ERRATUM: MONOTONICITY IN INVERSE MEDIUM SCATTERING ON UNBOUNDED DOMAINS*

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Abstract. We correct a mistake in the proof of Theorem 5.3 in [R. Griesmaier and B. Harrach. SIAM J. Appl. Math., 78(5):2533–2557, 2018].

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1. An error in the proof of Theorem 5.3 in [3]. At the end of the proof of Theorem 5.3 in [3] "Applying Theorem 4.5 with $D = B_R(0) \setminus \overline{O}$, $q_1 = 0$, and $q_2 = q$ ..." is not possible, because the assumption of Theorem 4.5 in [3] that $q_1(x) = q_2(x)$ for a.e. $x \in \mathbb{R}^d \setminus \overline{D}$ is not satisfied for this choice of D, q_1 and q_2.

To fix this issue we will extend the results on localized wave functions from Section 4 of [3] in Section 2 below. Then, in Section 3 we will reformulate Theorem 5.3 of [3], making stronger assumptions on the domains and on the index of refraction, and we will correct the final argument in the original proof in [3].

2. Simultaneously localized wave functions. We establish the existence of simultaneously localized wave functions that have arbitrarily large norm on some prescribed region $E \subseteq \mathbb{R}^d$ while at the same time having arbitrarily small norm in a different region $M \subseteq \mathbb{R}^d$, assuming among others that $\mathbb{R}^d \setminus (\overline{E} \cup \overline{M})$ is connected. The result generalizes Theorem 4.1 in [3] in the sense that we not only control the total field but also the incident field. Similar results have recently been established for the Schrödinger equation in [4, Thm. 3.11] and for the Helmholtz obstacle scattering problem in [1, Thm. 4.5].

THEOREM 2.1. *Suppose that* $q \in L^\infty_{0,+}(\mathbb{R}^d)$*, and let* $E, M \subseteq \mathbb{R}^d$ *be open and Lipschitz bounded such that* $\operatorname{supp}(q) \subseteq \overline{E} \cup \overline{M}$*,* $\mathbb{R}^d \setminus (\overline{E} \cup \overline{M})$ *is connected, and* $E \cap M = \emptyset$*. Assume furthermore that there is a connected subset* $\Gamma \subseteq \partial E \setminus \overline{M}$ *that is relatively open and* $C^{1,1}$ *smooth.*

Then for any finite dimensional subspace $V \subseteq L^2(S^{d-1})$ *there exists a sequence* $(g_m)_{m\in\mathbb{N}} \subseteq V^\perp$ *such that*

$$\int_E |u_{q,g_m}|^2 \,\mathrm{d}x \to \infty \qquad \text{and} \qquad \int_M \left(|u_{q,g_m}|^2 + |u^i_{g_m}|^2\right) \mathrm{d}x \to 0 \qquad \text{as } m \to \infty\,,$$

where $u^i_{g_m}, u_{q,g_m} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ *are given by* (2.8a)–(2.8b) *in* [3] *with* $g = g_m$*.*

The proof of Theorem 2.1 relies on the following three lemmas.

LEMMA 2.2. *Suppose that* $q \in L^\infty_{0,+}(\mathbb{R}^d)$*, let* $n^2 = 1 + q$*, and assume that* $D \subseteq \mathbb{R}^d$ *is open and bounded. We define*

$$L_{q,D} : L^2(S^{d-1}) \to H^1(D)\,, \quad g \mapsto u_{q,g}|_D\,,$$

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where $u_{q,g} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ *is given by* (2.8b) *in* [3]*. Then* $L_{q,D}$ *is a linear operator and its adjoint is given by*

$$L^*_{q,D} : H^1(D)^* \to L^2(S^{d-1}) \,, \quad f \mapsto \mathcal{S}^*_q w^\infty \,,$$

where $H^1(D)^*$ *is the dual of* $H^1(D)$*,* $\mathcal{S}^*_q$ *denotes the adjoint of the scattering operator from* (2.7) *in* [3]*, and* $w^\infty \in L^2(S^{d-1})$ *is the far field pattern of the radiating solution* $w \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ *to*

$$\Delta w + k^2 n^2 w \,=\, -f \qquad \text{in } \mathbb{R}^d \,. \tag{2.1}$$

Proof. This follows from the same arguments that have been used in the proof of Lemma 4.2 in [3]. ∎

Lemma 2.3. *Suppose that* $q \in L^\infty_{0,+}(\mathbb{R}^d)$*, and let* $E, M \subseteq \mathbb{R}^d$ *be open and Lipschitz bounded such that* $\mathrm{supp}(q) \subseteq \overline{E} \cup \overline{M}$*,* $\mathbb{R}^d \setminus (\overline{E} \cup \overline{M})$ *is connected, and* $E \cap M = \emptyset$*. Assume furthermore that there is a connected subset* $\Gamma \subseteq \partial E \setminus \overline{M}$ *that is relatively open and* $C^{1,1}$ *smooth. Then,*

$$\mathcal{R}(L^*_{q,E}) \not\subseteq \mathcal{R}\big(\begin{pmatrix} L^*_{q,M} & L^*_{0,M} \end{pmatrix}\big)$$

and there exists an infinite dimensional subspace $Z \subseteq \mathcal{R}(L^*_{q,E})$ *such that*

$$Z \cap \mathcal{R}\big(\begin{pmatrix} L^*_{q,M} & L^*_{0,M} \end{pmatrix}\big) \,=\, \{0\} \,.$$

Proof. Let $h \in \mathcal{R}(L^*_{q,E}) \cap \mathcal{R}\big(\begin{pmatrix} L^*_{q,M} & L^*_{0,M} \end{pmatrix}\big)$. Then Lemma 2.2 shows that there exist $f_{q,E} \in H^1(E)^*$ and $f_{q,M}, f_{0,M} \in H^1(M)^*$ such that the far field patterns $w^\infty_{q,E}, w^\infty_{q,M}, w^\infty_{0,M}$ of the radiating solutions $w_{q,E}, w_{q,M}, w_{0,M} \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ to

$$\begin{aligned} \Delta w_{q,E} + k^2(1+q) w_{q,E} \,&=\, -f_{q,E} && \text{in } \mathbb{R}^d \,, \\ \Delta w_{q,M} + k^2(1+q) w_{q,M} \,&=\, -f_{q,M} && \text{in } \mathbb{R}^d \,, \\ \Delta w_{0,M} + k^2 w_{0,M} \,&=\, -f_{0,M} && \text{in } \mathbb{R}^d \,, \end{aligned}$$

satisfy

$$h \,=\, \mathcal{S}^*_q w^\infty_{q,E} \,=\, w^\infty_{0,M} + \mathcal{S}^*_q w^\infty_{q,M} \,.$$

Here we used that $\mathcal{S}_0$ is the identity operator. Accordingly, using the definition of the scattering operator in (2.7) of [3], we find that

$$\begin{aligned} 0 \,&=\, w^\infty_{q,E} - w^\infty_{q,M} - \mathcal{S}_q w^\infty_{0,M} \\ &=\, w^\infty_{q,E} - w^\infty_{q,M} - w^\infty_{0,M} - 2\mathrm{i}k|C_d|^2 F_q w^\infty_{0,M} \\ &=\, w^\infty_{q,E} - (w^\infty_{q,M} + w^\infty_{0,M} + v^\infty_q) \,, \end{aligned}$$

where v^∞_q is the far field of a radiating solution $v_q \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ to

$$\Delta v_q + k^2(1+q) v_q \,=\, 0 \qquad \text{in } \mathbb{R}^d \,.$$

Since $\mathrm{supp}(q) \subseteq \overline{E} \cup \overline{M}$ and $\mathbb{R}^d \setminus (\overline{E} \cup \overline{M})$ is connected, Rellich's lemma and unique continuation guarantee that

$$w_{q,E} - (w_{q,M} + w_{0,M} + v_q) \,=\, 0 \qquad \text{in } \mathbb{R}^d \setminus (\overline{E} \cup \overline{M}) \tag{2.2}$$

(cf., e.g., [2, Thm. 2.14]).

Next we discuss the regularity of the traces of $w_{q,E}$ and $w_{q,M} + w_{0,M} + v_q$ at the boundary segment $\Gamma \subseteq \partial E \setminus \overline{M}$. W.l.o.g. we may assume that Γ is bounded away from $\overline{M}$. Since $\operatorname{supp}(f_{q,M} + f_{0,M}) \subseteq \overline{M}$, interior regularity results (see, e.g., [7, Thm. 4.18]) show that $(w_{q,M} + w_{0,M} + v_q)\big|_\Gamma \in H^{\frac{3}{2}}(\Gamma)$. Thus (2.2) implies that $w_{q,E}\big|_\Gamma^+ \in H^{\frac{3}{2}}(\Gamma)$ as well.

On the other hand, let $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ be the closure of $\mathcal{D}(\Gamma)$ in $H^{\frac{1}{2}}(\Gamma)$ (see, e.g., [7, p. 99]). We will construct sources $f \in H^1(E)^*$ such that $L_{q,E}^* f \notin \mathcal{R}\big(\big(L_{q,M}^* \quad L_{0,M}^*\big)\big)$. Given any $g \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$, we denote by $\widetilde{g} \in H^{\frac{1}{2}}(\partial E)$ its extension to ∂E by zero. Accordingly, let $u^+ \in H^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{E})$ be the radiating solution to the exterior Dirichlet problem

$$\Delta u^+ + k^2 n^2 u^+ = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{E}\,, \qquad u^+ = \widetilde{g} \quad \text{on } \partial E\,. \tag{2.3}$$

Similarly, we define $u^- \in H^1(E)$ as the solution to the interior Dirichlet problem

$$\Delta u^- = 0 \quad \text{in } E\,, \qquad u^- = \widetilde{g} \quad \text{on } \partial E\,.$$

Therewith we introduce $u \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ by

$$u := \begin{cases} u^- & \text{in } E\,,\\ u^+ & \text{in } \mathbb{R}^d \setminus \overline{E}\,, \end{cases}$$

and $f \in H^1(E)^*$ by

$$f := -k^2 n^2 u^- - \gamma^* \Big(\frac{\partial u}{\partial \nu}\Big|_{\partial E}^+ - \frac{\partial u}{\partial \nu}\Big|_{\partial E}^- \Big)\,,$$

where $\gamma^* : H^{-\frac{1}{2}}(\partial E) \to H^1(E)^*$ denotes the adjoint of the interior trace operator $\gamma : H^1(E) \to H^{\frac{1}{2}}(\partial E)$. Then $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ (see, e.g., [8, Lmm. 5.3]), and

$$\Delta u + k^2 n^2 u = -f \quad \text{in } \mathbb{R}^d$$

(see, e.g., [7, Lmm. 6.9]). Accordingly, $L_{q,E}^* f = \mathcal{S}_q^* u^\infty$, where $u^\infty \in L^2(S^{d-1})$ coincides with the far field of the radiating solution u^+ to the exterior Dirichlet problem (2.3). If $\widetilde{g} \notin H^{\frac{3}{2}}(\partial E)$, then our regularity considerations above show that $L_{q,E}^* f \notin \mathcal{R}\big(\big(L_{q,M}^* \quad L_{0,M}^*\big)\big)$.

Now let $X \subseteq \widetilde{H}^{\frac{1}{2}}(\Gamma)$ be an infinite dimensional subspace of $\widetilde{H}^{\frac{1}{2}}(\Gamma)$ such that $X \cap H^{\frac{3}{2}}(\Gamma) = \{0\}$ (e.g., the subspace of piecewise linear functions on Γ that vanish on $\partial\Gamma$ as considered in the proof of Lemma 4.6 in [1]). Let $G_E : H^{\frac{1}{2}}(\Gamma) \to L^2(S^{d-1})$ be the operator that maps $g \in H^{\frac{1}{2}}(\Gamma)$ to the far field pattern of the radiating solution u^+ of (2.3), where $\widetilde{g} \in H^{\frac{1}{2}}(\partial E)$ is again the extension of g to ∂E by zero. Then G_E is one-to-one (see, e.g., [1, Thm. 3.2]), and thus $Z := \mathcal{S}_q^* G_E(X)$ is infinite dimensional. Furthermore, we have just shown that

$$Z \subseteq \mathcal{R}(L_{q,E}^*) \qquad \text{and} \qquad Z \cap \mathcal{R}\big(\big(L_{q,M}^* \quad L_{0,M}^*\big)\big) = \{0\}\,.$$

□

In the next lemma we quote a special case of Lemma 2.5 in [6].

LEMMA 2.4. *Let X, Y and Z be Hilbert spaces, and let $A : X \to Y$ and $B : X \to Z$ be bounded linear operators. Then,*

$$\exists C > 0 : \ \|Ax\| \leq C\|Bx\| \quad \forall x \in X \qquad \text{if and only if} \qquad \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)\,.$$

Now we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $V \subseteq L^2(S^{d-1})$ be a finite dimensional subspace. We denote by $P_V : L^2(S^{d-1}) \to L^2(S^{d-1})$ the orthogonal projection on V. Combining Lemma 2.3 with a simple dimensionality argument (see [5, Lmm. 4.7]) shows that

$$Z \not\subseteq \mathcal{R}\big(\big(L_{q,M}^* \quad L_{0,M}^*\big)\big) + V \ = \ \mathcal{R}(\big(L_{q,M}^* \quad L_{0,M}^* \quad P_V\big))\,,$$

where $Z \subseteq \mathcal{R}(L_{q,E}^*)$ denotes the subspace in Lemma 2.3. Thus,

$$\mathcal{R}(L_{q,E}^*) \not\subseteq \mathcal{R}\big(\big(L_{q,M}^* \quad L_{0,M}^*\big)\big) + V \ = \ \mathcal{R}(\big(L_{q,M}^* \quad L_{0,M}^* \quad P_V\big))\,,$$

and accordingly Lemma 2.4 implies that there is no constant $C > 0$ such that

$$\begin{aligned}\|L_{q,E}g\|_{L^2(E)}^2 \ &\leq \ C^2 \left\| \begin{pmatrix} L_{q,M} \\ L_{0,M} \\ P_V \end{pmatrix} g \right\|_{L^2(M)\times L^2(M)\times L^2(S^{d-1})}^2 \\ &= \ C^2 \big(\|L_{q,M}g\|_{L^2(M)}^2 + \|L_{0,M}g\|_{L^2(M)}^2 + \|P_V g\|_{L^2(S^{d-1})}^2 \big)\end{aligned}$$

for all $g \in L^2(S^{d-1})$. Hence, there exists as sequence $(\widetilde{g}_m)_{m\in\mathbb{N}} \subseteq L^2(S^{d-1})$ such that

$$\begin{aligned} \|L_{q,E}\widetilde{g}_m\|_{L^2(E)} &\to \infty\,, \\ \|L_{q,M}\widetilde{g}_m\|_{L^2(M)} + \|L_{0,M}\widetilde{g}_m\|_{L^2(M)} + \|P_V\widetilde{g}_m\|_{L^2(S^{d-1})} &\to 0 \qquad \text{as } m \to \infty\,. \end{aligned}$$

Setting $g_m := \widetilde{g}_m - P_V\widetilde{g}_m \in V^\perp \subseteq L^2(S^{d-1})$ for any $m \in \mathbb{N}$, we finally obtain

$$\|L_{q,E}g_m\|_{L^2(E)} \ \geq \ \|L_{q,E}\widetilde{g}_m\|_{L^2(E)} - \|L_{q,E}\|\|P_V\widetilde{g}_m\|_{L^2(S^{d-1})} \ \to \ \infty \qquad \text{as } m \to \infty\,,$$

and

$$\begin{aligned}\|L_{q,M}g_m\|_{L^2(M)} + \|L_{0,M}g_m\|_{L^2(M)} \ &\leq \ \|L_{q,M}\widetilde{g}_m\|_{L^2(M)} + \|L_{0,M}\widetilde{g}_m\|_{L^2(M)} \\ &\quad + (\|L_{q,M}\| + \|L_{0,M}\|)\|P_V\widetilde{g}_m\|_{L^2(S^{d-1})} \ \to \ 0 \qquad \text{as } m \to \infty\,.\end{aligned}$$

Since $L_{q,E}g_m = u_{q,g_m}|_E$, $L_{q,M}g_m = u_{q,g_m}|_M$, and $L_{0,M}g_m = u_{g_m}^i|_M$, this ends the proof. ∎

3. Correction of the statement and of the proof of Theorem 5.3 in [3].

THEOREM 3.1. *Let $B, D \subseteq \mathbb{R}^d$ be open and Lipschitz bounded such that ∂D is piecewise $C^{1,1}$ smooth, and $\mathbb{R}^d \setminus \overline{B}$ as well as $\mathbb{R}^d \setminus \overline{D}$ are connected. Let $q \in L_{0,+}^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(q) = \overline{D}$, and suppose that $-1 < q_{\min} \leq q \leq q_{\max} < \infty$ a.e. on D for some constants $q_{\min}, q_{\max} \in \mathbb{R}$.*

Furthermore, we assume that for any point $x \in \partial D$ on the boundary of D, there exists a connected unbounded neighborhood $O \subseteq \mathbb{R}^d$ of x such that, for $E := O \cap D$,

$$q|_E \geq q_{\min,E} > 0 \qquad \text{or} \qquad q|_E \leq q_{\max,E} < 0 \tag{3.1}$$

for some constants $q_{\min,E}, q_{\max,E} \in \mathbb{R}$.

(a) If $D \subseteq B$, then there exists a constant $C > 0$ such that

$$\alpha T_B \leq_{\text{fin}} \text{Re}(F_q) \leq_{\text{fin}} \beta T_B \quad \textit{for all } \alpha \leq \min\{0, q_{\min}\}\,,\ \beta \geq \max\{0, C q_{\max}\}\,.$$

(b) If $D \not\subseteq B$, then

$$\alpha T_B \not\leq_{\text{fin}} \text{Re}(F_q) \quad \textit{for any } \alpha \in \mathbb{R} \quad \textit{or} \quad \text{Re}(F_q) \not\leq_{\text{fin}} \beta T_B \quad \textit{for any } \beta \in \mathbb{R}\,.$$

REMARK 3.2. The assumptions on B and D as well as the *local definiteness assumption* (3.1) in Theorem 3.1 are stronger than in the original version of Theorem 5.3 in [3]. ◊

Proof of Theorem 3.1. If $D \subseteq B$, then Corollary 3.4 and Theorem 4.5 in [3] with $q_1 = 0$ and $q_2 = q$ show that there exists a constant $C > 0$ and a finite dimensional subspace $V \subseteq L^2(S^{d-1})$ such that, for all $g \in V^\perp$ and any $\beta \geq \max\{0, C q_{\max}\}$,

$$\begin{aligned}\text{Re}\Big(\int_{S^{d-1}} g\, \overline{F_q g}\, \mathrm{d}s\Big) &\leq k^2 \int_D q |u_{q,g}|^2 \,\mathrm{d}x \leq k^2 q_{\max} \int_D |u_{q,g}|^2 \,\mathrm{d}x \\ &\leq k^2 C q_{\max} \int_D |u_g^i|^2 \,\mathrm{d}x \leq k^2 \beta \int_B |u_g^i|^2 \,\mathrm{d}x\,.\end{aligned}$$

Similarly, Theorem 3.2 in [3] with $q_1 = 0$ and $q_2 = q$ shows that there exists a finite dimensional subspace $V \subseteq L^2(S^{d-1})$ such that, for all $g \in V^\perp$ and any $\alpha \leq \min\{0, q_{\min}\}$,

$$\text{Re}\Big(\int_{S^{d-1}} g\, \overline{F_q g}\, \mathrm{d}s\Big) \geq k^2 \int_D q |u_g^i|^2 \,\mathrm{d}x \geq k^2 q_{\min} \int_D |u_g^i|^2 \,\mathrm{d}x \geq k^2 \alpha \int_B |u_g^i|^2 \,\mathrm{d}x\,,$$

and part (a) is proven.

We prove part (b) by contradiction. Since $D \not\subseteq B$, $U := D \setminus B$ is not empty, and there exists $x \in \overline{U} \cap \partial D$ as well as a connected unbounded open neighborhood $O \subseteq \mathbb{R}^d$ of x with $O \cap D \subseteq U$ and $O \cap B = \emptyset$, such that (3.1) is satisfied with $E := O \cap D$. Furthermore, let $R > 0$ be large enough such that $B, D \subseteq B_R(0)$. Without loss of generality we assume that $O \cap B_R(0)$, and $B_R(0) \setminus \overline{O}$ are connected.

We first assume that $q|_E \geq q_{\min,E} > 0$, and that $\text{Re}(F_q) \leq_{\text{fin}} \beta T_B$ for some $\beta \in \mathbb{R}$. Using the monotonicity relation (3.1) in Theorem 3.2 of [3] with $q_1 = 0$ and $q_2 = q$, we find that there exists a finite dimensional subspace $V \subseteq L^2(S^{d-1})$ such that, for any $g \in V^\perp$,

$$\begin{aligned}0 &\geq \int_{S^{d-1}} g(\overline{\text{Re}(F_q)g - \beta T_B g})\,\mathrm{d}s \geq k^2 \int_{B_R(0)} (q - \beta\chi_B)|u_g^i|^2 \,\mathrm{d}x \\ &= k^2 \int_{B_R(0)\setminus\overline{O}} (q - \beta\chi_B)|u_g^i|^2 \,\mathrm{d}x + k^2 \int_{B_R(0)\cap O} (q - \beta\chi_B)|u_g^i|^2 \,\mathrm{d}x \\ &\geq -k^2(\|q\|_{L^\infty(\mathbb{R}^d)} + |\beta|) \int_{B_R(0)\setminus\overline{O}} |u_g^i|^2 \,\mathrm{d}x + k^2 q_{\min,E} \int_E |u_g^i|^2 \,\mathrm{d}x\,.\end{aligned}$$

However, this contradicts Theorem 4.1 in [3] with $B = E$, $D = B_R(0) \setminus \overline{O}$, and $q = 0$, which guarantees the existence of a sequence $(g_m)_{m\in\mathbb{N}} \subseteq V^\perp$ with

$$\int_E |u_{g_m}^i|^2 \,\mathrm{d}x \to \infty \qquad \text{and} \qquad \int_{B_R(0)\setminus\overline{O}} |u_{g_m}^i|^2 \,\mathrm{d}x \to 0 \qquad \text{as } m \to \infty\,.$$

Consequently, $\mathrm{Re}(F_q) \not\leq_{\mathrm{fin}} \beta T_B$ for all $\beta \in \mathbb{R}$.

On the other hand, if $q|_E \leq q_{\max,E} < 0$, and if $\alpha T_B \leq_{\mathrm{fin}} \mathrm{Re}(F_q)$ for some $\alpha \in \mathbb{R}$, then the monotonicity relation (3.3) in Corollary 3.4 of [3] with $q_1 = 0$ and $q_2 = q$ shows that there exists a finite dimensional subspace $V \subseteq L^2(S^{d-1})$ such that, for any $g \in V^\perp$,

$$\begin{aligned} 0 \;&\leq\; \int_{S^{d-1}} g\overline{(\mathrm{Re}(F_q)g - \alpha T_B g)}\,\mathrm{d}s \;\leq\; k^2 \int_{B_R(0)} (q|u_{q,g}|^2 - \alpha\chi_B|u_g^i|^2)\,\mathrm{d}x \\ &= k^2 \int_{B_R(0)\setminus\overline{O}} (q|u_{q,g}|^2 - \alpha\chi_B|u_g^i|^2)\,\mathrm{d}x + k^2 \int_{B_R(0)\cap O} (q|u_{q,g}|^2 - \alpha\chi_B|u_g^i|^2)\,\mathrm{d}x \\ &\leq k^2 q_{\max} \int_{B_R(0)\setminus\overline{O}} |u_{q,g}|^2\,\mathrm{d}x + k^2|\alpha| \int_{B_R(0)\setminus\overline{O}} |u_g^i|^2\,\mathrm{d}x + k^2 q_{\max,E} \int_E |u_{q,g}|^2\,\mathrm{d}x\,. \end{aligned}$$

Let $M := B_R(0) \setminus \overline{O}$. Since ∂D is piecewise $C^{1,1}$ smooth, there is a connected subset $\Gamma \subseteq \partial E \setminus \overline{M}$ that is relatively open and $C^{1,1}$ smooth. Applying Theorem 2.1 we find that there exists a sequence $(g_m)_{m\in\mathbb{N}} \subseteq V^\perp$ such that

$$\int_E |u_{q,g_m}|^2\,\mathrm{d}x \to \infty \qquad \text{and} \qquad \int_{B_R(0)\setminus\overline{O}} |u_{q,g_m}|^2 + |u_{g_m}^i|^2\,\mathrm{d}x \to 0 \qquad \text{as } m \to \infty\,.$$

However, since $q_{\max,E} < 0$, this gives a contradiction. Consequently, $\alpha T_B \not\leq_{\mathrm{fin}} \mathrm{Re}(F_q)$ for all $\alpha \in \mathbb{R}$, which ends the proof of part (b). ∎

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