ERRATUM: MONOTONICITY IN INVERSE MEDIUM SCATTERING ON UNBOUNDED DOMAINS

ROLAND GRIESMAIER† AND BASTIAN HARRACH‡


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1. An error in the proof of Theorem 5.3 in [3]. At the end of the proof of Theorem 5.3 in [3] “Applying Theorem 4.5 with \( D = B_R(0) \setminus \overline{\Omega} \), \( q_1 = 0 \), and \( q_2 = q \ldots \)” is not possible, because the assumption of Theorem 4.5 in [3] that \( q_1(x) = q_2(x) \) for a.e. \( x \in \mathbb{R}^d \setminus \overline{D} \) is not satisfied for this choice of \( D \), \( q_1 \) and \( q_2 \).

To fix this issue we will extend the results on localized wave functions from Section 4 of [3] in Section 2 below. Then, in Section 3 we will reformulate Theorem 5.3 of [3], making stronger assumptions on the domains and on the index of refraction, and we will correct the final argument in the original proof in [3].

2. Simultaneously localized wave functions. We establish the existence of simultaneously localized wave functions that have arbitrarily large norm on some prescribed region \( E \subseteq \mathbb{R}^d \) while at the same time having arbitrarily small norm in a different region \( M \subseteq \mathbb{R}^d \), assuming among others that \( \mathbb{R}^d \setminus (E \cup M) \) is connected. The result generalizes Theorem 4.1 in [3] in the sense that we not only control the total field but also the incident field. Similar results have recently been established for the Schrödinger equation in [4, Thm. 3.11] and for the Helmholtz obstacle scattering problem in [1, Thm. 4.5].

Theorem 2.1. Suppose that \( q \in L^\infty_0(\mathbb{R}^d) \), and let \( E, M \subseteq \mathbb{R}^d \) be open and Lipschitz bounded such that \( \text{supp}(q) \subseteq E \cup M \), \( \mathbb{R}^d \setminus (E \cup M) \) is connected, and \( E \cap M = \emptyset \). Assume furthermore that there is a connected subset \( \Gamma \subseteq \partial E \setminus M \) that is relatively open and \( C^{1,1} \) smooth.

Then for any finite dimensional subspace \( V \subseteq L^2(S^{d-1}) \) there exists a sequence \( (g_m)_{m \in \mathbb{N}} \subseteq V^\perp \) such that

\[
\int_E |u_{q,g_m}|^2 \, dx \to \infty \quad \text{and} \quad \int_M (|u_{q,g_m}|^2 + |u^i_{g_m}|^2) \, dx \to 0 \quad \text{as } m \to \infty ,
\]

where \( u^i_{g_m}, u_{q,g_m} \in H^{1}_{\text{loc}}(\mathbb{R}^d) \) are given by (2.8a)–(2.8b) in [3] with \( g = g_m \).

The proof of Theorem 2.1 relies on the following three lemmas.

Lemma 2.2. Suppose that \( q \in L^\infty_0(\mathbb{R}^d) \), let \( n^2 = 1 + q \), and assume that \( D \subseteq \mathbb{R}^d \) is open and bounded. We define

\[
L_{q,D} : L^2(S^{d-1}) \to H^1(D) , \quad g \mapsto u_{q,g}|_D ,
\]

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†Institut für Angewandte und Numerische Mathematik, Karlsruher Institut für Technologie, 76049 Karlsruhe, Germany (roland.griesmaier@kit.edu).

‡Institut für Mathematik Universität Frankfurt, 60325 Frankfurt am Main, Germany (harrach@math.uni-frankfurt.de).
where $u_{q,g} \in H^1_{loc}(\mathbb{R}^d)$ is given by (2.8b) in [3]. Then $L_{q,D}$ is a linear operator and its adjoint is given by

$$L^*_{q,D} : H^1(D)^* \to L^2(S^{d-1}), \quad f \mapsto S^*_q w^\infty,$$

where $H^1(D)^*$ is the dual of $H^1(D)$, $S^*_q$ denotes the adjoint of the scattering operator from (2.7) in [3], and $w^\infty \in L^2(S^{d-1})$ is the far field pattern of the radiating solution $w \in H^1_{loc}(\mathbb{R}^d)$ to

$$\Delta w + k^2 n^2 w = -f \quad \text{in } \mathbb{R}^d.$$  

**Proof.** This follows from the same arguments that have been used in the proof of Lemma 4.2 in [3]. \qed

**Lemma 2.3.** Suppose that $q \in L^\infty_{loc}(\mathbb{R}^d)$, and let $E, M \subseteq \mathbb{R}^d$ be open and Lipschitz bounded such that supp$(q) \subseteq \overline{E \cup M}$, $\mathbb{R}^d \setminus (E \cup M)$ is connected, and $E \cap M = \emptyset$. Assume furthermore that there is a connected subset $\Gamma \subseteq \partial E \setminus \overline{M}$ that is relatively open and $C^{1,\alpha}$ smooth. Then,

$$\mathcal{R}(L^*_{q,E}) \not\subseteq \mathcal{R}((L^*_{q,M} L^*_{0,M}))$$

and there exists an infinite dimensional subspace $Z \subseteq \mathcal{R}(L^*_{q,E})$ such that

$$Z \cap \mathcal{R}((L^*_{q,M} L^*_{0,M})) = \{0\}.$$ 

**Proof.** Let $h \in \mathcal{R}(L^*_{q,E}) \cap \mathcal{R}((L^*_{q,M} L^*_{0,M}))$. Then Lemma 2.2 shows that there exist $f_{q,E} \in H^1(E)^*$ and $f_{q,M}, f_{0,M} \in H^1(M)^*$ such that the far field patterns $w_{q,E}, w_{q,M}, w_{0,M}$ of the radiating solutions $w_{q,E}, w_{q,M}, w_{0,M} \in H^1_{loc}(\mathbb{R}^d)$ to

$$\begin{align*}
\Delta w_{q,E} + k^2(1 + q) w_{q,E} &= -f_{q,E} \quad \text{in } \mathbb{R}^d, \\
\Delta w_{q,M} + k^2(1 + q) w_{q,M} &= -f_{q,M} \quad \text{in } \mathbb{R}^d, \\
\Delta w_{0,M} + k^2 w_{0,M} &= -f_{0,M} \quad \text{in } \mathbb{R}^d,
\end{align*}$$

satisfy

$$h = S^*_q w_{q,E} = w_{0,M} + S^*_q w_{q,M}.$$ 

Here we used that $S_0$ is the identity operator. Accordingly, using the definition of the scattering operator in (2.7) of [3], we find that

$$\begin{align*}
0 &= w_{q,E} - w_{q,M} - S_q w_{0,M} \\
&= w_{q,E} - w_{q,M} - w_{0,M} - 2i k |C_d|^2 F_q w_{0,M} \\
&= w_{q,E} - (w_{q,M}^\infty + w_{0,M}^\infty + v_q^\infty),
\end{align*}$$

where $v_q^\infty$ is the far field of a radiating solution $v_q \in H^1_{loc}(\mathbb{R}^d)$ to

$$\Delta v_q + k^2(1 + q) v_q = 0 \quad \text{in } \mathbb{R}^d.$$ 

Since supp$(q) \subseteq \overline{E \cup M}$ and $\mathbb{R}^d \setminus (E \cup M)$ is connected, Rellich’s lemma and unique continuation guarantee that

$$w_{q,E} - (w_{q,M} + w_{0,M} + v_q) = 0 \quad \text{in } \mathbb{R}^d \setminus (E \cup M).$$
(cf., e.g., [2, Thm. 2.14]).

Next we discuss the regularity of the traces of \( w_{q,E} \) and \( w_{q,M} + w_{0,M} + v_q \) at the boundary segment \( \Gamma \subseteq \partial E \setminus \overline{M} \). W.l.o.g. we may assume that \( \Gamma \) is bounded away from \( \overline{M} \). Since \( \text{supp}(f_{q,M} + f_{0,M}) \subseteq \overline{M} \), interior regularity results (see, e.g., [7, Thm. 4.18]) show that \( (w_{q,M} + w_{0,M} + v_q)_{|\Gamma} \in H^2(\Gamma) \). Thus (2.2) implies that \( w_{q,E}\big|_{\Gamma} \in H^2(\Gamma) \) as well.

On the other hand, let \( \tilde{H}^2(\Gamma) \) be the closure of \( \mathcal{D}(\Gamma) \) in \( H^2(\Gamma) \) (see, e.g., [7, p. 99]). We will construct sources \( (2.3) \) \( \Delta u^+ + k^2 n^2 u^+ = 0 \) in \( \mathbb{R}^d \setminus \overline{E} \), \( u^+ = \tilde{g} \) on \( \partial E \).

Similarly, we define \( u^- \in H^1(E) \) as the solution to the interior Dirichlet problem

\[
\Delta u^- = 0 \quad \text{in } E, \quad u^- = \tilde{g} \quad \text{on } \partial E.
\]

Therewith we introduce \( u \in L^2_{loc}(\mathbb{R}^d) \) by

\[
u := \begin{cases} u^- \text{ in } E, \\ u^+ \text{ in } \mathbb{R}^d \setminus \overline{E}, \end{cases}
\]

and \( f \in H^1(E)^* \) by

\[
f := -k^2 n^2 u^- - \gamma^* \left( \frac{\partial u^+}{\partial \nu} - \frac{\partial u^-}{\partial \nu} \right),
\]

where \( \gamma^* : H^{-\frac{1}{2}}(\partial E) \rightarrow H^1(E)^* \) denotes the adjoint of the interior trace operator \( \gamma : H^1(E) \rightarrow H^\frac{1}{2}(\partial E) \). Then \( u \in H^1_{loc}(\mathbb{R}^d) \) (see, e.g., [8, Lmm. 5.3]), and

\[
\Delta u + k^2 n^2 u = -f \quad \text{in } \mathbb{R}^d
\]

(see, e.g., [7, Lmm. 6.9]). Accordingly, \( L_{q,E}^* f = S_{q} u^\infty \), where \( u^\infty \in L^2(S^{d-1}) \) coincides with the far field of the radiating solution \( u^+ \) to the exterior Dirichlet problem (2.3). If \( \tilde{g} \notin H^\frac{1}{2}(\partial E) \), then our regularity considerations above show that \( L_{q,E}^* f \notin \mathcal{R}\left( (L_{q,M}^* L_{0,M}^*) \right) \).

Now let \( X \subseteq \tilde{H}^2(\Gamma) \) be an infinite dimensional subspace of \( \tilde{H}^2(\Gamma) \) such that \( X \cap \tilde{H}^2(\Gamma) = \{0\} \) (e.g., the subspace of piecewise linear functions on \( \Gamma \) that vanish on \( \partial \Gamma \) as considered in the proof of Lemma 4.6 in [1]). Let \( G_E : \tilde{H}^2(\Gamma) \rightarrow L^2(S^{d-1}) \) be the operator that maps \( g \in \tilde{H}^2(\Gamma) \) to the far field pattern of the radiating solution \( u^+ \) of (2.3), where \( \tilde{g} \in H^\frac{1}{2}(\partial E) \) is again the extension of \( g \) to \( \partial E \) by zero. Then \( G_E \) is one-to-one (see, e.g., [1, Thm. 3.2]), and thus \( Z := S_{q} G_E(X) \) is infinite dimensional.

Furthermore, we have just shown that

\[
Z \subseteq \mathcal{R}(L_{q,E}^*) \quad \text{and} \quad Z \cap \mathcal{R}( (L_{q,M}^* L_{0,M}^*) ) = \{0\}.
\]

In the next lemma we quote a special case of Lemma 2.5 in [6].
LEMMA 2.4. Let $X, Y$ and $Z$ be Hilbert spaces, and let $A : X \to Y$ and $B : X \to Z$ be bounded linear operators. Then,

$$\exists C > 0 : \|Ax\| \leq C\|Bx\| \quad \forall x \in X \quad \text{if and only if} \quad \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*).$$

Now we give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let $V \subseteq L^2(S^{d-1})$ be a finite dimensional subspace. We denote by $P_V : L^2(S^{d-1}) \to L^2(S^{d-1})$ the orthogonal projection on $V$. Combining Lemma 2.3 with a simple dimensionality argument (see [5, Lmm. 4.7]) shows that

$$Z \not\subseteq \mathcal{R}\left((L^*_q, M) + V = \mathcal{R}\left((L^*_q, M) P_V\right)\right),$$

where $Z \subseteq \mathcal{R}(L^*_q, E)$ denotes the subspace in Lemma 2.3. Thus,

$$\mathcal{R}(L^*_q, E) \not\subseteq \mathcal{R}\left((L^*_q, M) + V = \mathcal{R}\left((L^*_q, M) P_V\right)\right),$$

and accordingly Lemma 2.4 implies that there is no constant $C > 0$ such that

$$\|L_{q,E}g\|_{L^2(E)}^2 \leq C^2 \left\|\begin{pmatrix} L_{q,M} & 0 \\ 0 & L_{0,M} \\ P_V \end{pmatrix} g \right\|_{L^2(M) \times L^2(M) \times L^2(S^{d-1})}^2 = C^2 \left(\|L_{q,M}g\|_{L^2(M)}^2 + \|L_{0,M}g\|_{L^2(M)}^2 + \|P_Vg\|_{L^2(S^{d-1})}^2\right)$$

for all $g \in L^2(S^{d-1})$. Hence, there exists as sequence $(\tilde{g}_m)_{m\in\mathbb{N}} \subseteq L^2(S^{d-1})$ such that

$$\|L_{q,E}\tilde{g}_m\|_{L^2(E)} \to \infty,$$

$$\|L_{q,M}\tilde{g}_m\|_{L^2(M)} + \|L_{0,M}\tilde{g}_m\|_{L^2(M)} + \|P_V\tilde{g}_m\|_{L^2(S^{d-1})} \to 0 \quad \text{as} \quad m \to \infty.$$ 

Setting $g_m := \tilde{g}_m - P_V\tilde{g}_m \in V^\perp \subseteq L^2(S^{d-1})$ for any $m \in \mathbb{N}$, we finally obtain

$$\|L_{q,E}g_m\|_{L^2(E)} \geq \|L_{q,E}\tilde{g}_m\|_{L^2(E)} - \|L_{q,E}\||P_V\tilde{g}_m\|_{L^2(S^{d-1})} \to \infty \quad \text{as} \quad m \to \infty,$$

and

$$\|L_{q,M}g_m\|_{L^2(M)} + \|L_{0,M}g_m\|_{L^2(M)} \leq \|L_{q,M}\tilde{g}_m\|_{L^2(M)} + \|L_{0,M}\tilde{g}_m\|_{L^2(M)} + \|L_{q,E}\tilde{g}_m\|_{L^2(E)} - \|L_{q,E}\|\|P_V\tilde{g}_m\|_{L^2(S^{d-1})} \to 0 \quad \text{as} \quad m \to \infty.$$ 

Since $L_{q,E}g_m = u_{q,g_m}|_E$, $L_{q,M}g_m = u_{q,g_m}|_M$, and $L_{0,M}g_m = u^i_{g_m}|_M$, this ends the proof. 

3. **Correction of the statement and of the proof of Theorem 5.3 in [3].**

**Theorem 3.1.** Let $B, D \subseteq \mathbb{R}^d$ be open and Lipschitz bounded such that $\partial D$ is piecewise $C^{1,1}$ smooth, and $\mathbb{R}^d \setminus \overline{D}$ as well as $\mathbb{R}^d \setminus \overline{\partial D}$ are connected. Let $q \in L^\infty_0(\mathbb{R}^d)$ with $\text{supp}(q) = \overline{D}$, and suppose that $-1 < q_{\min} \leq q \leq q_{\max} < \infty$ a.e. on $D$ for some constants $q_{\min}, q_{\max} \in \mathbb{R}$.

Furthermore, we assume that for any point $x \in \partial D$ on the boundary of $D$, there exists a connected unbounded neighborhood $\Omega \subseteq \mathbb{R}^d$ of $x$ such that, for $E := \Omega \cap D$,

$$q|_E \geq q_{\min,E} > 0 \quad \text{or} \quad q|_E \leq q_{\max,E} < 0$$

for some constants $q_{\min,E}, q_{\max,E} \in \mathbb{R}$. 

(a) If $D \subseteq B$, then there exists a constant $C > 0$ such that
$$\alpha T_B \leq \sup \Re(F_q) \leq \inf \beta T_B \quad \text{for all } \alpha \leq \min\{0, q_{\min}\}, \beta \geq \max\{0, Cq_{\max}\}.$$ 

(b) If $D \nsubseteq B$, then
$$\alpha T_B \nleq \sup \Re(F_q) \quad \text{for any } \alpha \in \mathbb{R} \text{ or } \Re(F_q) \nleq \inf \beta T_B \quad \text{for any } \beta \in \mathbb{R}.$$ 

Remark 3.2. The assumptions on $B$ and $D$ as well as the local definiteness assumption (3.1) in Theorem 3.1 are stronger than in the original version of Theorem 5.3 in [3].

Proof of Theorem 3.1. If $D \subseteq B$, then Corollary 3.4 and Theorem 4.5 in [3] with $q_1 = 0$ and $q_2 = q$ show that there exists a constant $C > 0$ and a finite dimensional subspace $V \subseteq L^2(S^{d-1})$ such that, for all $g \in V^\perp$ and any $\beta \geq \max\{0, Cq_{\max}\}$,
$$\Re\left(\int_{S^{d-1}} g \overline{F_q g} \, ds\right) \leq k^2 \int_D q |u_{q,g}|^2 \, dx \leq k^2 q_{\max} \int_D |u_{q,g}|^2 \, dx \leq k^2 Cq_{\max} \int_D |u_{q,g}|^2 \, dx \leq k^2 \beta \int_B |u_{q,g}|^2 \, dx.$$ 

Similarly, Theorem 3.2 in [3] with $q_1 = 0$ and $q_2 = q$ shows that there exists a finite dimensional subspace $V \subseteq L^2(S^{d-1})$ such that, for all $g \in V^\perp$ and any $\alpha \leq \min\{0, q_{\min}\}$,
$$\Re\left(\int_{S^{d-1}} g \overline{F_q g} \, ds\right) \geq k^2 \int_D q |u_{q,g}|^2 \, dx \geq k^2 q_{\min} \int_D |u_{q,g}|^2 \, dx \geq k^2 \alpha \int_B |u_{q,g}|^2 \, dx,$$ 

and part (a) is proven.

We prove part (b) by contradiction. Since $D \nsubseteq B$, $U := D \setminus B$ is not empty, and there exists $x \in \overline{U} \cap \partial D$ as well as a connected unbounded open neighborhood $O \subseteq \mathbb{R}^d$ of $x$ with $O \cap D \subseteq U$ and $O \cap B = \emptyset$, such that (3.1) is satisfied with $E := O \cap D$. Furthermore, let $R > 0$ be large enough such that $B, D \subseteq B_R(0)$. Without loss of generality we assume that $O \cap B_R(0)$, and $B_R(0) \setminus \overline{O}$ are connected.

We first assume that $q |E| \geq q_{\min, E} > 0$, and that $\Re(F_q) \leq \inf \beta T_B$ for some $\beta \in \mathbb{R}$. Using the monotonicity relation (3.1) in Theorem 3.2 of [3] with $q_1 = 0$ and $q_2 = q$, we find that there exists a finite dimensional subspace $V \subseteq L^2(S^{d-1})$ such that, for any $g \in V^\perp$,
$$0 \geq \int_{S^{d-1}} g \Re(F_q |g - \beta T_B g|) \, ds \geq k^2 \int_{B_R(0)} (q - \beta \chi_B) |u_{q,g}|^2 \, dx \geq k^2 \int_{B_R(0) \setminus \overline{O}} (q - \beta \chi_B) |u_{q,g}|^2 \, dx \geq k^2 \int_{B_R(0) \cap \overline{O}} (q - \beta \chi_B) |u_{q,g}|^2 \, dx \geq k^2 |q|_{L^\infty(\mathbb{R}^d)} \int_{B_R(0) \setminus \overline{O}} |u_{q,g}|^2 \, dx + k^2 q_{\min, E} \int_E |u_{q,g}|^2 \, dx.$$ 

However, this contradicts Theorem 4.1 in [3] with $B = E$, $D = B_R(0) \setminus \overline{O}$, and $q = 0$, which guarantees the existence of a sequence $(g_m)_{m \in \mathbb{N}} \subseteq V^\perp$ with
$$\int_E |u_{g_m}|^2 \, dx \to \infty \quad \text{and} \quad \int_{B_R(0) \setminus \overline{O}} |u_{g_m}|^2 \, dx \to 0 \quad \text{as } m \to \infty.$$
Consequently, \( \Re(F_q) \not\leq_{\text{fin}} \beta T_B \) for all \( \beta \in \mathbb{R} \).

On the other hand, if \( q|E \leq q_{\text{max},E} < 0 \), and if \( \alpha T_B \leq_{\text{fin}} \Re(F_q) \) for some \( \alpha \in \mathbb{R} \), then the monotonicity relation (3.3) in Corollary 3.4 of [3] with \( q_1 = 0 \) and \( q_2 = q \) shows that there exists a finite dimensional subspace \( V \subseteq L^2(S^{d-1}) \) such that, for any \( g \in V \),

\[
0 \leq \int_{S^{d-1}} g (\Re(F_q)g - \alpha T_B g) \, ds \leq k^2 \int_{B_R(0)} (q|u_{q,g}|^2 - \alpha \chi_B|u_g|^2) \, dx \\
= k^2 \int_{B_R(0) \setminus \Omega} (q|u_{q,g}|^2 - \alpha \chi_B|u_g|^2) \, dx + k^2 \int_{B_R(0) \cap \Omega} (q|u_{q,g}|^2 - \alpha \chi_B|u_g|^2) \, dx \\
\leq k^2 q_{\text{max}} \int_{B_R(0) \setminus \Omega} |u_{q,g}|^2 \, dx + k^2 |\alpha| \int_{B_R(0) \cap \Omega} |u_g|^2 \, dx + k^2 q_{\text{max},E} \int_E |u_{q,g}|^2 \, dx.
\]

Let \( M := B_R(0) \setminus \Omega \). Since \( \partial D \) is piecewise \( C^{1,1} \) smooth, there is a connected subset \( \Gamma \subseteq \partial E \setminus M \) that is relatively open and \( C^{1,1} \) smooth. Applying Theorem 2.1 we find that there exists a sequence \((g_m)_{m \in \mathbb{N}} \subseteq V \) such that

\[
\int_E |u_{q,g_m}|^2 \, dx \to \infty \quad \text{and} \quad \int_{B_R(0) \setminus \Omega} |u_{q,g_m}|^2 + |u_g|^2 \, dx \to 0 \quad \text{as} \quad m \to \infty.
\]

However, since \( q_{\text{max},E} < 0 \), this gives a contradiction. Consequently, \( \alpha T_B \not\leq_{\text{fin}} \Re(F_q) \) for all \( \alpha \in \mathbb{R} \), which ends the proof of part (b). \( \square \)

REFERENCES