{q_1}^* F_{q_2} - F_{q_1}^* F_{q_1}),$$

and accordingly,

$$\operatorname{Re}(\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})) = \operatorname{Re}(F_{q_2} - F_{q_1} - 2ik|C_d|^2 F_{q_1}^* F_{q_2}).$$

Therefore the real part of the first two terms on the left-hand side of (3.9) fulfills

$$\begin{aligned} & \operatorname{Re}\left(\int_{S^{d-1}} (g \overline{F_{q_2} g} - \bar{g} F_{q_1} g) \, ds + 2ik|C_d|^2 \int_{S^{d-1}} F_{q_1} g \overline{F_{q_2} g} \, ds\right) \\ (3.10) \quad & = \operatorname{Re}\left(\int_{S^{d-1}} g \overline{(F_{q_2} - F_{q_1} - 2ik|C_d|^2 F_{q_1}^* F_{q_2})} g \, ds\right) \\ & = \operatorname{Re}\left(\int_{S^{d-1}} g \overline{\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})} g \, ds\right). \end{aligned}$$

The operator  $\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})$  is compact and normal (see [38, Lem. 1]).  $\square$

Next we consider the right-hand side of (3.9), and we show that it is nonnegative if  $g$  belongs to the complement of a certain finite dimensional subspace  $V \subseteq L^2(S^{d-1})$ . To that end we denote by  $I : H^1(B_R(0)) \rightarrow H^1(B_R(0))$  the identity operator, by  $J : H^1(B_R(0)) \rightarrow L^2(B_R(0))$  the compact embedding, and accordingly we define, for any  $q \in L_{0,+}^\infty(\mathbb{R}^d)$  and any ball  $B_R(0)$  containing  $\operatorname{supp}(q)$ , the operator  $K : H^1(B_R(0)) \rightarrow H^1(B_R(0))$  by

$$Kv := J^* Jv,$$

and  $K_q : H^1(B_R(0)) \rightarrow H^1(B_R(0))$  by

$$K_q v := J^*((1+q)Jv).$$

Then  $K$  and  $K_q$  are compact self-adjoint linear operators, and, for any  $v \in H^1(B_R(0))$ ,

$$\langle (I - K - k^2 K_q)v, v \rangle_{H^1(B_R(0))} = \int_{B_R(0)} (|\nabla v|^2 - k^2(1+q)|v|^2) \, dx.$$

For  $0 < \varepsilon < R$  we denote by  $N_\varepsilon : H^1(B_R(0)) \rightarrow L^2(\partial B_R(0))$  the bounded linear operator that maps  $v \in H^1(B_R(0))$  to the normal derivative  $\partial v_\varepsilon / \partial \nu$  on  $\partial B_R(0)$  of the radiating solution to the exterior boundary value problem

$$\Delta v_\varepsilon + k^2 v_\varepsilon = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B_{R-\varepsilon}(0)}, \quad v_\varepsilon = v \quad \text{on } \partial B_{R-\varepsilon}(0),$$

and  $\Lambda : L^2(\partial B_R(0)) \rightarrow L^2(\partial B_R(0))$  denotes the compact exterior Neumann-to-Dirichlet operator that maps  $\psi \in L^2(\partial B_R(0))$  to the trace  $w|_{\partial B_R(0)}$  of the radiating solution to

$$\Delta w + k^2 w = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B_R(0)}, \quad \frac{\partial w}{\partial \nu} = \psi \quad \text{on } \partial B_R(0)$$

(see, e.g., [9, pp. 51–55]). Then,

$$N_\varepsilon v = \frac{\partial v}{\partial \nu} \Big|_{\partial B_R(0)} \quad \text{and} \quad \Lambda N_\varepsilon = v|_{\partial B_R(0)},$$

and accordingly

$$\langle N_\varepsilon^* \Lambda N_\varepsilon v, v \rangle_{H^1(B_R(0))} = \langle \Lambda N_\varepsilon v, N_\varepsilon v \rangle_{L^2(\partial B_R(0))} = \int_{\partial B_R(0)} \bar{v} \frac{\partial v}{\partial \nu} \, ds$$

for any  $v \in H^1(B_R(0))$  that can be extended to a radiating solution of the Helmholtz equation

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B_{R-\varepsilon}(0)}.$$

In particular this holds for  $v = u_{q_2,g}^s - u_{q_1,g}^s$  if the ball  $B_{R-\varepsilon}(0)$  contains  $\text{supp}(q_1)$  and  $\text{supp}(q_2)$ .

LEMMA 3.9. *Let  $q_1, q_2 \in L^\infty_{0,+}(\mathbb{R}^d)$  and let  $B_R(0)$  be a ball containing  $\text{supp}(q_1) \cup \text{supp}(q_2)$ . Then there exists a finite dimensional subspace  $V \subset L^2(S^{d-1})$  such that*

$$\int_{B_R(0)} (|\nabla(u_{q_2,g}^s - u_{q_1,g}^s)|^2 - k^2 n_2^2 |u_{q_2,g}^s - u_{q_1,g}^s|^2) \, dx - \text{Re} \left( \int_{\partial B_R(0)} \overline{(u_{q_2,g}^s - u_{q_1,g}^s)} \frac{\partial(u_{q_2,g}^s - u_{q_1,g}^s)}{\partial \nu} \, ds \right) \geq 0 \quad \text{for all } g \in V^\perp.$$

*Proof.* Let  $\varepsilon > 0$  be sufficiently small, so that  $\text{supp}(q_1) \cup \text{supp}(q_2) \subset B_{R-\varepsilon}(0)$ . Then

$$\begin{aligned} & \int_{B_R(0)} (|\nabla(u_{q_2,g}^s - u_{q_1,g}^s)|^2 - k^2 n_2^2 |u_{q_2,g}^s - u_{q_1,g}^s|^2) \, dx \\ & - \text{Re} \left( \int_{\partial B_R(0)} \overline{(u_{q_2,g}^s - u_{q_1,g}^s)} \frac{\partial(u_{q_2,g}^s - u_{q_1,g}^s)}{\partial \nu} \, ds \right) \\ & = \langle (I - K - k^2 K_{q_2} - \text{Re}(N_\varepsilon^* \Lambda N_\varepsilon))(S_2 - S_1)g, (S_2 - S_1)g \rangle_{H^1(B_R(0))}, \end{aligned}$$

where for  $j = 1, 2$  we denote by  $S_j : L^2(S^{d-1}) \rightarrow H^1(B_R(0))$  the bounded linear operator that maps  $g \in L^2(S^{d-1})$  to the restriction of the scattered field  $u_{q_j,g}^s$  on  $B_R(0)$ .

Let  $W$  be the sum of eigenspaces of the compact self-adjoint operator  $K + k^2 K_{q_2} + \text{Re}(N_\varepsilon^* \Lambda N_\varepsilon)$  associated to eigenvalues larger than 1. Then  $W$  is finite dimensional and

$$\langle (I - K - k^2 K_{q_2} - \operatorname{Re}(N_\varepsilon^* \Lambda N_\varepsilon))w, w \rangle_{H^1(B_R(0))} \geq 0 \quad \text{for all } w \in W^\perp.$$

Since, for any  $g \in L^2(S^{d-1})$ ,

$$(S_2 - S_1)g \in W^\perp \quad \text{if and only if} \quad g \in ((S_2 - S_1)^*W)^\perp,$$

and of course  $\dim((S_2 - S_1)^*W) \leq \dim(W) < \infty$ , choosing  $V := (S_2 - S_1)^*W$  ends the proof.  $\square$

*Proof of Theorem 3.2.* Taking the real part of (3.9) and applying (3.10), the result follows immediately from Lemma 3.9.  $\square$

**4. Localized wave functions.** In this section we establish the existence of *localized wave functions* that have arbitrarily large norm on some prescribed region  $B \subseteq \mathbb{R}^d$  while at the same time having arbitrarily small norm in a different region  $D \subseteq \mathbb{R}^d$ , assuming that  $\mathbb{R}^d \setminus \overline{D}$  is connected. These will be utilized to establish a rigorous characterization of the support of scattering objects in terms of the far field operator using the monotonicity relations from Theorem 3.2 and Corollary 3.4 in section 5 below.

**THEOREM 4.1.** *Suppose that  $q \in L^\infty_{0,+}(\mathbb{R}^d)$ , and let  $B, D \subseteq \mathbb{R}^d$  be open and bounded such that  $\mathbb{R}^d \setminus \overline{D}$  is connected.*

*If  $B \not\subseteq D$ , then for any finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  there exists a sequence  $(g_m)_{m \in \mathbb{N}} \subseteq V^\perp$  such that*

$$\int_B |u_{q,g_m}|^2 dx \rightarrow \infty \quad \text{and} \quad \int_D |u_{q,g_m}|^2 dx \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

where  $u_{q,g_m} \in H^1_{\text{loc}}(\mathbb{R}^d)$  is given by (2.8b) with  $g = g_m$ .

The proof of Theorem 4.1 relies on the following lemmas.

**LEMMA 4.2.** *Suppose that  $q \in L^\infty_{0,+}(\mathbb{R}^d)$ , let  $n^2 = 1 + q$ , and assume that  $D \subseteq \mathbb{R}^d$  is open and bounded. We define*

$$L_{q,D} : L^2(S^{d-1}) \rightarrow L^2(D), \quad g \mapsto u_{q,g}|_D,$$

where  $u_{q,g} \in H^1_{\text{loc}}(\mathbb{R}^d)$  is given by (2.8b). Then  $L_{q,D}$  is a compact linear operator and its adjoint is given by

$$L^*_{q,D} : L^2(D) \rightarrow L^2(S^{d-1}), \quad f \mapsto \mathcal{S}_q^* w^\infty,$$

where  $\mathcal{S}_q$  denotes the scattering operator from (2.7), and  $w^\infty \in L^2(S^{d-1})$  is the far field pattern of the radiating solution  $w \in H^1_{\text{loc}}(\mathbb{R}^d)$  to<sup>2</sup>

$$(4.1) \quad \Delta w + k^2 n^2 w = -f \quad \text{in } \mathbb{R}^d.$$

*Proof.* The representation formula for the total field in (2.8b) shows that  $L_{q,D}$  is a Fredholm integral operator with square integrable kernel and therefore compact and linear from  $L^2(S^{d-1})$  to  $L^2(D)$ .

The existence and uniqueness of a radiating solution  $w \in H^1_{\text{loc}}(\mathbb{R}^d)$  of (4.1) follows again from [9, Thm. 8.7] (see also [34, Thm. 6.9]). To determine the adjoint of  $L_{q,D}$  we first observe that, for any ball  $B_R(0)$ , this solution satisfies, for any  $v \in H^1(B_R(0))$ ,

$$(4.2) \quad \int_{B_R(0)} (\nabla w \cdot \nabla v - k^2 n^2 wv) dx = \int_{B_R(0)} f v dx + \int_{\partial B_R(0)} v \frac{\partial w}{\partial \nu} ds.$$

<sup>2</sup>Throughout, we identify  $f \in L^2(D)$  with its continuation to  $\mathbb{R}^d$  by zero whenever appropriate.

We choose  $R > 0$  large enough such that  $\text{supp}(q)$  and  $D$  are contained in  $B_R(0)$ . Applying (4.2), Green’s formula, and the representation formula for the far field pattern  $w^\infty$  of  $w$  analogous to (2.5) we find that, for any  $g \in L^2(S^{d-1})$  and  $f \in L^2(D)$ ,

$$\begin{aligned}
 \int_D (L_{q,D}g)\bar{f} \, dx &= \int_{B_R(0)} (\nabla u_{q,g} \cdot \overline{\nabla w} - k^2 n^2 u_{q,g} \overline{w}) \, dx - \int_{\partial B_R(0)} u_{q,g} \overline{\frac{\partial w}{\partial \nu}} \, ds \\
 &= \int_{\partial B_R(0)} \left( \frac{\partial u_{q,g}}{\partial \nu} \overline{w} - u_{q,g} \overline{\frac{\partial w}{\partial \nu}} \right) \, ds \\
 &= \int_{S^{d-1}} g(\theta) \int_{\partial B_R(0)} \left( \frac{\partial e^{ik\theta \cdot y}}{\partial \nu_y} \overline{w(y)} - e^{ik\theta \cdot y} \overline{\frac{\partial w}{\partial \nu}(y)} \right) \, ds(y) \, ds(\theta) \\
 (4.3) \quad &+ \int_{S^{d-1}} g(\theta) \int_{\partial B_R(0)} \left( \frac{\partial u_{q,g}^s}{\partial \nu_y}(y; \theta) \overline{w(y)} - u_{q,g}^s(y; \theta) \overline{\frac{\partial w}{\partial \nu}(y)} \right) \, ds(y) \, ds(\theta) \\
 &= \int_{S^{d-1}} g(\theta) \overline{w^\infty(\theta)} \, ds(\theta) \\
 &+ \int_{S^{d-1}} g(\theta) \int_{\partial B_R(0)} \left( \frac{\partial u_{q,g}^s}{\partial \nu_y}(y; \theta) \overline{w(y)} - u_{q,g}^s(y; \theta) \overline{\frac{\partial w}{\partial \nu}(y)} \right) \, ds(y) \, ds(\theta).
 \end{aligned}$$

Using the radiation condition (2.2c) and the farfield expansion (2.3) we obtain that, as  $R \rightarrow \infty$ ,

$$\begin{aligned}
 \int_{\partial B_R(0)} \left( \frac{\partial u_{q,g}^s}{\partial \nu_y}(y; \theta) \overline{w(y)} - u_{q,g}^s(y; \theta) \overline{\frac{\partial w}{\partial \nu}(y)} \right) \, ds(y) \\
 &= 2ik \int_{\partial B_R(0)} u_{q,g}^s(y; \theta) \overline{w(y)} \, ds(y) + o(1) \\
 &= 2ik |C_d|^2 \int_{S^{d-1}} u_{q,g}^\infty(\hat{y}; \theta) \overline{w^\infty(\hat{y})} \, ds(\hat{y}) + o(1).
 \end{aligned}$$

Accordingly, substituting this into (4.3) and using (2.6) and (2.7) gives

$$\begin{aligned}
 \int_D (L_{q,D}g)\bar{f} \, dx \\
 &= \int_{S^{d-1}} g(\theta) \overline{w^\infty(\theta)} \, ds(\theta) + 2ik |C_d|^2 \int_{S^{d-1}} g(\theta) \int_{S^{d-1}} u_{q,g}^\infty(\hat{y}; \theta) \overline{w^\infty(\hat{y})} \, ds(\hat{y}) \, ds(\theta) \\
 &= \int_{S^{d-1}} g(\theta) \overline{w^\infty(\theta)} \, ds(\theta) + 2ik |C_d|^2 \int_{S^{d-1}} (F_q g)(\hat{y}) \overline{w^\infty(\hat{y})} \, ds(\hat{y}) = \int_{S^{d-1}} g \overline{\mathcal{S}_q^* w^\infty} \, ds.
 \end{aligned}$$

□

LEMMA 4.3. *Suppose that  $q \in L^\infty_{0,+}(\mathbb{R}^d)$ . Let  $B, D \subseteq \mathbb{R}^d$  be open and bounded such that  $\mathbb{R}^d \setminus (\overline{B} \cup \overline{D})$  is connected and  $\overline{B} \cap \overline{D} = \emptyset$ . Then,*

$$\mathcal{R}(L_{q,B}^*) \cap \mathcal{R}(L_{q,D}^*) = \{0\},$$

and  $\mathcal{R}(L_{q,B}^*), \mathcal{R}(L_{q,D}^*) \subseteq L^2(S^{d-1})$  are both dense.

*Proof.* To start with, we show the injectivity of  $L_{q,B}$ , and we note that the injectivity of  $L_{q,D}$  follows analogously. Let  $R > 0$  such that  $\text{supp}(q) \subseteq B_R(0)$ . Then the solution  $u_{q,g}$  of (2.2) from (2.8b) satisfies the Lippmann–Schwinger equation

$$(4.4) \quad u_{q,g}(x) = u_g^i(x) + k^2 \int_{\mathbb{R}^d} q(y) \Phi(x-y) u_{q,g}(y) \, dy, \quad x \in B_R(0),$$

where  $\Phi$  denotes the fundamental solution to the Helmholtz equation (cf., e.g., [9, Thm. 8.3]). By unique continuation,  $L_{q,B}g = u_{q,g}|_B = 0$  implies that  $u_{q,g} = 0$  in  $\mathbb{R}^d$  (cf., e.g., [24, sect. 2.3]). Substituting this into (4.4), we find that the Herglotz wave function  $u_g^i = 0$  in  $B_R(0)$  and thus by analyticity on all of  $\mathbb{R}^d$ . This implies that  $g = 0$  (cf., e.g., [9, Thm. 3.19]); i.e.,  $L_{q,B}$  is injective.

The injectivity of  $L_{q,B}$  and  $L_{q,D}$  immediately yields that  $\mathcal{R}(L_{q,B}^*)$  and  $\mathcal{R}(L_{q,D}^*)$  are dense in  $L^2(S^{d-1})$ . Next suppose that  $h \in \mathcal{R}(L_{q,B}^*) \cap \mathcal{R}(L_{q,D}^*)$ . Then Lemma 4.2 shows that there exist  $f_B \in L^2(B)$ ,  $f_D \in L^2(D)$ , and  $w_B, w_D \in H_{\text{loc}}^1(\mathbb{R}^d)$  such that the far field patterns  $w_B^\infty$  and  $w_D^\infty$  of the radiating solutions to

$$\Delta w_B + k^2(1+q)w_B = -f_B \quad \text{and} \quad \Delta w_D + k^2(1+q)w_D = -f_D \quad \text{in } \mathbb{R}^d$$

satisfy

$$w_B^\infty = w_D^\infty = \mathcal{S}_q h.$$

Rellich’s lemma and unique continuation guarantee that  $w_B = w_D$  in  $\mathbb{R}^d \setminus (\overline{B} \cup \overline{D})$  (cf., e.g., [9, Thm. 2.14]). Hence we may define  $w \in H_{\text{loc}}^1(\mathbb{R}^d)$  by

$$w := \begin{cases} w_B = w_D & \text{in } \mathbb{R}^d \setminus (\overline{B} \cup \overline{D}), \\ w_B & \text{in } D, \\ w_D & \text{in } B, \end{cases}$$

and  $w$  is the unique radiating solution to

$$\Delta w + k^2(1+q)w = 0 \quad \text{in } \mathbb{R}^d.$$

Thus  $w = 0$  in  $\mathbb{R}^d$ , and since the scattering operator is unitary, this shows that  $h = \mathcal{S}_q^* w^\infty = 0$ . □

In the next lemma we quote a special case of Lemma 2.5 in [26].

LEMMA 4.4. *Let  $X, Y$ , and  $Z$  be Hilbert spaces, and let  $A : X \rightarrow Y$  and  $B : X \rightarrow Z$  be bounded linear operators. Then,*

$$\exists C > 0 : \|Ax\| \leq C\|Bx\| \quad \forall x \in X \quad \text{if and only if} \quad \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*).$$

Now we give the proof of Theorem 4.1.

*Proof of Theorem 4.1.* Suppose that  $q \in L_{0,+}^\infty(\mathbb{R}^d)$ , let  $B, D \subseteq \mathbb{R}^d$  be open such that  $\mathbb{R}^d \setminus \overline{D}$  is connected, and let  $V \subseteq L^2(S^{d-1})$  be a finite dimensional subspace. We first note that without loss of generality we may assume that  $\overline{B} \cap \overline{D} = \emptyset$  and that  $\mathbb{R}^d \setminus (\overline{B} \cup \overline{D})$  is connected (otherwise we replace  $B$  by a sufficiently small ball  $\tilde{B} \subseteq B \setminus \overline{D}_\varepsilon$ , where  $D_\varepsilon$  denotes a sufficiently small neighborhood of  $D$ ).

We denote by  $P_V : L^2(S^{d-1}) \rightarrow L^2(S^{d-1})$  the orthogonal projection on  $V$ . Lemma 4.3 shows that  $\mathcal{R}(L_{q,B}^*) \cap \mathcal{R}(L_{q,D}^*) = \{0\}$  and that  $\mathcal{R}(L_{q,B}^*)$  is infinite dimensional. Using a simple dimensionality argument (see [24, Lem. 4.7]) it follows that

$$\mathcal{R}(L_{q,B}^*) \not\subseteq \mathcal{R}(L_{q,D}^*) + V = \mathcal{R}\left(\begin{pmatrix} L_{q,D}^* & P_V^* \end{pmatrix}\right) = \mathcal{R}\left(\left(\begin{pmatrix} L_{q,D} \\ P_V \end{pmatrix}\right)^*\right).$$

Accordingly, Lemma 4.4 implies that there is no constant  $C > 0$  such that

$$\|L_{q,B}g\|_{L^2(B)}^2 \leq C^2 \left\| \begin{pmatrix} L_{q,D} \\ P_V \end{pmatrix} g \right\|_{L^2(D) \times L^2(S^{d-1})}^2 = C^2 (\|L_{q,D}g\|_{L^2(D)}^2 + \|P_Vg\|_{L^2(S^{d-1})}^2)$$

for all  $g \in L^2(S^{d-1})$ . Hence, there exists a sequence  $(\tilde{g}_m)_{m \in \mathbb{N}} \subseteq L^2(S^{d-1})$  such that

$$\|L_{q,B}\tilde{g}_m\|_{L^2(B)} \rightarrow \infty \quad \text{and} \quad \|L_{q,D}\tilde{g}_m\|_{L^2(D)} + \|P_V\tilde{g}_m\|_{L^2(S^{d-1})} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Setting  $g_m := \tilde{g}_m - P_V\tilde{g}_m \in V^\perp \subseteq L^2(S^{d-1})$  for any  $m \in \mathbb{N}$ , we finally obtain

$$\begin{aligned} \|L_{q,B}g_m\|_{L^2(B)} &\geq \|L_{q,B}\tilde{g}_m\|_{L^2(B)} - \|L_{q,B}\| \|P_V\tilde{g}_m\|_{L^2(S^{d-1})} \rightarrow \infty \quad \text{as } m \rightarrow \infty, \\ \|L_{q,D}g_m\|_{L^2(D)} &\leq \|L_{q,D}\tilde{g}_m\|_{L^2(D)} + \|L_{q,D}\| \|P_V\tilde{g}_m\|_{L^2(S^{d-1})} \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Since  $L_{q,B}g_m = u_{q,g_m}|_B$  and  $L_{q,D}g_m = u_{q,g_m}|_D$ , this ends the proof. □

The next result is a simple consequence of the Lemmas 4.2 and 4.4.

**THEOREM 4.5.** *Suppose that  $q_1, q_2 \in L_{0,+}^\infty(\mathbb{R}^d)$ , and let  $D \subseteq \mathbb{R}^d$  be open and bounded. If  $q_1(x) = q_2(x)$  for a.e.  $x \in \mathbb{R}^d \setminus \overline{D}$ , then there exist constants  $c, C > 0$  such that*

$$c \int_D |u_{q_1,g}|^2 dx \leq \int_D |u_{q_2,g}|^2 dx \leq C \int_D |u_{q_1,g}|^2 dx \quad \text{for all } g \in L^2(S^{d-1}),$$

where  $u_{q_j,g} \in H_{\text{loc}}^1(\mathbb{R}^d)$ ,  $j = 1, 2$ , is given by (2.8b) with  $q = q_j$ .

*Proof.* Let  $q_1, q_2 \in L_{0,+}^\infty(\mathbb{R}^d)$ . We denote by  $L_{q_1,D}$  and  $L_{q_2,D}$  the operators from Lemma 4.2 with  $q = q_1$  and  $q = q_2$ , respectively. We showed in Lemma 4.2 that for any  $f \in L^2(D)$

$$(4.5) \quad L_{q_1,D}^* f = \mathcal{S}_{q_1}^* w_1^\infty \quad \text{and} \quad L_{q_2,D}^* f = \mathcal{S}_{q_2}^* w_2^\infty,$$

where  $w_j^\infty$ ,  $j = 1, 2$ , are the far field patterns of the radiating solutions to

$$\Delta w_j + k^2(1 + q_j)w_j = -f \quad \text{in } \mathbb{R}^d.$$

This implies that

$$(4.6a) \quad \Delta w_1 + k^2(1 + q_2)w_1 = -(f + k^2(q_1 - q_2)w_1) \quad \text{in } \mathbb{R}^d,$$

$$(4.6b) \quad \Delta w_2 + k^2(1 + q_1)w_2 = -(f + k^2(q_2 - q_1)w_2) \quad \text{in } \mathbb{R}^d.$$

Since  $q_1 - q_2$  vanishes a.e. outside  $D$ , we find that

$$\mathcal{S}_{q_2}^* w_1^\infty = L_{q_2,D}^*(f + k^2(q_1 - q_2)w_1) \quad \text{and} \quad \mathcal{S}_{q_1}^* w_2^\infty = L_{q_1,D}^*(f + k^2(q_2 - q_1)w_2).$$

Combining (4.5) and (4.6), we obtain that  $\mathcal{R}(\mathcal{S}_{q_1}^* L_{q_1,D}^*) = \mathcal{R}(\mathcal{S}_{q_2}^* L_{q_2,D}^*)$ . Since  $\mathcal{S}_{q_1}$  and  $\mathcal{S}_{q_2}$  are unitary operators, the assertion follows from Lemma 4.4. □

As a first application of Theorem 4.1 we establish a converse of (3.2) in Theorem 3.2.

**THEOREM 4.6.** *Suppose that  $q_1, q_2 \in L_{0,+}^\infty(\mathbb{R}^d)$  with  $\text{supp}(q_j) \subseteq B_R(0)$ . If  $O \subseteq \mathbb{R}^d$  is an unbounded domain such that*

$$q_1 \leq q_2 \quad \text{a.e. in } O,$$

and if  $B \subseteq B_R(0) \cap O$  is open with

$$(4.7) \quad q_1 \leq q_2 - c \quad \text{a.e. in } B \text{ for some } c > 0,$$

then

$$\operatorname{Re}(\mathcal{S}_{q_1}^* F_{q_1}) \not\leq_{\text{fin}} \operatorname{Re}(\mathcal{S}_{q_1}^* F_{q_2});$$

i.e., the operator  $\operatorname{Re}(\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1}))$  has infinitely many positive eigenvalues. In particular, this implies that  $F_{q_1} \neq F_{q_2}$ .

*Proof.* We prove the result by contradiction and assume that

$$(4.8) \quad \operatorname{Re}(\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})) \leq_{\text{fin}} 0.$$

Using the monotonicity relation (3.1) in Theorem 3.2, we find that there exists a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that

$$(4.9) \quad \operatorname{Re} \left( \int_{S^{d-1}} g \overline{\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})g} \, ds \right) \geq k^2 \int_{B_R(0)} (q_2 - q_1) |u_{q_1,g}|^2 \, dx \quad \text{for all } g \in V^\perp.$$

Combining (4.8), (4.9), and (4.7) we obtain that there exists a finite dimensional subspace  $\tilde{V} \subseteq L^2(S^{d-1})$  such that, for any  $g \in \tilde{V}^\perp$ ,

$$\begin{aligned} 0 &\geq \operatorname{Re} \left( \int_{S^{d-1}} g \overline{\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})g} \, ds \right) \geq k^2 \int_{B_R(0)} (q_2 - q_1) |u_{q_1,g}|^2 \, dx \\ &= k^2 \int_{O \cap B_R(0)} (q_2 - q_1) |u_{q_1,g}|^2 \, dx + k^2 \int_{B_R(0) \setminus \bar{O}} (q_2 - q_1) |u_{q_1,g}|^2 \, dx \\ &\geq ck^2 \int_B |u_{q_1,g}|^2 \, dx - Ck^2 \int_{B_R(0) \setminus \bar{O}} |u_{q_1,g}|^2 \, dx, \end{aligned}$$

where  $C := \|q_1\|_{L^\infty(\mathbb{R}^d)} + \|q_2\|_{L^\infty(\mathbb{R}^d)}$ . However, this contradicts Theorem 4.1 with  $D = B_R(0) \setminus \bar{O}$  and  $q = q_1$ , which guarantees the existence of  $(g_m)_{m \in \mathbb{N}} \subseteq \tilde{V}^\perp$  with

$$\int_B |u_{q_1,g_m}|^2 \, dx \rightarrow \infty \quad \text{and} \quad \int_{B_R(0) \setminus \bar{O}} |u_{q_1,g_m}|^2 \, dx \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Consequently,  $\operatorname{Re}(\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})) \not\leq_{\text{fin}} 0$ . □

**5. Monotonicity based shape reconstruction.** Given any open and bounded subset  $B \subseteq \mathbb{R}^d$ , we define the operator  $T_B : L^2(S^{d-1}) \rightarrow L^2(S^{d-1})$  by

$$(5.1) \quad T_B g := k^2 H_B^* H_B g,$$

where  $H_B : L^2(S^{d-1}) \rightarrow L^2(B)$  denotes the *Herglotz operator* given by

$$(H_B g)(x) := \int_{S^{d-1}} e^{ikx \cdot \theta} g(\theta) \, ds(\theta), \quad x \in B.$$

Accordingly,

$$\int_{S^{d-1}} g \overline{T_B g} \, ds = k^2 \int_B |u_g^i|^2 \, dx \quad \text{for all } g \in L^2(S^{d-1}),$$

where  $u_g^i$  denotes the Herglotz wave function with density  $g$  from (2.8a). The operator  $T_B$  is bounded, compact, and self-adjoint, and it coincides with the Born approximation of the far field operator  $F_q$  with contrast function  $q = \chi_B$ , where  $\chi_B$  denotes the characteristic function of  $B$  (see, e.g., [35]).

In the following we discuss criteria to determine the support  $\text{supp}(q)$  of an unknown scattering object in terms of the corresponding far field operator  $F_q$ . To begin with we discuss the case when the contrast function  $q$  is positive a.e. on its support.

**THEOREM 5.1.** *Let  $B, D \subseteq \mathbb{R}^d$  be open and bounded such that  $\mathbb{R}^d \setminus \overline{D}$  is connected, and let  $q \in L_{0,+}^\infty(\mathbb{R}^d)$  with  $\text{supp}(q) = \overline{D}$ . Suppose that  $0 \leq q_{\min} \leq q \leq q_{\max} < \infty$  a.e. in  $D$  for some constants  $q_{\min}, q_{\max} \in \mathbb{R}$ .*

(a) *If  $B \subseteq D$ , then*

$$\alpha T_B \leq_{\text{fin}} \text{Re}(F_q) \quad \text{for all } \alpha \leq q_{\min}.$$

(b) *If  $B \not\subseteq D$ , then*

$$\alpha T_B \not\leq_{\text{fin}} \text{Re}(F_q) \quad \text{for any } \alpha > 0;$$

*i.e., the operator  $\text{Re}(F_q) - \alpha T_B$  has infinitely many negative eigenvalues for all  $\alpha > 0$ .*

*Proof.* From Theorem 3.2 with  $q_1 = 0$  and  $q_2 = q$  we obtain that there exists a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that

$$\text{Re}\left(\int_{S^{d-1}} g \overline{F_q g} \, ds\right) \geq k^2 \int_D q |u_g^i|^2 \, dx \quad \text{for all } g \in V^\perp.$$

Moreover, if  $B \subseteq D$  and  $\alpha \leq q_{\min}$ , then

$$\alpha \int_{S^{d-1}} g \overline{T_B g} \, ds = k^2 \int_B \alpha |u_g^i|^2 \, dx \leq k^2 \int_D q |u_g^i|^2 \, dx,$$

which shows part (a).

We prove part (b) by contradiction. Let  $B \not\subseteq D$ ,  $\alpha > 0$ , and assume that

$$(5.2) \quad \alpha T_B \leq_{\text{fin}} \text{Re}(F_q).$$

Using the monotonicity relation (3.3) in Corollary 3.4 with  $q_1 = 0$  and  $q_2 = q$ , we find that there exists a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that

$$(5.3) \quad \text{Re}\left(\int_{S^{d-1}} g \overline{F_q g} \, ds\right) \leq k^2 \int_D q |u_{q,g}|^2 \, dx \quad \text{for all } g \in V^\perp.$$

Combining (5.2) and (5.3), we obtain that there exists a finite dimensional subspace  $\tilde{V} \in L^2(S^{d-1})$  such that

$$k^2 \alpha \int_B |u_g^i|^2 \, dx \leq k^2 \int_D q |u_{q,g}|^2 \, dx \leq k^2 q_{\max} \int_D |u_{q,g}|^2 \, dx \quad \text{for all } g \in \tilde{V}^\perp.$$

Applying Theorem 4.5 with  $q_1 = 0$  and  $q_2 = q$ , this implies that there exists a constant  $C > 0$  such that

$$k^2 \alpha \int_B |u_g^i|^2 \, dx \leq C k^2 q_{\max} \int_D |u_g^i|^2 \, dx \quad \text{for all } g \in \tilde{V}^\perp.$$

However, this contradicts Theorem 4.1 with  $q = 0$ , which guarantees the existence of a sequence  $(g_m)_{m \in \mathbb{N}} \subseteq \tilde{V}^\perp$  with

$$\int_B |u_{g_m}^i|^2 dx \rightarrow \infty \quad \text{and} \quad \int_D |u_{g_m}^i|^2 dx \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Hence,  $\text{Re}(F_q) - \alpha T_B$  must have infinitely many negative eigenvalues.  $\square$

The next result is analogous to Theorem 5.2 but with contrast functions being negative on the support of the scattering objects, instead of being positive.

**THEOREM 5.2.** *Let  $B, D \subseteq \mathbb{R}^d$  be open and bounded such that  $\mathbb{R}^d \setminus \overline{D}$  is connected, and let  $q \in L_{0,+}^\infty(\mathbb{R}^d)$  with  $\text{supp}(q) = \overline{D}$ . Suppose that  $-1 < q_{\min} \leq q \leq q_{\max} \leq 0$  a.e. in  $D$  for some constants  $q_{\min}, q_{\max} \in \mathbb{R}$ .*

(a) *If  $B \subseteq D$ , then there exists a constant  $C > 0$  such that*

$$\alpha T_B \geq_{\text{fin}} \text{Re}(F_q) \quad \text{for all } \alpha \geq C q_{\max}.$$

(b) *If  $B \not\subseteq D$ , then*

$$\alpha T_B \not\geq_{\text{fin}} \text{Re}(F_q) \quad \text{for any } \alpha < 0;$$

*i.e., the operator  $\text{Re}(F_q) - \alpha T_B$  has infinitely many positive eigenvalues.*

*Proof.* If  $B \subseteq D$ , then Corollary 3.4 and Theorem 4.5 with  $q_1 = 0$  and  $q_2 = q$  show that there exists a constant  $C > 0$  and a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that, for any  $g \in V^\perp$ ,

$$\begin{aligned} \text{Re} \left( \int_{S^{d-1}} g \overline{F_q g} ds \right) &\leq k^2 \int_D q |u_{q,g}|^2 dx \\ &\leq k^2 q_{\max} \int_D |u_{q,g}|^2 dx \leq C k^2 q_{\max} \int_D |u_g^i|^2 dx. \end{aligned}$$

In particular,

$$\text{Re}(F_q) \leq_{\text{fin}} \alpha T_B \quad \text{for all } \alpha \geq C q_{\max},$$

and part (a) is proven.

We prove part (b) by contradiction. Let  $B \not\subseteq D$ ,  $\alpha < 0$ , and assume that

$$(5.4) \quad \alpha T_B \geq_{\text{fin}} \text{Re}(F_q).$$

Using the monotonicity relation (3.1) in Theorem 3.2 with  $q_1 = 0$  and  $q_2 = q$ , we find that there exists a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that

$$(5.5) \quad \text{Re} \left( \int_{S^{d-1}} g \overline{F_q g} ds \right) \geq k^2 \int_D q |u_g^i|^2 dx \quad \text{for all } g \in V^\perp.$$

Combining (5.4) and (5.5) shows that there exists a finite dimensional subspace  $\tilde{V} \in L^2(S^{d-1})$  such that

$$k^2 \alpha \int_B |u_g^i|^2 dx \geq k^2 \int_D q |u_g^i|^2 dx \geq k^2 q_{\min} \int_D |u_g^i|^2 dx \quad \text{for all } g \in \tilde{V}^\perp.$$

However, since  $\alpha < 0$ , this contradicts Theorem 4.1 with  $q = 0$ , which guarantees the existence of a sequence  $(g_m)_{m \in \mathbb{N}} \subseteq \tilde{V}^\perp$  such that

$$\alpha \int_B |u_{g_m}^i|^2 dx \rightarrow -\infty \quad \text{and} \quad \int_D |u_{g_m}^i|^2 dx \rightarrow 0.$$

Hence,  $\text{Re}(F_q) - \alpha T_B$  must have infinitely many positive eigenvalues for all  $\alpha < 0$ .  $\square$

Next we consider the general case; i.e., the contrast function  $q$  is no longer required to be either positive or negative a.e. on the support of all scattering objects. While in the *sign definite case* the criteria developed in Theorems 5.1 and 5.2 determine whether a certain *probing domain*  $B$  is contained in the support  $D$  of the scattering objects or not, the criterion for the *indefinite case* established in Theorem 5.3 below characterizes whether a certain probing domain  $B$  contains the support  $D$  of the scattering objects or not.

**THEOREM 5.3.** *Let  $B, D \subseteq \mathbb{R}^d$  be open and bounded such that  $\mathbb{R}^d \setminus \overline{D}$  is connected, and let  $q \in L^\infty_{0,+}(\mathbb{R}^d)$  with  $\text{supp}(q) = \overline{D}$ . Suppose that  $-1 < q_{\min} \leq q \leq q_{\max} < \infty$  a.e. on  $D$  for some constants  $q_{\min}, q_{\max} \in \mathbb{R}$ .*

*Furthermore, we assume that for any point  $x \in \partial D$  on the boundary of  $D$ , and for any neighborhood  $U \subseteq D$  of  $x$  in  $D$ , there exists an unbounded neighborhood  $O \subseteq \mathbb{R}^d$  of  $x$  with  $O \cap D \subseteq U$ , and an open subset  $E \subseteq O \cap D$ , such that<sup>3</sup>*

$$(5.6) \quad q|_O \geq 0 \text{ and } q|_E \geq q_{\min,E} > 0 \quad \text{or} \quad q|_O \leq 0 \text{ and } q|_E \leq q_{\max,E} < 0$$

for some constants  $q_{\min,E}, q_{\max,E} \in \mathbb{R}$ .

(a) *If  $D \subseteq B$ , then there exists a constant  $C > 0$  such that*

$$\alpha T_B \leq_{\text{fin}} \text{Re}(F_q) \leq_{\text{fin}} \beta T_B \quad \text{for all } \alpha \leq \min\{0, q_{\min}\}, \beta \geq \max\{0, Cq_{\max}\}.$$

(b) *If  $D \not\subseteq B$ , then*

$$\alpha T_B \not\leq_{\text{fin}} \text{Re}(F_q) \quad \text{for any } \alpha \in \mathbb{R} \quad \text{or} \quad \text{Re}(F_q) \not\leq_{\text{fin}} \beta T_B \quad \text{for any } \beta \in \mathbb{R}.$$

*Remark 5.4.* The *local definiteness property* (5.6) in Theorem 5.3 is, e.g., always satisfied, if the contrast function is piecewise analytic (see Appendix A of [26]) or if the supports of the positive part and of the negative part of the contrast function are well-separated from each other.  $\square$

*Proof of Theorem 5.3.* If  $D \subseteq B$ , then Corollary 3.4 and Theorem 4.5 with  $q_1 = 0$  and  $q_2 = q$  show that there exists a constant  $C > 0$  and a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that, for all  $g \in V^\perp$  and any  $\beta \geq \max\{0, Cq_{\max}\}$ ,

$$\begin{aligned} \text{Re}\left(\int_{S^{d-1}} g \overline{F_q g} \, ds\right) &\leq k^2 \int_D q |u_{q,g}|^2 \, dx \leq k^2 q_{\max} \int_D |u_{q,g}|^2 \, dx \\ &\leq k^2 C q_{\max} \int_D |u_g^i|^2 \, dx \leq k^2 \beta \int_B |u_g^i|^2 \, dx. \end{aligned}$$

Similarly, Theorem 3.2 with  $q_1 = 0$  and  $q_2 = q$  shows that there exists a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that, for all  $g \in V^\perp$  and any  $\alpha \leq \min\{0, q_{\min}\}$ ,

$$\text{Re}\left(\int_{S^{d-1}} g \overline{F_q g} \, ds\right) \geq k^2 \int_D q |u_g^i|^2 \, dx \geq k^2 q_{\min} \int_D |u_g^i|^2 \, dx \geq k^2 \alpha \int_B |u_g^i|^2 \, dx,$$

and part (a) is proven.

We prove part (b) by contradiction. Since  $D \not\subseteq B$ ,  $U := D \setminus B$  is not empty, and there exists  $x \in \overline{U} \cap \partial D$  as well as an unbounded open neighborhood  $O \subseteq \mathbb{R}^d$  of  $x$  with  $O \cap D \subseteq U$ , and an open subset  $E \subseteq O \cap D$  such that (5.6) is satisfied. Furthermore, let  $R > 0$  be large enough such that  $B, D \subseteq B_R(0)$ .

<sup>3</sup>As usual, the inequalities in (5.6) are to be understood pointwise a.e.

We first assume that  $q|_O \geq 0$  and  $q|_B \geq q_{\min,E} > 0$  and that  $\operatorname{Re}(F_q) \leq_{\text{fin}} \beta T_B$  for some  $\beta \in \mathbb{R}$ . Using the monotonicity relation (3.1) in Theorem 3.2 with  $q_1 = 0$  and  $q_2 = q$ , we find that there exists a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that, for any  $g \in V^\perp$ ,

$$\begin{aligned} 0 &\geq \int_{S^{d-1}} g \overline{(\operatorname{Re}(F_q)g - \beta T_B g)} \, ds \geq k^2 \int_{B_R(0)} (q - \beta \chi_B) |u_g^i|^2 \, dx \\ &= k^2 \int_{B_R(0) \setminus \overline{O}} (q - \beta \chi_B) |u_g^i|^2 \, dx + k^2 \int_{B_R(0) \cap O} (q - \beta \chi_B) |u_g^i|^2 \, dx \\ &\geq -k^2 (\|q\|_{L^\infty(\mathbb{R}^d)} + |\beta|) \int_{B_R(0) \setminus \overline{O}} |u_g^i|^2 \, dx + k^2 q_{\min,E} \int_E |u_g^i|^2 \, dx. \end{aligned}$$

However, this contradicts Theorem 4.1 with  $B = E$ ,  $D = B_R(0) \setminus \overline{O}$ , and  $q = 0$ , which guarantees the existence of a sequence  $(g_m)_{m \in \mathbb{N}} \subseteq V^\perp$  with

$$\int_E |u_{g_m}^i|^2 \, dx \rightarrow \infty \quad \text{and} \quad \int_{B_R(0) \setminus \overline{O}} |u_{g_m}^i|^2 \, dx \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Consequently,  $\operatorname{Re}(F_q) \not\leq_{\text{fin}} \beta T_B$  for all  $\beta \in \mathbb{R}$ .

On the other hand, if  $q|_O \leq 0$  and  $q|_E \leq q_{\max,E} < 0$ , and if  $\alpha T_B \leq_{\text{fin}} \operatorname{Re}(F_q)$  for some  $\alpha \in \mathbb{R}$ , then the monotonicity relation (3.3) in Corollary 3.4 with  $q_1 = 0$  and  $q_2 = q$  shows that there exists a finite dimensional subspace  $V \subseteq L^2(S^{d-1})$  such that, for any  $g \in V^\perp$ ,

$$\begin{aligned} 0 &\leq \int_{S^{d-1}} g \overline{(\operatorname{Re}(F_q)g - \alpha T_B g)} \, ds \leq k^2 \int_{B_R(0)} (q |u_{q,g}|^2 - \alpha \chi_B |u_g^i|^2) \, dx \\ &= k^2 \int_{B_R(0) \setminus \overline{O}} (q |u_{q,g}|^2 - \alpha \chi_B |u_g^i|^2) \, dx + k^2 \int_{B_R(0) \cap O} (q |u_{q,g}|^2 - \alpha \chi_B |u_g^i|^2) \, dx \\ &\leq k^2 q_{\max} \int_{B_R(0) \setminus \overline{O}} |u_{q,g}|^2 \, dx + k^2 |\alpha| \int_{B_R(0) \setminus \overline{O}} |u_g^i|^2 \, dx + k^2 q_{\max,E} \int_E |u_{q,g}|^2 \, dx. \end{aligned}$$

Applying Theorem 4.5 with  $D = B_R(0) \setminus \overline{O}$ ,  $q_1 = 0$ , and  $q_2 = q$  we find that there exists a constant  $C > 0$  such that

$$0 \leq k^2 (C q_{\max} + |\alpha|) \int_{B_R(0) \setminus \overline{O}} |u_g^i|^2 \, dx + k^2 C q_{\max,E} \int_E |u_g^i|^2 \, dx.$$

However, since  $q_{\max,E} < 0$ , this contradicts Theorem 4.1 with  $B = E$ ,  $D = B_R(0) \setminus \overline{O}$ , and  $q = 0$ , which guarantees the existence of a sequence  $(g_m)_{m \in \mathbb{N}} \subseteq V^\perp$  with

$$\int_E |u_{g_m}^i|^2 \, dx \rightarrow \infty \quad \text{and} \quad \int_{B_R(0) \setminus \overline{O}} |u_{g_m}^i|^2 \, dx \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Consequently,  $\alpha T_B \not\leq_{\text{fin}} \operatorname{Re}(F_q)$  for all  $\alpha \in \mathbb{R}$ , which ends the proof of part (b). □

**6. Numerical examples.** In the following we discuss two numerical examples for the two-dimensional sign definite case to illustrate the theoretical results developed in Theorems 5.1 and 5.2. The algorithm suggested below is preliminary and does not immediately extend to the indefinite case considered in Theorem 5.3.

We assume that far field observations  $u^\infty(\hat{x}_l; \theta_m)$  are available for  $N$  equidistant observation and incident directions

$$(6.1) \quad \hat{x}_l, \theta_m \in \{(\cos \phi_n, \sin \phi_n) \in S^1 \mid \phi_n = (n - 1)2\pi/N, n = 0, \dots, N - 1\},$$

$1 \leq l, m \leq N$ . Accordingly, the matrix

$$(6.2) \quad \mathbf{F}_q = \frac{2\pi}{N} [u^\infty(\hat{x}_l; \theta_m)]_{1 \leq l, m \leq N} \in \mathbb{C}^{N \times N}$$

approximates the far field operator  $F_q$  from (2.6). If the support of the contrast function  $q$ , i.e., of the scattering objects, is contained in the ball  $B_R(0)$  for some  $R > 0$ , then it is appropriate to choose

$$(6.3) \quad N \gtrsim 2kR,$$

where as before  $k$  denotes the wave number, to fully resolve the relevant information contained in the far field patterns (see, e.g., [15]).

We consider an equidistant grid on the region of interest

$$(6.4) \quad [-R, R]^2 = \bigcup_{j=1}^J P_j, \quad R > 0,$$

with quadratic pixels  $P_j = z_j + [-\frac{h}{2}, \frac{h}{2}]^2$ ,  $1 \leq j \leq J$ , where  $z_j \in \mathbb{R}^2$  denotes the center of  $P_j$  and  $h$  is its side length. In this case a short computation shows that for each pixel  $P_j$  the operator  $T_{P_j}$  from (5.1) is approximated by the matrix

$$\mathbf{T}_{P_j} = \frac{2\pi}{N} \left[ (kh)^2 e^{ikz_j \cdot (\theta_m - \theta_l)} \operatorname{sinc}\left(\frac{kh}{2}(\theta_m - \theta_l)_1\right) \operatorname{sinc}\left(\frac{kh}{2}(\theta_m - \theta_l)_2\right) \right]_{1 \leq l, m \leq N} \in \mathbb{C}^{N \times N}.$$

Therewith, we compute the eigenvalues  $\lambda_1^{(j)}, \dots, \lambda_N^{(j)} \in \mathbb{R}$  of the self-adjoint matrix

$$(6.5) \quad \mathbf{A}_{P_j} = \operatorname{sign}(q)(\operatorname{Re}(\mathbf{F}_q) - \alpha \mathbf{T}_{P_j}), \quad 1 \leq j \leq J.$$

For numerical stabilization, we discard those eigenvalues whose absolute values are smaller than some threshold. This number depends on the quality of the data. If there are good reasons to believe that  $\mathbf{A}_{P_j}$  is known up to a perturbation of size  $\delta > 0$  (with respect to the spectral norm), then we can only trust in those eigenvalues with magnitude larger than  $\delta$  (see, e.g., [14, Thm. 7.2.2]). To obtain a reasonable estimate for  $\delta$ , we use the magnitude of the nonunitary part of  $\mathbf{S}_q := (\mathbf{I}_N + i/(4\pi)\mathbf{F}_q)$ ; i.e. we take  $\delta = \|\mathbf{S}_q^* \mathbf{S}_q - \mathbf{I}_N\|_2$ , since this quantity should be zero for exact data and be of the order of the data error, otherwise.

Assuming that the contrast function  $q$  is either larger or smaller than zero a.e. in  $\operatorname{supp}(q)$ , and that the parameter  $\alpha \in \mathbb{R}$  satisfies the conditions in part (a) of Theorems 5.1 or 5.2, respectively, we then simply count for each pixel  $P_j$  the number of negative eigenvalues of  $\mathbf{A}_{P_j}$ , and define the *indicator function*  $I_\alpha : [-R, R]^2 \rightarrow \mathbb{N}$ ,

$$(6.6) \quad I_\alpha(x) = \#\{\lambda_n^{(j)} \mid \lambda_n^{(j)} < -\delta, 1 \leq n \leq N\} \quad \text{if } x \in P_j.$$

Theorems 5.1 and 5.2 suggest that  $I_\alpha$  is larger on pixels  $P_j$  that do not intersect the support  $\operatorname{supp}(q)$  of the scattering object than on pixels  $P_j$  contained in  $\operatorname{supp}(q)$ .

*Example 6.1.* We consider two penetrable scatterers, a kite and an ellipse, with positive constant contrast functions  $q = 1$  (kite) and  $q = 2$  (ellipse) as sketched in Figure 6.1 (dashed lines), and simulate the corresponding far field matrix  $\mathbf{F}_q \in \mathbb{C}^{64 \times 64}$  for  $N = 64$  observation and incident directions as in (6.1) using a Nyström method for a boundary integral formulation of the scattering problem with three different wave numbers  $k = 1, 2$ , and  $k = 5$ .

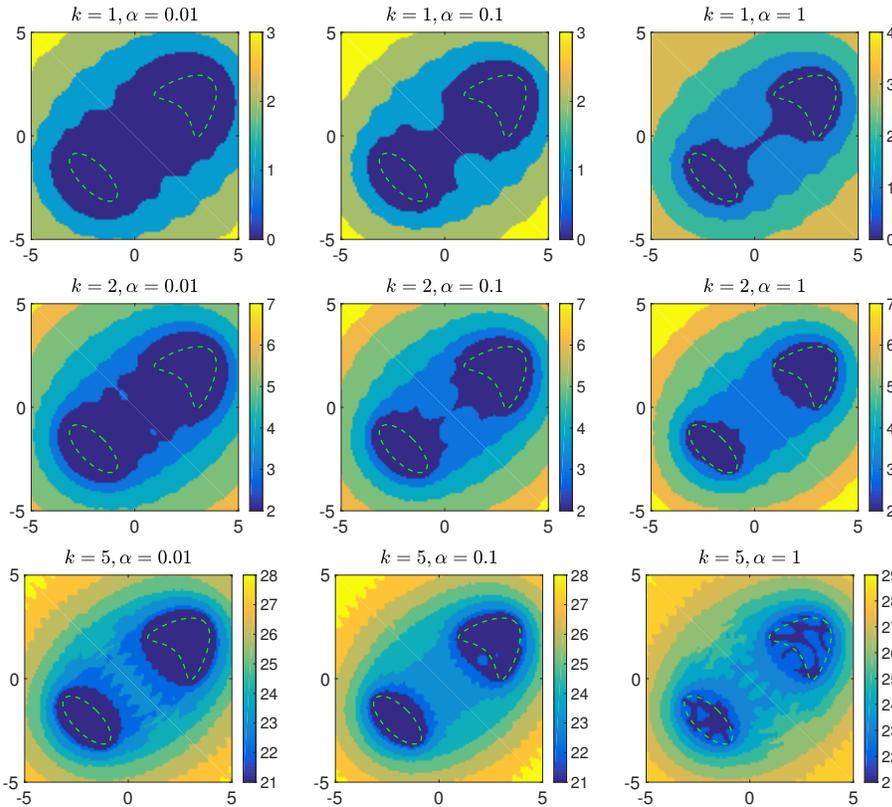


FIG. 6.1. Scatterers with positive contrast functions: Visualization of the indicator function  $I_\alpha$  for three different parameters  $\alpha = 0.01, 0.1$ , and  $\alpha = 1$  (left to right) and three different wave numbers  $k = 1, 2$ , and  $k = 5$  (top down). Exact shape of the scatterers is shown as dashed lines.

In Figure 6.1, we show color coded plots of the indicator function  $I_\alpha$  from (6.6) with threshold parameter  $\delta = 10^{-14}$  (i.e., the number of negative eigenvalues smaller than  $-\delta = -10^{-14}$  of the matrix  $\mathbf{A}_{P_j}$  from (6.5) on each pixel  $P_j$ ) in the region of interest  $[-5, 5]^2 \subseteq \mathbb{R}^2$  for three different parameters  $\alpha = 0.01, 0.1$ , and  $\alpha = 1$  (left to right) and three different wave numbers  $k = 1, 2$ , and  $k = 5$  (top down). The equidistant rectangular sampling grid on the region of interest from (6.4) consists of 100 pixels in each direction.

Overall, the number of negative eigenvalues of the matrix  $\mathbf{A}_{P_j}$  increases with increasing wave number, and it is larger on pixels  $P_j$  sufficiently far away from the support of the scatterers than on pixels  $P_j$  inside, as suggested by Theorems 5.1 and 5.2. The lower value always coincides with the number of negative eigenvalues of the real part  $\text{Re}(\mathbf{F}_q)$  of the far field matrix from (6.2) that are smaller than the threshold  $-\delta$ . The number of eigenvalues of  $\mathbf{A}_{P_j}$ ,  $j = 1, \dots, J$ , whose absolute values are larger than  $\delta$  is approximately (on average) 25 (for  $k = 1$ ), 36 (for  $k = 2$ ), and 62 (for  $k = 5$ ), independent of  $\alpha$ .

If the parameter  $\alpha$  is suitably chosen, depending on the wave number, then the lowest level set of the indicator function  $I_\alpha$  nicely approximates the support of the two scatterers.  $\square$

*Example 6.2.* In the second example, we consider three penetrable scatterers, a kite, an ellipse and a nut-shaped scatterer, with negative constant contrasts  $q = -0.8$

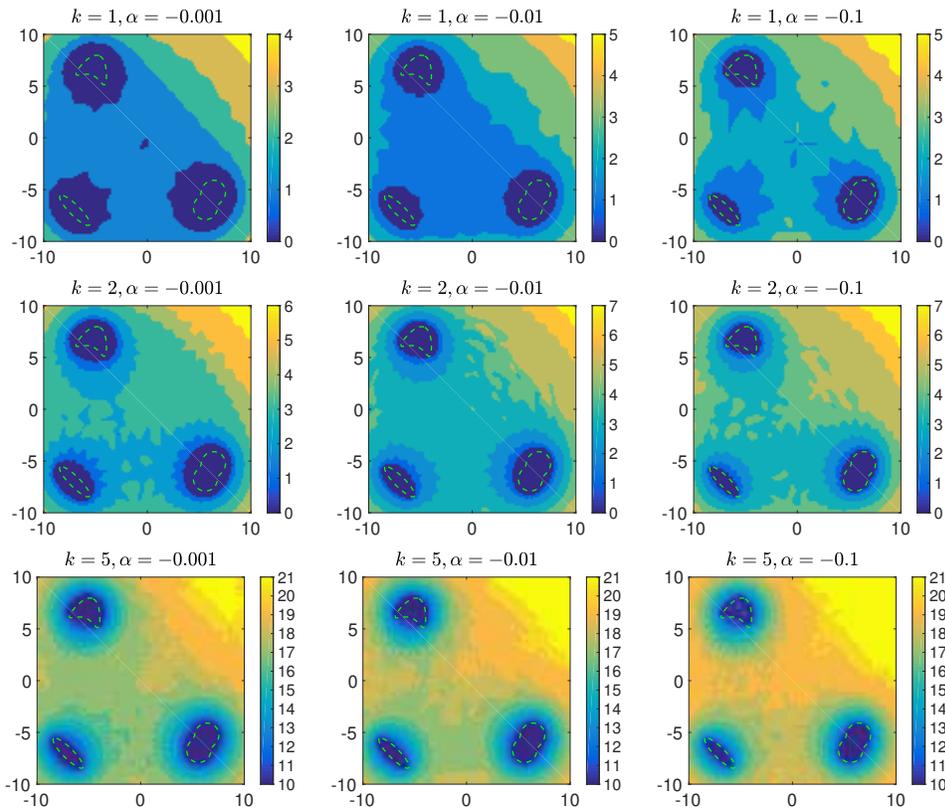


FIG. 6.2. Scatterers with negative contrast functions: Visualization of the indicator function  $I_\alpha$  for three different parameters  $\alpha = -0.001, -0.01$ , and  $\alpha = -0.1$  (left to right) and three different wave numbers  $k = 1, 2$ , and  $k = 5$  (top down). Exact shape of the scatterers is shown as dashed lines.

(kite),  $q = -0.4$  (nut), and  $q = -0.2$  (ellipse) as sketched in Figure 6.2 (dashed lines) and simulate the corresponding far field matrix  $\mathbf{F}_q \in \mathbb{C}^{128 \times 128}$  for  $N = 128$  observation and incident directions for three different wave numbers  $k = 1, 2$ , and  $k = 5$ . We increase the number of discretization points because the diameter of the support of this configuration of scattering objects is roughly twice as large as in the previous example (i.e., to fulfill the sampling condition (6.3)).

In Figure 6.1, we show color coded plots of the indicator function  $I_\alpha$  from (6.6) with threshold parameter  $\delta = 10^{-14}$  in the region of interest  $[-10, 10]^2 \subseteq \mathbb{R}^2$  for three different parameters  $\alpha = -0.001, -0.01$ , and  $\alpha = -0.1$  (left to right) and three different wave numbers  $k = 1, 2$ , and  $k = 5$  (top down). The equidistant rectangular sampling grid on the region of interest from (6.4) on this region of interest consists of 100 pixels in each direction.

Again, the number of negative eigenvalues of the matrix  $\mathbf{A}_{P_j}$  increases with increasing wave number, and it is larger on pixels  $P_j$  sufficiently far away from the support of the scatterers than on pixels  $P_j$  inside, in compliance with Theorems 5.1 and 5.2. The lower value always coincides with the number of positive eigenvalues of the matrix  $\text{Re}(\mathbf{F}_q)$  from (6.2) that are larger than the threshold  $\delta = 10^{-14}$ . The number of eigenvalues of  $\mathbf{A}_{P_j}$ ,  $j = 1, \dots, J$ , whose absolute values are larger than  $\delta$  is approximately (on average) 39 (for  $k = 1$ ), 60 (for  $k = 2$ ), and 115 (for  $k = 5$ ), independent of  $\alpha$ .

If the parameter  $\alpha$  is suitably chosen, depending on the wave number, then the support of the indicator function  $I_\alpha$  approximates the support of the three scatterers rather well.  $\square$

An efficient and suitably regularized numerical implementation of the theoretical results developed in Theorems 5.1–5.3 is beyond the scope of this article, and the preliminary algorithm discussed in this section cannot be considered competitive when compared against state-of-the-art implementations of linear sampling or factorization methods. The numerical results in the Examples 6.1 and 6.2 have been obtained for highly accurate simulated far field data. Further numerical tests showed that the algorithm is rather sensitive to noise in the data.

**Conclusions.** We have derived new monotonicity relations for the far field operator for the inverse medium scattering problem with compactly supported scattering objects, and we used them to provide novel monotonicity tests to determine the support of unknown scattering objects from far field observations of scattered waves corresponding to infinitely many plane wave incident fields. Along the way we have shown the existence of localized wave functions that have arbitrarily large norm in some prescribed region while having arbitrarily small norm in some other prescribed region.

When compared to traditional qualitative reconstructions methods, advantages of these new characterizations are that they apply to indefinite scattering configurations. Moreover, these characterizations are independent of transmission eigenvalues. However, although we presented some preliminary numerical examples for the sign definite case, a stable numerical implementation of these monotonicity tests still needs to be developed.

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