Inverse medium scattering for a nonlinear Helmholtz equation

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Abstract

We discuss a time-harmonic inverse scattering problem for a nonlinear Helmholtz equation with compactly supported inhomogeneous scattering objects that are described by a nonlinear refractive index in unbounded free space. Assuming the knowledge of a nonlinear far field operator, which maps Herglotz incident waves to the far field patterns of corresponding solutions of the nonlinear scattering problem, we show that the nonlinear index of refraction is uniquely determined. We also generalize two reconstruction methods, a factorization method and a monotonicity method, to recover the support of such nonlinear scattering objects. Numerical results illustrate our theoretical findings.

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1 Introduction

The linear Helmholtz equation is used to model the propagation of sound waves or electromagnetic waves of small amplitude in inhomogeneous isotropic media in the time-harmonic regime (see, e.g., [9]). However, if the magnitudes are large, then intensity-dependent material laws might be required, and nonlinear Helmholtz equations are often more appropriate. A prominent example are Kerr-type nonlinear media (see, e.g., [3, 34] for the physical background). Optical Kerr effects are studied in various applications from laser optics (see, e.g., [1, 6]) both from a theoretical and applied point of view. In this theoretical study we consider an inverse medium scattering problem for a class of nonlinear Helmholtz equations that covers for instance generalized Kerr-type nonlinear media of arbitrary order.

To begin with, we discuss the well-posedness of the direct scattering problem. We consider compactly supported scatterers that are described by a nonlinear refractive index, which we basically assume to be well approximated by a linear refractive index at low intensities. Rewriting the scattering problem in terms of a nonlinear Lippmann-Schwinger equation we use a contraction argument together with resolvent estimates for the linearized problem to establish the existence and uniqueness of solutions for incident waves that are sufficiently small relative to the size of the nonlinearity. However, since the parameters in nonlinear material laws are typically extremely small (see, e.g., [3, p. 212]), this assumption is usually not too restrictive. As a byproduct we also give a priori estimates for the solution of the nonlinear scattering problem as

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well as estimates for the linearization error, which are instrumental for the rest of the work. The main reason for considering incident waves that are small relative to the size of the nonlinearity here is that we later use linearization techniques to solve the corresponding inverse problem. However, we note that a more general existence result for the direct scattering problem that avoids any smallness assumption on the incident field has recently been established in [7] (see also [14, 33]).

We define a nonlinear far field operator that maps densities of Herglotz incident fields to the far field patterns of the corresponding solutions of the direct scattering problem. In the linear case such far field operators are used to describe the scattering process for infinitely many incident fields, and their properties have been widely studied (see, e.g., [9]). Similar to [32] (see also [28] for the linear case) we derive a factorization of this operator into three simpler operators. Here it is important to note that only the second operator in this factorization is nonlinear. We derive estimates for the corresponding linearization error.

Restricting the discussion to a class of generalized Kerr-type nonlinearities of arbitrary order, we then turn to the associated inverse scattering problem. We show that the knowledge of the nonlinear far field operator uniquely determines the nonlinear refractive index. This generalizes earlier results for the inverse medium scattering problem for nonlinear Helmholtz equations from [11, 23]. In comparison to these works we consider a less regular and more general class of nonlinear refractive indices. Our proof relies on linearization to determine the terms in the generalized Kerr-type nonlinearity recursively, and it uses the classical uniqueness result for the corresponding linear inverse medium scattering problem (see, e.g., [4, 35, 36, 37]). Recently, a uniqueness proof that avoids the use of the linear result has been established for a more regular class of power-type nonlinearities than considered here in [10, 31, 16]. Earlier uniqueness results for semilinear elliptic inverse problems have, e.g., been obtained in [20, 21, 22, 43]. Furthermore, inverse scattering problems for nonlinear Schrödinger equations, which are closely related to the nonlinear Helmholtz equations considered in this work, have been studied using different techniques than those applied in this work in [15, 38, 39, 40, 41, 42].

We also generalize two popular methods for shape reconstruction for inverse scattering problems, the factorization method and the monotonicity method, to the nonlinear scattering problem. A related factorization method has been discussed in [32] for a class of weakly scattering objects and for scattering objects with small nonlinearity of linear growth. In comparison to this work we consider a larger class of nonlinearities without any smallness assumption on the nonlinearity, but on the other hand we assume that the incident fields are sufficiently small relative to the size of the nonlinearity. For linear scattering problems the factorization method has originally been developed in [24, 25, 26] (see also [8] and the monographs [5, 28]). Using estimates for the linearization error we show that the inf-criterion from [26] can be extended to the nonlinear case considered in this work. However, since the far field operator is nonlinear, the efficient numerical implementation of this criterion using spectral theory that is used for the linear scattering problem no longer applies. Instead we have to solve a nonlinear constrained optimization problem for each sampling point to decide whether it belongs to the support of the nonlinear scatterer or not. This leads to a numerical scheme that is considerably more time consuming than the traditional scheme for the linear case.

The situation is similar for the nonlinear monotonicity method. For linear scattering problems monotonicity based reconstruction methods have been proposed in [2, 13, 17, 18] (see also [19, 44] for earlier contributions in electrical impedance tomography). Using linearization techniques we show that the method can be extended to the nonlinear case considered in this work. Again the tools from spectral theory that have been used for the numerical implementation of the monotonicity criteria in [2, 13] are not available for the nonlinear scattering problem. However, we show that there is a close connection between the nonlinear monotonicity based shape characterization and the inf-criterion for the nonlinear factorization method, which we exploit to implement the nonlinear monotoncity based reconstruction method in terms of a similar constrained optimization problem as for the nonlinear factorization method.

We consider a numerical example with a scattering object that is described by a third-order nonlinear refractive index using optical coefficients for glass from [3]. Since the nonlinear part of the refractive index is extremely small, we work with incident fields of very high intensity such that there is a significant nonlinear contribution in the scattered field. The forward solver, which is based on the same fixed point iteration for the nonlinear Lippmann-Schwinger equation that we use to analyze the direct scattering problem, as well as the reconstruction methods work well. This suggests that the smallness assumptions on the intensity of the incident fields that we have to make in our theoretical results is not too restrictive.

The article is organized as follows. In Section 2 we introduce the nonlinear scattering problem, and we establish existence and uniqueness of solutions for the direct scattering problem. In Section 3 we turn to the inverse scattering problem to recover the nonlinear refractive index from observations of the corresponding nonlinear far field operator. Focusing on a class of generalized Kerr-type nonlinearities, we show that this inverse problem has a unique solution. In Sections 4 and 5 we derive and analyze a nonlinear inf-criterion and a nonlinear monotonicity method for reconstructing the support of nonlinear scatterers. In Section 6 we provide numerical examples.

2 The nonlinear scattering problem

The nonlinear wave equation

$$\frac{\partial^2\psi}{\partial t^2}(t,x) - \Delta\psi(t,x) \;=\; h(x,\psi(t,x))\,, \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d\,,$$

is used to model the interaction of acoustic or electromagnetic waves with a compactly supported inhomogeneous penetrable scattering object with nonlinear response in d-dimensional free space for $d=2,3$. In the following we restrict the discussion to nonlinearities of the form

$$h(x,\psi(t,x)) \;=\; k^2 q(x,|\psi(t,x)|)\psi(t,x)\,, \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d\,,$$

where $q:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ is real-valued. Specifying a *wave number* $k>0$, the time-periodic ansatz

$$\psi(x,t) \;=\; e^{-\mathrm{i}kt}u(x)\,, \qquad (x,t)\in\mathbb{R}^d\times\mathbb{R}\,,$$

gives the nonlinear Helmholtz equation

$$\Delta u + k^2 u \;=\; -k^2 q(x,|u|)u\,, \qquad x\in\mathbb{R}^d\,.$$

Denoting by $n^2 := 1+q$ the associated nonlinear *refractive index*, we make the following general assumptions throughout this work.

Assumption 2.1. *The nonlinear contrast function $q\in L^\infty(\mathbb{R}^d\times\mathbb{R})$ shall satisfy*

(i) $z\in\mathbb{C}\mapsto q(\,\cdot\,,|z|)\in L^\infty(\mathbb{R}^d)$ is continuous,

(ii) $q(\,\cdot\,,|z|)=0$ a.e. in $\mathbb{R}^d\setminus D$ for some bounded open set $D\subset\mathbb{R}^d$ and all $z\in\mathbb{C}$,

(iii) and there exist $q_0\in L^\infty(\mathbb{R}^d)$ with $\operatorname{ess\,inf} q_0>-1$ in $\mathbb{R}^d$ and $q_0=0$ a.e. in $\mathbb{R}^d\setminus D$, and a parameter $\alpha>0$ such that for any $z_1,z_2\in\mathbb{C}$ with $|z_1|,|z_2|\le 1$,

$$\bigl\|q(\,\cdot\,,|z_1|)z_1 - q(\,\cdot\,,|z_2|)z_2 - q_0(z_1-z_2)\bigr\|_{L^\infty(\mathbb{R}^d)} \;\le\; C_q(|z_1|^\alpha+|z_2|^\alpha)|z_1-z_2|\,. \qquad (2.1)$$

For later reference we note that (2.1) implies

$$\big\| q(\,\cdot\,,|z|)z - q_0 z \big\|_{L^\infty(\mathbb{R}^d)} \;\leq\; C_q |z|^{1+\alpha} \qquad \text{for all } z \in \mathbb{C}\,,\; |z| \leq 1\,. \tag{2.2}$$

Example 2.2. An example for a nonlinear material law that satisfies Assumption 2.1 is the *generalized Kerr-type material law*

$$q(x,|z|) \;=\; q_0(x) + \sum_{l=1}^{L} q_l(x)|z|^{\alpha_l} \qquad x \in \mathbb{R}^d\,,\; z \in \mathbb{C}\,, \tag{2.3}$$

for $q_0, \dots, q_L \in L^\infty(\mathbb{R}^d)$ that vanish a.e. in $\mathbb{R}^d \setminus D$, where the lowest order term satisfies $\operatorname{ess\,inf} q_0 > -1$ in $\mathbb{R}^d$, and the exponents fulfill $0 < \alpha_1 < \cdots < \alpha_L < \infty$. Using Lemma A.1 in the appendix we find that, for any $z_1, z_2 \in \mathbb{C}$ with $|z_1|, |z_2| \leq 1$,

$$\begin{aligned}\big\| q(\,\cdot\,,|z_1|)z_1 - q(\,\cdot\,,|z_2|)z_2 - q_0(z_1 - z_2)\big\|_{L^\infty(\mathbb{R}^d)} \;&=\; \bigg\| \sum_{l=1}^{L} q_l \big(|z_1|^{\alpha_l} z_1 - |z_2|^{\alpha_l} z_2 \big) \bigg\|_{L^\infty(\mathbb{R}^d)} \\ &\leq\; 2^{1+\alpha_L} \sum_{l=1}^{L} \|q_l\|_{L^\infty(\mathbb{R}^d)} \left(\frac{|z_1|}{2} + \frac{|z_2|}{2} \right)^{\alpha_1} |z_1 - z_2|\,.\end{aligned}$$

Distinguishing the cases $0 < \alpha_1 < 1$ and $\alpha_1 \geq 1$ we obtain that

$$\left(\frac{|z_1|}{2} + \frac{|z_2|}{2} \right)^{\alpha_1} \;\leq\; 2^{-\min\{1,\alpha_1\}} \big(|z_1|^{\alpha_1} + |z_2|^{\alpha_1} \big)\,.$$

Accordingly, (2.1) is satisfied with $\alpha = \alpha_1$ and $C_q = 2^{1+\alpha_L - \min\{1,\alpha_1\}} \sum_{l=1}^{L} \|q_l\|_{L^\infty(\mathbb{R}^d)}$. For the special case when $L = 1$ and $\alpha_1 = 2$ this gives the well-known Kerr nonlinearity (see, e.g., [3, 34]). ♢

We suppose that the wave motion is caused by an *incident field* u^i satisfying the linear Helmholtz equation

$$\Delta u^i + k^2 u^i \;=\; 0 \qquad \text{in } \mathbb{R}^d\,. \tag{2.4a}$$

The scattering problem that we consider consists in determining the *total field* $u = u^i + u^s$ such that

$$\Delta u + k^2 n^2(\,\cdot\,,|u|)u \;=\; 0 \qquad \text{in } \mathbb{R}^d\,, \tag{2.4b}$$

where the *scattered field* u^s satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \Big(\frac{\partial u^s}{\partial r}(x) - \mathrm{i}k u^s(x) \Big) \;=\; 0\,, \qquad r = |x|\,, \tag{2.4c}$$

uniformly with respect to all directions $x/|x| \in S^{d-1}$.

Remark 2.3. Throughout this work (nonlinear) Helmholtz equations are to be understood in the strong sense. For instance, $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ is a solution to (2.4b) if and only if it satisfies the equation weakly almost everywhere in $\mathbb{R}^d$. Elliptic regularity results show that u^i is smooth throughout $\mathbb{R}^d$, and that u and thus also u^s are smooth in $\mathbb{R}^d \setminus \overline{D}$. In particular the radiation condition (2.4c) is well-defined. As usual we call a solution to a (nonlinear) Helmholtz equation on an unbounded domain that satisfies the Sommerfeld radiation condition a *radiating solution.* ♢

Next we show that the scattering problem (2.4) is equivalent to the problem of solving the *nonlinear Lippmann-Schwinger equation*

$$u(x) \;=\; u^i(x) + k^2 \int_D \Phi_k(x-y) q(y,|u(y)|) u(y) \,\mathrm{d}y\,, \qquad x \in D\,, \tag{2.5}$$

in $L^\infty(D)$. Here Φ_k is the outgoing free space fundamental solution to the Helmholtz equation, i.e., for $x, y \in \mathbb{R}^d$, $x \neq y$, we have $\Phi_k(x-y) = (\mathrm{i}/4)\, H_0^{(1)}(k|x-y|)$ if $d = 2$ and $\Phi_k(x-y) = e^{\mathrm{i}k|x-y|}/(4\pi|x-y|)$ if $d = 3$. The arguments that we use to prove this are the same as in the linear case (see, e.g., [27, Thm. 7.12]).

Lemma 2.4. *Let $q \in L^\infty(\mathbb{R}^d \times \mathbb{R})$ such that $z \in \mathbb{C} \mapsto q(\,\cdot\,,|z|) \in L^\infty(\mathbb{R}^d)$ is continuous and $q(\,\cdot\,,|z|) = 0$ a.e. in $\mathbb{R}^d \setminus D$ for some bounded open set $D \subset \mathbb{R}^d$ and all $z \in \mathbb{C}$. If $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ is a solution of (2.4), then $u|_D$ is a solution of (2.5). Conversely, if $u \in L^\infty(D)$ is a solution of (2.5) then u can be extended to a solution $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ of (2.4).*

Proof. Let $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ be a solution of (2.4). Then $q(\,\cdot\,,|u|)u|_D \in L^\infty(D)$, and the volume potential $v := k^2 \Phi_k * (q(\,\cdot\,,|u|)u) \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ is a radiating solution of

$$\Delta v + k^2 v \;=\; -k^2 q(\,\cdot\,,|u|)u \qquad \text{in } \mathbb{R}^d \tag{2.6}$$

(see, e.g., [27, Thm. 7.11]). Accordingly, $u^s - v$ is a radiating solution of $\Delta(u^s - v) + k^2(u^s - v) = 0$ in $\mathbb{R}^d$. Thus $v = u^s$ (see, e.g., [9, p. 24]), which proves the first part.

Conversely, let $u \in L^\infty(D)$ be a solution of (2.5). Defining $v := k^2 \Phi_k * (q(\,\cdot\,,|u|)u)$ in $\mathbb{R}^d$, we find that $u = u^i + v$ in D. Moreover, $v \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ satisfies (2.6), and if we extend u by $u^i + v$ to all of $\mathbb{R}^d$, then u solves (2.4). □

In the following we consider this problem for more general source terms and study radiating solutions $v \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ of

$$\Delta v + k^2 v \;=\; -k^2 q(\,\cdot\,,|v+f|)(v+f) \qquad \text{in } \mathbb{R}^d\,, \tag{2.7}$$

where $f \in L^\infty(D)$. In this situation, f represents the incident field and v the corresponding scattered field. As in Lemma 2.4 we find that this is equivalent to the problem of solving the nonlinear integral equation

$$v(x) \;=\; k^2 \int_D \Phi_k(x-y) q(y,|v(y)+f(y)|)(v(y)+f(y)) \,\mathrm{d}y\,, \qquad x \in D\,, \tag{2.8}$$

in $L^\infty(D)$.

Remark 2.5. In the linear case, i.e., when $q = q_0$, the scattering problem (2.7) reduces to

$$\Delta v_0 + k^2 v_0 \;=\; -k^2 q_0(v_0 + f) \qquad \text{in } \mathbb{R}^d\,, \tag{2.9}$$

and the corresponding linear Lippmann-Schwinger equation reads

$$v_0(x) \;=\; k^2 \int_D \Phi_k(x-y) q_0(y)(v_0(y) + f(y)) \,\mathrm{d}y\,, \qquad x \in D\,. \tag{2.10}$$

We note that $I - k^2 \Phi_k * (q_0\,\cdot\,)$ is an isomorphism on $L^2(D)$ (see [27, Thm. 7.13] for the corresponding result in the case when D is a ball $B_R(0)$). On the other hand, $k^2 \Phi_k * (q_0\,\cdot\,)$ maps $L^\infty(D)$ into $H^2(B_R(0))$ for $B_R(0)$ containing D (see [27, Thm. 7.11]), and the restricted embedding of $H^2(B_R(0))$ into $L^\infty(D)$ is compact (see [12, Thm. 7.26]). Combining Riesz' theorem (see

[30, Thm. 3.4]) with the uniqueness of radiating solutions to (2.9) (see [27, Thm. 7.8]) shows that $I - k^2\Phi_k * (q_0\,\cdot\,)$ is an isomorphism on $L^\infty(D)$ as well. In particular we have

$$\big\|\big(I - k^2\Phi_k * (q_0\,\cdot\,)\big)^{-1}g\big\|_{L^2(D)} \,\leq\, C_{LS,2}\|g\|_{L^2(D)}\,, \qquad g \in L^2(D)\,, \tag{2.11a}$$

$$\big\|\big(I - k^2\Phi_k * (q_0\,\cdot\,)\big)^{-1}g\big\|_{L^\infty(D)} \,\leq\, C_{LS,\infty}\|g\|_{L^\infty(D)}\,, \qquad g \in L^\infty(D)\,. \tag{2.11b}$$

Accordingly, the unique solution v_0 of (2.10) is given by

$$v_0 \,=\, \big(I - k^2\Phi_k * (q_0\,\cdot\,)\big)^{-1}\big(k^2\Phi_k * (q_0 f)\big) \qquad \text{in } D\,, \tag{2.12}$$

and we denote by V_0 the linear operator that maps f to v_0. The solution v_0 can be extended by the right hand side of (2.10) to a radiating solution of (2.9) in all of $\mathbb{R}^d$, which we also denote by $v_0 = V_0 f$. For later reference we note that (2.11) implies

$$\|V_0 f\|_{L^2(D)} \,\leq\, C_{V_0,2}\|f\|_{L^2(D)}\,, \qquad f \in L^2(D)\,, \tag{2.13a}$$

$$\|V_0 f\|_{L^\infty(D)} \,\leq\, C_{V_0,\infty}\|f\|_{L^\infty(D)}\,, \qquad f \in L^\infty(D)\,, \tag{2.13b}$$

where $C_{V_0,\infty} = k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}\|q_0\|_{L^\infty(D)}$ and $C_{V_0,2} = k^2 C_{LS,2}\|\Phi_k\|_{L^1(B_{2R}(0))}\|q_0\|_{L^\infty(D)}$. Here and in the following $R > 0$ is chosen such that $D \subseteq B_R(0)$. ♢

In Proposition 2.6 below we establish well-posedness of (2.7). Writing

$$U_\delta \,:=\, \big\{v \in L^\infty(D) \,\big|\, \|v\|_{L^\infty(D)} \leq \delta\big\}\,, \qquad \delta > 0\,,$$

we show that for any $f \in U_\delta$ with $\delta > 0$ sufficiently small there exists a unique solution v of (2.8) in $L^\infty(D)$ such that the difference $w := v - v_0$ with v_0 from (2.12) satisfies $w \in U_\delta$. We call this v the *unique small solution* of (2.8). Denoting by V the nonlinear operator that maps f to v, we shall see that V is Fréchet-differentiable at zero and $V'(0) = V_0$. The mere existence of such an operator is well-known, see for instance, [7, Thm. 1.2], [14, Thm. 1], or [33, Thm. 1].

Proposition 2.6. *Suppose that Assumption 2.1 is satisfied. There exists $\delta > 0$ such that for any given $f \in U_\delta$ the nonlinear integral equation (2.8) has a unique solution $v = V(f) \in L^\infty(D)$ satisfying $v - V_0 f \in U_\delta$, and there exists a constant[1] $C > 0$ such that, for all such f,*

$$\|V(f)\|_{L^\infty(D)} \,\leq\, C\|f\|_{L^\infty(D)}\,, \tag{2.14a}$$

$$\|V(f)\|_{L^2(D)} \,\leq\, C\|f\|_{L^2(D)}\,, \tag{2.14b}$$

$$\|V(f) - V_0 f\|_{L^\infty(D)} \,\leq\, C\|f\|_{L^\infty(D)}^{1+\alpha}\,, \tag{2.14c}$$

$$\|V(f) - V_0 f\|_{L^2(D)} \,\leq\, C\|f\|_{L^\infty(D)}^{\alpha}\|f\|_{L^2(D)}\,. \tag{2.14d}$$

Remark 2.7. The proof of Proposition 2.6 below shows that the upper bound $\delta > 0$ has to be such that the product $C_q\delta > 0$, where C_q is the upper bound on the nonlinearity from Assumption 2.1, is sufficiently small. This means that there is a tradeoff between the size of the nonlinearity and the intensity of the incident fields and scattered fields that are covered by this well-posedness result. ♢

Proof of Proposition 2.6. For any given $f \in L^\infty(D)$ let $v_0 := V_0 f \in L^\infty(D)$ as in (2.12). Then, $v \in L^\infty(D)$ solves (2.8) if and only if $w := v - v_0$ satisfies

$$w - k^2\Phi_k * (q_0 w) \,=\, k^2\Phi_k * \big(q_N(\,\cdot\,, |w + v_0 + f|)(w + v_0 + f)\big) \qquad \text{in } D\,,$$

[1] Throughout C denotes a generic constant, the values of which might change from line to line.

where $q_N := q - q_0$ denotes the nonlinear part of the contrast function. This is equivalent to w being a fixed point of the nonlinear map $G : L^\infty(D) \to L^\infty(D)$,

$$G(w) := \big(I - k^2\Phi_k * (q_0\,\cdot)\big)^{-1}\Big(k^2\Phi_k * \big(q_N(\,\cdot\,,|w+v_0+f|)(w+v_0+f)\big)\Big). \tag{2.15}$$

Using (2.11b), Young's inequality, (2.2), and (2.13b) we have for any $f \in U_\delta$ and $w \in U_\delta$ that

$$\begin{aligned}
\|G(w)\|_{L^\infty(D)} &\le C_{LS,\infty}\big\|k^2\Phi_k * \big(q_N(\,\cdot\,,|w+v_0+f|)(w+v_0+f)\big)\big\|_{L^\infty(D)}\\
&\le k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}\big\|q_N(\,\cdot\,,|w+v_0+f|)(w+v_0+f)\big\|_{L^\infty(D)}\\
&\le k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}C_q\|w+v_0+f\|_{L^\infty(D)}^{1+\alpha}\\
&\le k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}C_q\big(\delta + (C_{V_0,\infty}+1)\delta\big)^{1+\alpha}.
\end{aligned}$$

Here, $R > 0$ was chosen such that $D \subseteq B_R(0)$. Similarly, applying (2.1) we obtain for any $f \in U_\delta$ and $w_1, w_2 \in U_\delta$ that

$$\begin{aligned}
&\|G(w_1) - G(w_2)\|_{L^\infty(D)}\\
&\le k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}C_q\big(\|w_1+v_0+f\|_{L^\infty(D)}^\alpha + \|w_2+v_0+f\|_{L^\infty(D)}^\alpha\big)\|w_1-w_2\|_{L^\infty(D)}\\
&\le k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}C_q\,2\big(\delta + (C_{V_0,\infty}+1)\delta\big)^\alpha\|w_1-w_2\|_{L^\infty(D)}.
\end{aligned}$$

Choosing $\delta > 0$ such that $C_q\delta > 0$ is sufficiently small, we find that

$$\|G(w)\|_{L^\infty(D)} \le \delta, \qquad \|G(w_1) - G(w_2)\|_{L^\infty(D)} \le \frac{1}{2}\|w_1 - w_2\|_{L^\infty(D)}.$$

So $G : U_\delta \to U_\delta$ is a contraction, and Banach's fixed point theorem yields the existence of a uniquely determined fixed point $w \in U_\delta$ of G such that $v = V(f) := w + V_0 f$ solves (2.8).

It remains to show (2.14a)–(2.14d). This follows from (2.15), (2.11b), Young's inequality, and (2.2) because

$$\begin{aligned}
&\|V(f) - V_0 f\|_{L^\infty(D)} = \|G(V(f) - V_0 f)\|_{L^\infty(D)}\\
&= \Big\|\big(I - k^2\Phi_k * (q_0\,\cdot)\big)^{-1}\Big(k^2\Phi_k * \big(q_N(\,\cdot\,,|V(f)+f|)(V(f)+f)\big)\Big)\Big\|_{L^\infty(D)}\\
&\le k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}\big\|q_N(\,\cdot\,,|V(f)+f|)(V(f)+f)\big\|_{L^\infty(D)}\\
&\le k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}C_q\|V(f)+f\|_{L^\infty(D)}^{1+\alpha}\\
&\le k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}C_q\big(\|V(f)-V_0 f\|_{L^\infty(D)} + \|V_0 f + f\|_{L^\infty(D)}\big)^\alpha\|V(f)+f\|_{L^\infty(D)}.
\end{aligned} \tag{2.16}$$

Hence, (2.13b) yields

$$\begin{aligned}
\|V(f)\|_{L^\infty(D)} \le\ & C_{V_0,\infty}\|f\|_{L^\infty(D)}\\
&+ k^2 C_{LS,\infty}\|\Phi_k\|_{L^1(B_{2R}(0))}C_q\big(\delta + (C_{V_0,\infty}+1)\delta\big)^\alpha\big(\|V(f)\|_{L^\infty(D)} + \|f\|_{L^\infty(D)}\big).
\end{aligned}$$

Given that $C_q\delta > 0$ is sufficiently small as in the first part of the proof we thus obtain (2.14a). Therewith, (2.16) shows (2.14c). Finally, using (2.15), (2.11a), Young's inequality, and (2.2) we get

$$\begin{aligned}
&\|V(f) - V_0 f\|_{L^2(D)}\\
&= \|G(V(f) - V_0 f)\|_{L^2(D)}\\
&= \Big\|\big(I - k^2\Phi_k * (q_0\,\cdot)\big)^{-1}\Big(k^2\Phi_k * \big(q_N(\,\cdot\,,|V(f)+f|)(V(f)+f)\big)\Big)\Big\|_{L^2(D)}\\
&\le k^2 C_{LS,2}\|\Phi_k\|_{L^1(B_{2R}(0))}\big\|q_N(\,\cdot\,,|V(f)+f|)(V(f)+f)\big\|_{L^2(D)}\\
&\le k^2 C_{LS,2}\|\Phi_k\|_{L^1(B_{2R}(0))}C_q\||V(f)+f|^{1+\alpha}\|_{L^2(D)}\\
&\le k^2 C_{LS,2}\|\Phi_k\|_{L^1(B_{2R}(0))}C_q\|V(f)+f\|_{L^\infty(D)}^\alpha\|V(f)+f\|_{L^2(D)}.
\end{aligned}$$

Proceeding as before this implies (2.14b) when $C_q\delta > 0$ is sufficiently small, and thus also (2.14d). $\square$

After extending the right hand side of (2.8) to all of $\mathbb{R}^d$, Proposition 2.6 guarantees the existence of a unique small radiating solution of the generalized scattering problem (2.7) for any $f \in L^\infty(D)$ that is sufficiently small. We denote this extension by $v = V(f)$ as well. In particular, Proposition 2.6 tells us that for all $L^\infty(D)$-small incoming waves u^i we have a unique small solution $u = u^i + V(u^i|_D)$ of the nonlinear forward problem (2.4). Here small means that $\|V(u^i|_D) - V_0(u^i|_D)\|_{L^\infty(D)} \leq \delta$ with $\delta > 0$ from Proposition 2.6. Substituting the far field asymptotics of the fundamental solution (see, e.g., [9, p. 24 and p. 89]) into the extension of the integral representation (2.8) to all of $\mathbb{R}^d$, we obtain the following result.

Proposition 2.8. *Suppose that Assumption 2.1 is satisfied, let $\delta > 0$ be as in Proposition 2.6, and let $f \in U_\delta$. Then the unique small radiating solution $v = V(f) \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ of (2.7) admits the asymptotic behavior*

$$v(x) \,=\, C_d e^{\mathrm{i}k|x|}|x|^{\frac{1-d}{2}} v^\infty(\widehat{x}) + O(|x|^{-\frac{d+1}{2}})\,, \qquad |x| \to \infty\,,$$

uniformly in all directions $\widehat{x} := x/|x| \in S^{d-1}$, where

$$C_d \,=\, e^{\mathrm{i}\pi/4}/\sqrt{8\pi k} \quad \textit{if } d = 2 \qquad \textit{and} \qquad C_d \,=\, 1/(4\pi) \quad \textit{if } d = 3\,.$$

The far field pattern *$v^\infty = (V(f))^\infty \in L^2(S^{d-1})$ is given by*

$$v^\infty(\widehat{x}) \,=\, k^2 \int_D q\bigl(y, |v(y) + f(y)|\bigr)\bigl(v(y) + f(y)\bigr) e^{-\mathrm{i}k\widehat{x}\cdot y}\,\mathrm{d}y\,, \qquad \widehat{x} \in S^{d-1}\,. \tag{2.17}$$

In the following we will we restrict the discussion to incident fields that are superpositions of plane waves. We define the *Herglotz operator* $H : L^2(S^{d-1}) \to L^2(D)$,

$$(H\psi)(x) \,:=\, \int_{S^{d-1}} \psi(\theta) e^{\mathrm{i}kx\cdot\theta}\,\mathrm{d}s(\theta)\,, \qquad x \in D\,, \tag{2.18}$$

and we note that its adjoint $H^* : L^2(D) \to L^2(S^{d-1})$ satisfies

$$(H^*\phi)(\widehat{x}) \,=\, \int_D \phi(y) e^{-\mathrm{i}k\widehat{x}\cdot y}\,\mathrm{d}y\,, \qquad \widehat{x} \in S^{d-1}\,. \tag{2.19}$$

The operators H and H^* are compact. Observing that

$$\|H\psi\|_{L^\infty(D)} \,\leq\, \omega_{d-1}^{1/2} \|\psi\|_{L^2(S^{d-1})}\,, \tag{2.20}$$

where ω_{d-1} denotes the area of the unit sphere, we define

$$\mathcal{D}(F) \,:=\, \bigl\{\psi \in L^2(S^{d-1}) \bigm| \|\psi\|_{L^2(S^{d-1})} \leq \delta/\omega_{d-1}^{1/2}\bigr\}\,,$$

where $\delta > 0$ is as in Proposition 2.6. Then any $f = Hg$ with $g \in \mathcal{D}(F)$ satisfies $f \in U_\delta$, and the unique small radiating solution $v = V(Hg)$ of (2.7) has the far field pattern $v^\infty = (V(Hg))^\infty$. Introducing the *nonlinear far field operator* $F : \mathcal{D}(F) \subseteq L^2(S^{d-1}) \to L^2(S^{d-1})$ by

$$F(g) \,:=\, (V(Hg))^\infty\,, \tag{2.21}$$

we obtain from (2.17) that

$$F(g) \,=\, H^*\bigl(k^2 q(\,\cdot\,, |v + Hg|)(v + Hg)\bigr)\,.$$

These facts are summarized as follows.

Proposition 2.9. *Suppose that Assumption 2.1 holds, and let $g \in \mathcal{D}(F)$. Then the far field pattern of the unique small radiating solution $V(f)$ of (2.7) with $f = Hg$ satisfies*

$$F(g) \;=\; H^*T(Hg)\,, \tag{2.22}$$

where $T : \mathcal{D}(T) \subseteq L^2(D) \to L^2(D)$ is defined by

$$T(f)(x) \;=\; k^2 q\big(x, |V(f)(x) + f(x)|\big)\big(V(f)(x) + f(x)\big)\,, \qquad x \in D\,. \tag{2.23}$$

Here $\mathcal{D}(T) := \overline{H(\mathcal{D}(F))}$.

Remark 2.10. In the linear case when $q = q_0$, the far field operator $F_0 : L^2(S^{d-1}) \to L^2(S^{d-1})$ is given by

$$F_0 g \;:=\; (V_0 Hg)^\infty\,.$$

The factorization (2.22) reads

$$F_0 g \;=\; H^* T_0 Hg\,, \qquad g \in L^2(S^{d-1})\,,$$

where $T_0 : L^2(D) \to L^2(D)$ is defined by

$$T_0 f \;:=\; k^2 q_0 (f + V_0 f)\,. \tag{2.24}$$

Then (2.2) implies that, for any $f \in \mathcal{D}(T)$,

$$\begin{aligned}
\|T(f) - T_0 f\|_{L^2(D)} \;&=\; k^2 \big\| q(\,\cdot\,, |V(f) + f|)(V(f) + f) - q_0(V_0 f + f) \big\|_{L^2(D)} \\
&\le\; k^2 \big\| q(\,\cdot\,, |V(f) + f|)(V(f) + f) - q_0(V(f) + f) \big\|_{L^2(D)} + k^2 \|q_0(V(f) - V_0 f)\|_{L^2(D)} \\
&\le\; k^2 C_q \big\| |V(f) + f|^{1+\alpha} \big\|_{L^2(D)} + k^2 \|q_0\|_{L^\infty(D)} \|V(f) - V_0 f\|_{L^2(D)}\,.
\end{aligned}$$

Applying (2.14d) and (2.14a)–(2.14b) gives

$$\begin{aligned}
\|T(f) - T_0 f\|_{L^2(D)} \;&\le\; k^2 C_q \|V(f) + f\|^\alpha_{L^\infty(D)} \|V(f) + f\|_{L^2(D)} + C\|f\|^\alpha_{L^\infty(D)} \|f\|_{L^2(D)} \\
&\le\; C\|f\|^\alpha_{L^\infty(D)} \|f\|_{L^2(D)}\,.
\end{aligned} \tag{2.25}$$

Similarly, using (2.14c) and (2.14a), we find that, for any $f \in \mathcal{D}(T)$,

$$\|T(f) - T_0 f\|_{L^\infty(D)} \;\le\; C\|f\|^{\alpha+1}_{L^\infty(D)}\,. \tag{2.26}$$

◊

3 Uniqueness for the inverse scattering problem

In this section we restrict the discussion to generalized Kerr-type nonlinearities q as in (2.3). We show that the knowledge of the nonlinear far field operator uniquely determines the associated nonlinear refractive index. A related result has recently been established for a different class of real analytic nonlinearities in [11].

Theorem 3.1. *For $j = 1, 2$ let*

$$q^{(j)}(x, |z|) \;=\; q_0^{(j)}(x) + \sum_{l=1}^{L} q_l^{(j)}(x)|z|^{\alpha_l} \qquad x \in \mathbb{R}^d\,,\ z \in \mathbb{C}\,,\ j = 1, 2\,, \tag{3.1}$$

be a generalized Kerr-type nonlinearity, where $q_0^{(j)}, \dots, q_L^{(j)} \in L^\infty(\mathbb{R}^d)$ with support in D, the lowest order term satisfies $\operatorname{ess\,inf} q_0^{(j)} > -1$, and the exponents fulfill $0 < \alpha_1 < \dots < \alpha_L < \infty$. If the associated nonlinear far field operators satisfy $F^{(1)} = F^{(2)}$, then $q^{(1)} = q^{(2)}$.

Proof. By linearization around zero we first show that $q_0^{(1)} = q_0^{(2)}$ in (3.1). We consider factorizations of the far field operators $F^{(j)} = H^* T^{(j)}(H)$, $j = 1, 2$, as in Proposition 2.9, where H and H^* are the Herglotz operator and its adjoint from (2.18) and (2.19), and the operator $T^{(j)}$ is as in (2.23) with q replaced by $q^{(j)}$. Furthermore, we denote by $T_0^{(j)}$ the bounded linear operator from (2.24) with q_0 replaced by $q_0^{(j)}$. Then (2.26) shows that $T^{(j)}(f) = T_0^{(j)} f + O(\|f\|_{L^\infty(D)}^{\alpha_1+1})$ as $\|f\|_{L^\infty(D)} \to 0$. Recalling (2.20), we obtain from $F^{(1)} = F^{(2)}$ that

$$F_0^{(1)} \;=\; H^* T_0^{(1)} H \;=\; H^* T_0^{(2)} H \;=\; F_0^{(2)}\,,$$

where $F_0^{(j)}$ is the linear far field operator corresponding to the contrast function $q_0^{(j)}$, $j = 1, 2$. The uniqueness of solutions to the inverse medium scattering problem for the linear Helmholtz equation (see, e.g., [27, Thm. 7.28] or [4, 35, 36, 37]) implies that $q_0^{(1)} = q_0^{(2)} =: q_0$. In particular we conclude that $T_0^{(1)} = T_0^{(2)} =: T_0$.

To prove the theorem by induction, we now assume $q_l^{(1)} = q_l^{(2)} =: q_l$ for $l = 0, \dots, m-1$, where $m \in \{1, \dots, L\}$. The nonlinear Lippmann-Schwinger equation (2.8) gives, for $f \in U_\delta$ and $j = 1, 2$,

$$V^{(j)}(f) \;=\; k^2 \Phi_k * \Big(q^{(j)}\big(\,\cdot\,, |V^{(j)}(f) + f|\big)(V^{(j)}(f) + f) \Big) \qquad \text{in } D\,.$$

Here, $V^{(j)}(f)$ stands for the solution map $V(f)$ from Proposition 2.6 with q replaced by $q^{(j)}$. Setting $\alpha_0 := 0$ and using (3.1) we obtain that

$$\begin{aligned}
&V^{(1)}(f) - V^{(2)}(f) \\
&\;=\; k^2 \Phi_k * \big(q^{(1)}(\,\cdot\,, |V^{(1)}(f) + f|)(V^{(1)}(f) + f) - q^{(2)}(\,\cdot\,, |V^{(2)}(f) + f|)(V^{(2)}(f) + f) \big) \\
&\;=\; k^2 \Phi_k * \Bigg(\sum_{l=0}^{m-1} q_l \Big(|V^{(1)}(f) + f|^{\alpha_l} (V^{(1)}(f) + f) - |V^{(2)}(f) + f|^{\alpha_l} (V^{(2)}(f) + f) \Big) \Bigg) \\
&\qquad + k^2 \Phi_k * \Bigg(\sum_{l=m}^{L} \Big(q_l^{(1)} |V^{(1)}(f) + f|^{\alpha_l} (V^{(1)}(f) + f) - q_l^{(2)} |V^{(2)}(f) + f|^{\alpha_l} (V^{(2)}(f) + f) \Big) \Bigg)
\end{aligned} \tag{3.2}$$

in D. Applying Lemma A.1 and (2.14a) we find that, for $l = 1, \dots, m-1$,

$$\begin{aligned}
\big| |V^{(1)}(f) + f|^{\alpha_l} (V^{(1)}(f) + f) &- |V^{(2)}(f) + f|^{\alpha_l} (V^{(2)}(f) + f) \big| \\
&\leq C \big(|f| + |V^{(1)}(f)| + |V^{(2)}(f)| \big)^{\alpha_l} \big| V^{(1)}(f) - V^{(2)}(f) \big| \\
&\leq C \|f\|_{L^\infty(D)}^{\alpha_l} \big| V^{(1)}(f) - V^{(2)}(f) \big|\,.
\end{aligned} \tag{3.3}$$

Accordingly,

$$\begin{aligned}
\sum_{l=0}^{m-1} q_l \Big(|V^{(1)}(f) + f|^{\alpha_l} (V^{(1)}(f) + f) - |V^{(2)}(f) + f|^{\alpha_l} (V^{(2)}(f) + f) \Big) \\
= \widetilde{q}_{f,m-1} \big(V^{(1)}(f) - V^{(2)}(f) \big)\,,
\end{aligned}$$

where $\widetilde{q}_{f,m-1} \in L^\infty(\mathbb{R}^d)$ is given by

$$\widetilde{q}_{f,m-1} := q_0 + \sum_{l=1}^{m-1} q_l \frac{|V^{(1)}(f) + f|^{\alpha_l} (V^{(1)}(f) + f) - |V^{(2)}(f) + f|^{\alpha_l} (V^{(2)}(f) + f)}{V^{(1)}(f) - V^{(2)}(f)} \mathbf{1}_{V^{(1)}(f) \neq V^{(2)}(f)}\,.$$

Therewith (3.2) can be rewritten as

$$\begin{aligned}
&\big(I - k^2\Phi_k * (\widetilde{q}_{f,m-1}\,\cdot)\big)\,\big(V^{(1)}(f) - V^{(2)}(f)\big)\\
&\quad = k^2\Phi_k * \Bigg(\sum_{l=m}^{L}\Big(q_l^{(1)}|V^{(1)}(f)+f|^{\alpha_l}(V^{(1)}(f)+f) - q_l^{(2)}|V^{(2)}(f)+f|^{\alpha_l}(V^{(2)}(f)+f)\Big)\Bigg)\,.
\end{aligned}$$

We note that $\widetilde{q}_{f,m-1}$ is supported in D and (3.3) implies that

$$\|\widetilde{q}_{f,m-1} - q_0\|_{L^\infty(D)} \,\leq\, C\|f\|_{L^\infty(D)}^{\alpha_1}\,. \tag{3.4}$$

Hence, for $f \in U_\delta$ such that $\|f\|_{L^\infty(D)}$ is sufficiently small, we conclude from (2.11b) that the operator $I - k^2\Phi_k * (\widetilde{q}_{f,m-1}\,\cdot) : L^\infty(D) \to L^\infty(D)$ is invertible with a uniform bound for the operator norm of the inverse (see, e.g., [30, Thm. 10.1]). Denoting

$$R(f) \,:=\, \sum_{l=m}^{L}\Big(q_l^{(1)}|V^{(1)}(f)+f|^{\alpha_l}(V^{(1)}(f)+f) - q_l^{(2)}|V^{(2)}(f)+f|^{\alpha_l}(V^{(2)}(f)+f)\Big) \tag{3.5}$$

we find from (3.2), Young's inequality, and (2.14a) that

$$\begin{aligned}
\big\|V^{(1)}(f) - V^{(2)}(f)\big\|_{L^\infty(D)} &= \big\|\big(I - k^2\Phi_k * (\widetilde{q}_{f,m-1}\,\cdot)\big)^{-1}(k^2\Phi_k * R(f))\big\|_{L^\infty(D)}\\
&\leq C\|k^2\Phi_k * R(f)\|_{L^\infty(D)}\\
&\leq C\|\Phi_k\|_{L^1(B_{2R}(0))}\|R(f)\|_{L^\infty(D)}\\
&\leq C\big(\|V^{(1)}(f)+f\|_{L^\infty(D)}^{\alpha_m+1} + \|V^{(2)}(f)+f\|_{L^\infty(D)}^{\alpha_m+1}\big)\\
&\leq C\|f\|_{L^\infty(D)}^{\alpha_m+1}\,.
\end{aligned} \tag{3.6}$$

Here, $R>0$ was chosen such that $D \subseteq B_R(0)$.

Next we want to use (3.6) in order to deduce $q_m^{(1)} = q_m^{(2)}$. Set $w_f := V^{(1)}(f) - V^{(2)}(f)$. By assumption we know that the far field of w_f vanishes whenever f is a sufficiently small Herglotz wave, i.e., when $f \in U_\delta \cap \mathcal{R}(H)$ with $\delta > 0$ as in Proposition 2.6. Moreover, we find as in the proof of Lemma 2.4 that (3.2) implies

$$\Delta w_f + k^2(1+\widetilde{q}_{f,m-1})w_f \,=\, -k^2R(f) \qquad \text{in } \mathbb{R}^d\,,$$

in particular $\Delta w_f + k^2 w_f = 0$ in $\mathbb{R}^d \setminus \overline{B_R(0)}$ and w_f is radiating. So Rellich's lemma (see, e.g., [9, Lmm. 2.12]) gives $w_f = 0$ in $\mathbb{R}^d \setminus \overline{B_R(0)}$. Now let $v \in H^2(B_R(0))$ be any solution of $\Delta v + k^2(1+q_0)v = 0$ in $B_R(0)$. Then, for all $f \in U_\delta \cap \mathcal{R}(H)$,

$$\begin{aligned}
0 &= \int_{\partial B_R(0)}\Big(w_f\frac{\partial v}{\partial\nu} - v\frac{\partial w_f}{\partial\nu}\Big)\,\mathrm{d}s\\
&= \int_{B_R(0)}\big(w_f\Delta v - v\Delta w_f\big)\,\mathrm{d}x\\
&= \int_{B_R(0)}\Big(w_f\big(-k^2(1+q_0)v\big) - v\big(-k^2(1+\widetilde{q}_{f,m-1})w_f - k^2R(f)\big)\Big)\,\mathrm{d}x\\
&= \int_{B_R(0)} v\big(k^2R(f) + k^2(\widetilde{q}_{f,m-1}-q_0)w_f\big)\,\mathrm{d}x\\
&= \int_D v\big(k^2R(f) + k^2(\widetilde{q}_{f,m-1}-q_0)w_f\big)\,\mathrm{d}x\,.
\end{aligned}$$

In the last equality we used that $R(f)$ and $\widetilde{q}_{f,m-1} - q_0$ are supported in D by our assumption on the nonlinear contrast function. In (3.6) we found that $\|w_f\|_{L^\infty(D)} \le C\|f\|_{L^\infty(D)}^{\alpha_m+1}$, and combining this with (3.4) gives

$$0 \;=\; \int_D v\, R(f)\;\mathrm{d}x + O\big(\|f\|_{L^\infty(D)}^{\alpha_m+\alpha_1+1}\big) \qquad \text{as } \|f\|_{L^\infty(D)} \to 0\,. \tag{3.7}$$

Next we identify the leading order term in

$$\begin{aligned} R(f) \;=\; &\big(q_m^{(1)} - q_m^{(2)}\big)|V_0(f)+f|^{\alpha_m}(V_0(f)+f)\\ &+ q_m^{(1)}\big(|V^{(1)}(f)+f|^{\alpha_m}(V^{(1)}(f)+f) - |V_0(f)+f|^{\alpha_m}(V_0(f)+f)\big)\\ &- q_m^{(2)}\big(|V^{(2)}(f)+f|^{\alpha_m}(V^{(2)}(f)+f) - |V_0(f)+f|^{\alpha_m}(V_0(f)+f)\big)\\ &+ \sum_{l=m+1}^{L}\Big(q_l^{(1)}|V^{(1)}(f)+f|^{\alpha_l}(V^{(1)}(f)+f) - q_l^{(2)}|V^{(2)}(f)+f|^{\alpha_l}(V^{(2)}(f)+f)\Big)\,. \end{aligned}$$

Using Lemma A.1 and (2.14a), (2.14c) we obtain that, for $j=1,2$,

$$\begin{aligned} &\big||V^{(j)}(f)+f|^{\alpha_m}(V^{(j)}(f)+f) - |V_0f+f|^{\alpha_m}(V_0f+f)\big|\\ &\le\; C\big(|f|+|V^{(j)}(f)|+|V_0f|\big)^{\alpha_m}\big|V^{(j)}(f)-V_0^{(j)}f\big|\\ &\le\; C\|f\|_{L^\infty(D)}^{\alpha_m+\alpha_1+1}\,. \end{aligned} \tag{3.8}$$

Similarly, we find that, for $j=1,2$ and $l=m+1,\dots,L$,

$$\big||V^{(j)}(f)+f|^{\alpha_l}(V^{(j)}(f)+f)\big| \;\le\; C\|f\|_{L^\infty(D)}^{\alpha_{m+1}+1}\,. \tag{3.9}$$

Substituting (3.5) into (3.7), and applying (3.8)–(3.9) gives

$$0 \;=\; \int_D v\,\big(q_m^{(1)}-q_m^{(2)}\big)|V_0f+f|^{\alpha_m}(V_0f+f)\;\mathrm{d}x + O\big(\|f\|_{L^\infty(D)}^{\alpha_m+\alpha_1+1}\big) + O\big(\|f\|_{L^\infty(D)}^{\alpha_{m+1}+1}\big)\,.$$

as $\|f\|_{L^\infty(D)} \to 0$. Hence, for all $f \in U_\delta \cap \mathcal{R}(H)$,

$$0 \;=\; \int_D v\,\big(q_m^{(1)}-q_m^{(2)}\big)|V_0f+f|^{\alpha_m}(V_0f+f)\;\mathrm{d}x\,.$$

Setting $f = f_1 + tf_2$ for $f_1, f_2 \in U_\delta \cap \mathcal{R}(H)$ and differentiating with respect to t at $t=0$ gives

$$0 \;=\; \int_D v\big(q_m^{(1)}-q_m^{(2)}\big)Z(f_1,f_2)\;\mathrm{d}x\,,$$

where

$$\begin{aligned} Z(f_1,f_2) \;:=\; &\Big(1+\frac{\alpha_m}{2}\Big)|V_0f_1+f_1|^{\alpha_m}(V_0f_2+f_2)\\ &+ \frac{\alpha_m}{2}|V_0f_1+f_1|^{\alpha_m-2}(V_0f_1+f_1)^2\,\overline{(V_0f_2+f_2)}\,. \end{aligned}$$

Since $\mathrm{i}f_1 \in U_\delta \cap \mathcal{R}(H)$, too, we even get

$$\begin{aligned} 0 \;&=\; \int_D v\,\big(q_m^{(1)}-q_m^{(2)}\big)\big(Z(f_1,f_2)+Z(\mathrm{i}f_1,f_2)\big)\;\mathrm{d}x\\ &=\; (2+\alpha_m)\int_D v\,\big(q_m^{(1)}-q_m^{(2)}\big)|V_0f_1+f_1|^{\alpha_m}(V_0f_2+f_2)\;\mathrm{d}x\,. \end{aligned} \tag{3.10}$$

Observing that V_0 is linear, (3.10) holds true for all $f_1, f_2 \in \mathcal{R}(H)$.

Next we recall that the span of all total fields $f + V_0 f$ that correspond to radiating solutions $V_0 f$ of the linear scattering problem (2.9) with Herglotz incident fields $f = Hg$, $g \in L^2(S^{d-1})$, is dense in the space of solutions to the linear Helmholtz equation in

$$\Delta\widetilde{v} + k^2(1+q_0)\widetilde{v} \,=\, 0 \qquad \text{in } B_R(0)$$

with respect to the $L^2(B_R(0))$-norm where $D \subset B_R(0)$ (see [27, Thm. 7.24], where this result has been shown for plane wave incident fields instead of Herglotz incident fields). Since $f_1, f_2 \in \mathcal{R}(H)$ have been arbitrary in (3.10), we get for all solutions $v, \widetilde{v} \in H^2(B_R(0))$ of $\Delta v + k^2(1+q_0)v = 0$ in $B_R(0)$ that

$$0 \,=\, \int_D v\widetilde{v}\big(q_m^{(1)} - q_m^{(2)}\big)|V_0 f_1 + f_1|^{\alpha_m} \,\mathrm{d}x\,.$$

This gives $(q_m^{(1)} - q_m^{(2)})|V_0 f_1 + f_1|^{\alpha_m} = 0$ for any $f_1 \in \mathcal{R}(H)$ (see, e.g., [27, Thm. 7.27] or [4, 35, 36, 37]). From this we infer $|q_m^{(1)} - q_m^{(2)}|^{\frac{1}{\alpha_m}}(V_0 f_1 + f_1) = 0$ for any $f_1 \in \mathcal{R}(H)$ and thus

$$\int_D |q_m^{(1)} - q_m^{(2)}|^{\frac{1}{\alpha_m}}(V_0 f_1 + f_1)(V_0 f_2 + f_2)\,\mathrm{d}x \,=\, 0$$

for all $f_1, f_2 \in \mathcal{R}(H)$. The density result used above shows $|q_m^{(1)} - q_m^{(2)}|^{\frac{1}{\alpha_m}} = 0$ a.e. in D, and thus $q_m^{(1)} = q_m^{(2)}$. So the claim is proven by induction. $\square$

4 The nonlinear inf-criterion

In this section we discuss a generalization of the factorization method to recover the shape of a nonlinear scattering object from observations of the corresponding nonlinear far field operator. We consider general nonlinear contrast functions $q \in L^\infty(\mathbb{R}^d \times \mathbb{R})$ as in Section 2, but here we make the following slightly stronger assumptions.

Assumption 4.1. *Let $D \subseteq \mathbb{R}^d$ be open and Lipschitz bounded such that $\mathbb{R}^d \setminus \overline{D}$ is connected. Then the nonlinear contrast function $q \in L^\infty(\mathbb{R}^d \times \mathbb{R})$ shall satisfy*

(i) $z \in \mathbb{C} \mapsto q(\,\cdot\,, |z|) \in L^\infty(\mathbb{R}^d)$ is continuous,

(ii) $q(\,\cdot\,, |z|) = 0$ a.e. in $\mathbb{R}^d \setminus D$ for all $z \in \mathbb{C}$,

(iii) there exists $q_0 \in L^\infty(\mathbb{R}^d)$ with $q_0 \geq q_{0,\min} > 0$ a.e. in D for some $q_{0,\min} > 0$, $q_0 = 0$ a.e. in $\mathbb{R}^d \setminus D$, and there exists a parameter $\alpha > 0$ such that for any $z_1, z_2 \in \mathbb{C}$ with $|z_1|, |z_2| \leq 1$,

$$\big\|q(\,\cdot\,, |z_1|)z_1 - q(\,\cdot\,, |z_2|)z_2 - q_0(z_1 - z_2)\big\|_{L^\infty(\mathbb{R}^d)} \,\leq\, C_q(|z_1|^\alpha + |z_2|^\alpha)|z_1 - z_2|\,,$$

(iv) and the wave number k^2 is such that the homogeneous linear transmission eigenvalue problem to determine $v, w \in L^2(D)$ with $v - w \in H^2(D)$ such that

$$\begin{aligned} \Delta v + k^2 v \,&=\, 0 \quad \text{in } D\,, & v \,&=\, w \quad \text{on } \partial D\,,\\ \Delta w + k^2(1+q_0)w \,&=\, 0 \quad \text{in } D\,, & \frac{\partial v}{\partial \nu} \,&=\, \frac{\partial w}{\partial \nu} \quad \text{on } \partial D\,, \end{aligned}$$

(see, e.g., [27, Def. 7.21]) has no nontrivial solution.

A factorization method for nonlinear weakly scattering objects and for scattering objects with small nonlinearity of linear growth has already been discussed in [32]. In constrast to this work, we consider a larger class of nonlinear refractive indices without any smallness assumption on the a priori unknown nonlinearity, but we assume that the incident fields that are used for the reconstruction are small relative to the size of the nonlinearity.

Let $\delta > 0$ be as in Proposition 2.6. We consider the nonlinear far field operator F from (2.21) with the factorization $F = H^*T(H)$ from Proposition 2.9. The next theorem is a nonlinear version of the abstract inf-criterion of the factorization method to describe the range of H^* in terms of F. A similar result has been established in [32, Thm. 2.1]. In contrast to [32, Thm. 2.1] we add a size constraint on the Herglotz densities to accommodate for the smallness of the incident fields relative to the size of the nonlinearity. The proof is essentially the same as in the linear case (see, e.g., [27, Lmm. 7.33]).

Theorem 4.2. *Let X and Y be Hilbert spaces, $\rho > 0$, and let*

$$\mathcal{F} : \mathcal{D}(\mathcal{F}) := \{g \in X \mid \|g\|_X \leq \rho\} \subseteq X \to X$$

be an operator (not necessarily linear). We assume that $\mathcal{F} = \mathcal{H}^*\mathcal{T}(\mathcal{H})$, where $\mathcal{H} : X \to Y$ is a compact linear operator and $\mathcal{T} : \mathcal{D}(\mathcal{T}) \subseteq Y \to Y$ with $\mathcal{D}(\mathcal{T}) = \overline{\mathcal{H}(\mathcal{D}(\mathcal{F}))}$ satisfies

$$\|\mathcal{T}(\mathcal{H}g)\|_Y \leq C_*\|\mathcal{H}g\|_Y$$

and

$$|\langle \mathcal{T}(\mathcal{H}g), \mathcal{H}g\rangle_Y| \geq c_*\|\mathcal{H}g\|_Y^2$$

for all $g \in \mathcal{D}(\mathcal{F})$ with $\|g\|_X \leq \rho$ and some $c_*, C_* > 0$. Then, for any $\phi \in X$, $\phi \neq 0$, and any $0 < \widetilde{\rho} \leq \rho$,

$$\phi \in \mathcal{R}(\mathcal{H}^*) \quad \iff \quad \inf\left\{ \frac{|\langle \mathcal{F}(g), g\rangle_X|}{|\langle g, \phi\rangle_X|^2} \;\middle|\; g \in \mathcal{D}(\mathcal{F}) \subseteq X\,,\ \|g\|_X = \widetilde{\rho}\,,\ \langle g, \phi\rangle_X \neq 0 \right\} > 0\,. \tag{4.1}$$

Proof. Let $0 \neq \phi = \mathcal{H}^*\psi \in \mathcal{R}(\mathcal{H}^*)$ for some $\psi \in Y$. Then $\psi \neq 0$, and for any $g \in \mathcal{D}(\mathcal{F}) \subseteq X$ with $\|g\|_X = \widetilde{\rho} \leq \rho$ and $\langle g, \phi\rangle_X \neq 0$ we find that

$$\begin{aligned}
|\langle \mathcal{F}(g), g\rangle_X| &= |\langle \mathcal{H}^*\mathcal{T}(\mathcal{H}g), g\rangle_X| = |\langle \mathcal{T}(\mathcal{H}g), \mathcal{H}g\rangle_Y| \geq c_*\|\mathcal{H}g\|_Y^2 \\
&= \frac{c_*}{\|\psi\|_Y^2}\|\mathcal{H}g\|_Y^2\|\psi\|_Y^2 \geq \frac{c_*}{\|\psi\|_Y^2}|\langle \mathcal{H}g, \psi\rangle_Y|^2 \\
&= \frac{c_*}{\|\psi\|_Y^2}|\langle g, \mathcal{H}^*\psi\rangle_X|^2 = \frac{c_*}{\|\psi\|_Y^2}|\langle g, \phi\rangle_X|^2
\end{aligned}$$

Thus we have found a positive lower bound for the infimum in (4.1).

Now let $0 \neq \phi \notin \mathcal{R}(\mathcal{H}^*)$. We first show that the subspace $\{\mathcal{H}g \mid g \in X\,,\ \langle g, \phi\rangle_X = 0\}$ is dense in $\mathcal{R}(\mathcal{H})$. Let $\psi \in \mathcal{R}(\mathcal{H})$ such that $0 = \langle \mathcal{H}g, \psi\rangle_Y = \langle g, \mathcal{H}^*\psi\rangle_X$ for all $g \in X$ with $\langle g, \phi\rangle_X = 0$. That means $\mathcal{H}^*\psi \in \operatorname{span}\{\phi\}$, and because $\phi \notin \mathcal{R}(\mathcal{H}^*)$, we conclude that $\mathcal{H}^*\psi = 0$. Therefore $\psi \in \mathcal{R}(\mathcal{H}) \cap \mathcal{N}(\mathcal{H}^*)$, i.e., $\psi = 0$, and using the Hahn-Banach theorem (see [46, p. 109]) we obtain that $\{\mathcal{H}g \mid g \in X\,,\ \langle g, \phi\rangle_X = 0\}$ is dense in $\mathcal{R}(\mathcal{H})$. Since $\mathcal{H}\phi/\|\phi\|_Y^2 \in \mathcal{R}(\mathcal{H})$, we can find a sequence $(\widetilde{g}_n)_n \subseteq \{g \in X \mid \langle g, \phi\rangle_X = 0\}$ such that $\mathcal{H}\widetilde{g}_n \to -\mathcal{H}\phi/\|\phi\|_X^2$. Setting $\widehat{g}_n := \widetilde{g}_n + \phi/\|\phi\|_X^2$ this yields $\langle \widehat{g}_n, \phi\rangle_X = 1$ and $\mathcal{H}\widehat{g}_n \to 0$ as $n \to \infty$. Thus, we define $g_n := \widetilde{\rho}\,\widehat{g}_n/\|\widehat{g}_n\|_X \in \mathcal{D}(\mathcal{F})$ to obtain

$$\frac{|\langle \mathcal{F}(g_n), g_n\rangle_X|}{|\langle g_n, \phi\rangle_X|^2} = \frac{|\langle \mathcal{T}(\mathcal{H}g_n), \mathcal{H}g_n\rangle_Y|}{|\langle g_n, \phi\rangle_X|^2} \leq \frac{C_*\|\mathcal{H}g_n\|_Y^2}{|\langle g_n, \phi\rangle_X|^2} = \frac{C_*\|\mathcal{H}\widehat{g}_n\|_Y^2}{|\langle \widehat{g}_n, \phi\rangle_X|^2} \to 0 \qquad \text{as } n \to \infty\,,$$

i.e., the infimum in (4.1) is zero. $\square$

Next we show that the operator T from Proposition 2.9 satisfies the assumptions in Theorem 4.2.

Proposition 4.3. *Suppose that Assumption 4.1 holds, and let $\delta > 0$ be as in Proposition 2.6. Then there are constants $c_*, C_*, C > 0$ such that*

$$\|T(f)\|_{L^2(D)} \leq C_*\big(1 + \|f\|^{\alpha}_{L^\infty(D)}\big)\|f\|_{L^2(D)}\,, \tag{4.2a}$$

$$|\langle T(f), f\rangle_{L^2(D)}| \geq c_*\big(1 - C\|f\|^{\alpha}_{L^\infty(D)}\big)\|f\|^2_{L^2(D)} \tag{4.2b}$$

for all $f \in \mathcal{D}(T)$.

Proof. Let $f \in \mathcal{D}(T)$. We first note that (2.24) and (2.13a) show that

$$\|T_0 f\|_{L^2(D)} \leq k^2\|q_0\|_{L^\infty(D)}(1 + C_{V_0,2})\|f\|_{L^2(D)}\,.$$

Combining this with (2.25) gives

$$\begin{aligned}
\|T(f)\|_{L^2(D)} &\leq \|T_0 f\|_{L^2(D)} + \|T(f) - T_0 f\|_{L^2(D)}\\
&\leq k^2\|q_0\|_{L^\infty(D)}(1 + C_{V_0,2})\|f\|_{L^2(D)} + C\|f\|^{\alpha}_{L^\infty(D)}\|f\|_{L^2(D)}\\
&\leq C_*\big(1 + \|f\|^{\alpha}_{L^\infty(D)}\big)\|f\|_{L^2(D)}
\end{aligned}$$

for some $C_* > 0$.

Next let $S_0 : L^2(D) \to L^2(D)$ be defined by

$$S_0\psi := \frac{1}{k^2 q_0}\psi - \Phi_k * \psi\,.$$

It has been shown in [27, Thm. 7.32] that S_0 is an isomorphism. Using (2.24) and (2.12) we find that, for any $h \in L^2(D)$,

$$\begin{aligned}
S_0 T_0 h &= \frac{1}{k^2 q_0}T_0 h - \Phi_k * (T_0 h) = (I + V_0)h - \Phi_k * \big(k^2 q_0(h + V_0 h)\big)\\
&= h + \big(I - k^2\Phi_k * (q_0\,\cdot)\big)(V_0 h) - k^2\Phi_k * (q_0 h) = h\,.
\end{aligned}$$

Thus, $T_0 = S_0^{-1}$.

If k^2 is not an interior transmission eigenvalue then it follows from [27, Lmm. 7.35] together with the identity relating H and S_0^* at the end of the proof of [27, Thm. 7.30] that there exists a constant $c > 0$ such that

$$|\langle S_0^{-1} f, f\rangle_{L^2(D)}| \geq c\|S_0^{-1} f\|^2_{L^2(D)} \qquad \text{for all } f \in \overline{\mathcal{R}(H)}\,.$$

Using the continuity of S_0 we obtain that there is a constant $c_* > 0$ with

$$|\langle T_0 f, f\rangle_{L^2(D)}| = |\langle S_0^{-1} f, f\rangle_{L^2(D)}| \geq c_*\|f\|^2_{L^2(D)} \qquad \text{for all } f \in \overline{\mathcal{R}(H)}\,.$$

Accordingly, combining this with (2.25) gives

$$\begin{aligned}
\big|\langle T(f), f\rangle_{L^2(D)}\big| &\geq \big|\langle T_0 f, f\rangle_{L^2(D)}\big| - \big|\langle T(f) - T_0 f, f\rangle_{L^2(D)}\big|\\
&\geq \big(c_* - C\|f\|^{\alpha}_{L^\infty(D)}\big)\|f\|^2_{L^2(D)} \geq c_*\big(1 - C\|f\|^{\alpha}_{L^\infty(D)}\big)\|f\|^2_{L^2(D)}
\end{aligned}$$

for any $f \in \mathcal{D}(T) = \overline{H(\mathcal{D}(F))}$. ∎

Combining (4.2) with (2.18) and applying Hölder's inequality gives the following corollary.

Corollary 4.4. *Suppose that Assumption 4.1 holds. Then there are constants $c_*, C_*, C > 0$ such that*

$$\|T(Hg)\|_{L^2(D)} \;\leq\; C_*\bigl(1 + C\omega_{d-1}^{\alpha/2}\|g\|^{\alpha}_{L^2(S^{d-1})}\bigr)\|Hg\|_{L^2(D)}\,, \tag{4.3a}$$

$$|\langle T(Hg), Hg\rangle_{L^2(D)}| \;\geq\; c_*\bigl(1 - C\omega_{d-1}^{\alpha/2}\|g\|^{\alpha}_{L^2(S^{d-1}))}\bigr)\|Hg\|^2_{L^2(D)} \tag{4.3b}$$

for all $g \in \mathcal{D}(F)$.

The following result has been shown in [28, Thm. 4.6]. We write out the proof for the convenience of the reader. We note that the test function ϕ_z in (4.4) below coincides with the far field pattern of the fundamental solution $\Phi_k(\,\cdot\, - z)$ with singularity at $z \in \mathbb{R}^d$.

Proposition 4.5. *Let $D \subseteq \mathbb{R}^d$ be open and Lipschitz bounded such that $\mathbb{R}^d \setminus \overline{D}$ is connected. For any $z \in \mathbb{R}^d$ we define the* test function *$\phi_z \in L^2(S^{d-1})$ by*

$$\phi_z(\widehat{x}) \;:=\; e^{-\mathrm{i}kz\cdot\widehat{x}}\,, \qquad \widehat{x} \in S^{d-1}\,. \tag{4.4}$$

Then $z \in D$ if and only if $\phi_z \in \mathcal{R}(H^*)$.

Proof. Let $z \in D$, and let $\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq D$. Choose $\eta \in C^\infty(\mathbb{R})$ with $0 \leq \eta \leq 1$, $\eta(t) = 0$ for $|t| \leq \frac{\varepsilon}{2}$, and $\eta(t) = 1$ for $|t| \geq \varepsilon$. We define $v \in C^\infty(\mathbb{R}^d)$ by

$$v(x) \;:=\; \eta(|x-z|)\Phi_k(x-z)\,, \qquad x \in \mathbb{R}^d\,.$$

Then v satisfies $v = \Phi_k(\,\cdot\, - z)$ in $\mathbb{R}^d \setminus \overline{D}$, and $\Delta v + k^2 v = 0$ for $|x - z| \geq \varepsilon$. Accordingly, Green's representation theorem in the interior (see, e.g., [29, Thm. 3.3]) gives for any $x \in D$ that

$$\begin{aligned} v(x) \;&=\; \int_{\partial D} \left(\Phi_k(x-y)\frac{\partial v(y)}{\partial \nu} - v(y)\frac{\partial \Phi_k(x-y)}{\partial \nu(y)}\right) \mathrm{d}s(y) \\ &\qquad - \int_D \bigl(\Delta v(y) + k^2 v(y)\bigr)\,\Phi_k(x-y)\;\mathrm{d}y \\ &=\; -\int_{B_\varepsilon(z)} \bigl(\Delta v(y) + k^2 v(y)\bigr)\,\Phi_k(x-y)\;\mathrm{d}y\,. \end{aligned}$$

The boundary integral is zero by Green's representation theorem in the exterior (see, e.g., [29, Thm. 3.6]). Since $v = \Phi_k(\,\cdot\, - z)$ in $\mathbb{R}^d \setminus \overline{D}$, we find that

$$\phi_z(\widehat{x}) \;=\; v^\infty(\widehat{x}) \;=\; -\int_{B_\varepsilon(z)} \bigl(\Delta v(y) + k^2 v(y)\bigr)\, e^{-\mathrm{i}k\widehat{x}\cdot y}\;\mathrm{d}y\,, \qquad \widehat{x} \in S^{d-1}\,.$$

Defining $\psi \in L^2(D)$ by

$$\psi(x) \;:=\; \begin{cases} -\Delta v(x) - k^2 v(x)\,, & x \in B_\varepsilon(z)\,, \\ 0\,, & x \in D \setminus B_\varepsilon(z)\,, \end{cases}$$

we obtain that $\phi_z = H^*\psi$.

Let now $z \notin D$ and assume to the contrary that $H^*\psi = \phi_z$ for some $\psi \in L^2(D)$. Then by Rellich's lemma (see, e.g., [9, Lmm. 2.12]) and the analyticity of solutions to the Helmholtz equation with constant coefficient,

$$\int_D \psi(y)\Phi_k(x-z)\;\mathrm{d}y \;=\; \Phi_k(x-z) \qquad \text{for all } x \in \mathbb{R}^d \setminus \bigl(\overline{D} \cup \{z\}\bigr)\,.$$

However, the left hand side is in $H^2_{\mathrm{loc}}(\mathbb{R}^d)$ (see, e.g., [27, Thm. 7.11]), while $\Phi_k(\,\cdot\, - z) \notin H^2(B)$ for any bounded set containing z in its interior (for $z \in \partial D$ one considers a bounded subset B of an open cone with vertex at z and $B \cap D = \emptyset$ and shows that $\Phi_k(\,\cdot\, - z) \notin H^2(B)$). ☐

Combining the results above, we obtain the main result of this section.

Theorem 4.6. *Suppose that Assumption 4.1 holds, and let $\delta > 0$ be as in Proposition 2.6. Let $C > 0$ be the constant in (4.3b), and let*

$$\rho := \min\left\{ \frac{\delta}{\omega_{d-1}^{1/2}} , \frac{1}{\omega_{d-1}^{1/2}} \Big(\frac{1}{2C} \Big)^{1/\alpha} \right\}$$

Then, for any $0 < \widetilde{\rho} \leq \rho$ and $z \in \mathbb{R}^d$,

$$z \in D \iff \inf\left\{ \frac{|\langle F(g), g \rangle_{L^2(S^{d-1})}|}{|\langle g, \phi_z \rangle_{L^2(S^{d-1})}|^2} \;\middle|\; g \in L^2(S^{d-1}) \,,\, \|g\|_{L^2(S^{d-1})} = \widetilde{\rho} \,,\, \langle g, \phi_z \rangle_{L^2(S^{d-1})} \neq 0 \right\} > 0 \,. \tag{4.5}$$

Proof. By Proposition 4.5 we know that $z \in D$ is equivalent to $\phi_z \in \mathcal{R}(H^*)$, which, by Theorem 4.2, is in turn equivalent to the condition on the right hand side of (4.5) provided that the nonlinear far field operator F admits the factorization $F = H^* T(H)$ for an operator T satisfying estimates as in Theorem 4.2. This has been shown in Corollary 4.4. Note that our choice of ρ guarantees the existence of the far field operator (see Proposition 2.6) as well as the coercivity estimate in Corollary 4.4. $\square$

We will comment on a numerical implementation of this criterion in Section 6 below. For numerical implementations in the linear case we refer, e.g., to [25, 28].

5 The nonlinear monotonicity method

In this section we consider general nonlinear contrast functions $q \in L^\infty(\mathbb{R}^d \times \mathbb{R})$ as in Section 4, but we waive the assumption on k^2 not being a transmission eigenvalue.

Assumption 5.1. *Let $D \subseteq \mathbb{R}^d$ be open and Lipschitz bounded such that $\mathbb{R}^d \setminus \overline{D}$ is connected. Then the nonlinear contrast function $q \in L^\infty(\mathbb{R}^d \times \mathbb{R})$ shall satisfy*

(i) $z \in \mathbb{C} \mapsto q(\,\cdot\,, |z|) \in L^\infty(\mathbb{R}^d)$ *is continuous,*

(ii) $q(\,\cdot\,, |z|) = 0$ *a.e. in* $\mathbb{R}^d \setminus D$ *for all* $z \in \mathbb{C}$,

(iii) there exists $q_0 \in L^\infty(\mathbb{R}^d)$ with $0 < q_{0,\min} \leq q_0 \leq q_{0,\max} < \infty$ a.e. in D for some $q_{0,\min}, q_{0,\max} > 0$, $q_0 = 0$ a.e. in $\mathbb{R}^d \setminus D$, and there exists a parameter $\alpha > 0$ such that for any $z_1, z_2 \in \mathbb{C}$ with $|z_1|, |z_2| \leq 1$,

$$\big\| q(\,\cdot\,, |z_1|) z_1 - q(\,\cdot\,, |z_2|) z_2 - q_0 (z_1 - z_2) \big\|_{L^\infty(\mathbb{R}^d)} \leq C_q (|z_1|^\alpha + |z_2|^\alpha) |z_1 - z_2| \,.$$

Given any open and bounded subset $B \subseteq \mathbb{R}^d$, we define the associated *probing operator* $P_B : L^2(S^{d-1}) \to L^2(S^{d-1})$ by

$$P_B g := k^2 H_B^* H_B g \,,$$

where $H_B : L^2(S^{d-1}) \to L^2(B)$ and $H_B^* : L^2(B) \to L^2(S^{d-1})$ are given as in (2.18) and (2.19) with D replaced by B. Accordingly, we find that for all $g \in L^2(S^{d-1})$,

$$\langle P_B g, g \rangle_{L^2(S^{d-1})} = k^2 \int_B |Hg|^2 \,\mathrm{d}x = k^2 \int_B \left| \int_{S^{d-1}} e^{\mathrm{i}k\theta\cdot x} g(\theta) \,\mathrm{d}s(\theta) \right|^2 \mathrm{d}x \,. \tag{5.1}$$

The operator P_B is bounded, compact, and self-adjoint.

Theorem 5.2. *Suppose that Assumption 5.1 holds, and let $\delta > 0$ be as in Proposition 2.6. Let $B \subseteq \mathbb{R}^d$ be open and bounded, and let*

$$\rho := \min\left\{ \frac{\delta}{\omega_{d-1}^{1/2}}, \frac{1}{\omega_{d-1}^{1/2}} \left(\frac{k^2 q_{0,\min}}{2\,C} \right)^{\frac{1}{\alpha}} \right\},$$

where $C > 0$ is the constant from (2.25). For any $0 < \widetilde{\rho} \leq \rho$ the following characterization of D holds.

(a) If $B \subseteq D$, then there exists a finite dimensional subspace $\mathcal{V} \subseteq L^2(S^{d-1})$ such that, for all $\beta \leq \frac{q_{0,\min}}{2}$,

$$\beta \langle P_B g, g\rangle_{L^2(S^{d-1})} \leq \mathrm{Re}\big(\langle F(g), g\rangle_{L^2(S^{d-1})}\big) \quad \text{for all } g \in \mathcal{D}(F) \cap \mathcal{V}^\perp \text{ with } \|g\|_{L^2(S^{d-1})} = \widetilde{\rho}. \tag{5.2}$$

(b) If $B \not\subseteq D$, then there is no finite dimensional subspace $\mathcal{V} \subseteq L^2(S^{d-1})$ and no $\beta > 0$ such that (5.2) holds.

Proof. We consider the factorization of the far field operator $F = H^* T(H)$ as in (2.22). Accordingly, the linear far field operator corresponding to the contrast function q_0 satisfies $F_0 = H^* T_0 H$, and we obtain from (2.25) that, for all $g \in \mathcal{D}(F)$,

$$\begin{aligned}
\mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{F(g)}\,\mathrm{d}s \Big) &= \mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{F_0 g}\,\mathrm{d}s \Big) + \mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{(F - F_0)(g)}\,\mathrm{d}s \Big) \\
&\geq \mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{F_0 g}\,\mathrm{d}s \Big) - C \|Hg\|_{L^\infty(D)}^\alpha \|Hg\|_{L^2(D)}^2 .
\end{aligned}$$

Applying [13, Thm. 3.2] with $q_1 = 0$ and $q_2 = q_0$ we find that there exists a finite dimensional subspace $\mathcal{V} \subseteq L^2(S^{d-1})$ such that, for all $g \in \mathcal{D}(F) \cap \mathcal{V}^\perp$,

$$\begin{aligned}
\mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{F(g)}\,\mathrm{d}s \Big) &\geq k^2 \int_D q_0 |Hg|^2\,\mathrm{d}x - C\|Hg\|_{L^\infty(D)}^\alpha \int_D |Hg|^2\,\mathrm{d}x \\
&\geq k^2 \Big(q_{0,\min} - \frac{C\omega_{d-1}^{\alpha/2}}{k^2} \|g\|_{L^2(S^{d-1})}^\alpha \Big) \int_D |Hg|^2\,\mathrm{d}x .
\end{aligned}$$

Assuming that $\|g\|_{L^2(S^{d-1})} = \widetilde{\rho}$ we obtain that

$$\mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{F(g)}\,\mathrm{d}s \Big) \geq k^2 \frac{q_{0,\min}}{2} \int_D |Hg|^2\,\mathrm{d}x .$$

Moreover, if $B \subseteq D$ and $\beta \leq \frac{q_{0,\min}}{2}$, then

$$\beta \int_{S^{d-1}} g\overline{P_B g}\,\mathrm{d}s = k^2 \beta \int_B |Hg|^2\,\mathrm{d}x \leq k^2 \frac{q_{0,\min}}{2} \int_D |Hg|^2\,\mathrm{d}x ,$$

which shows part (a).

We prove part (b) by contradiction. Let $B \not\subseteq D$, $\beta > 0$, and assume that

$$\beta \langle P_B g, g\rangle_{L^2(S^{d-1})} \leq \mathrm{Re}\big(\langle F(g), g\rangle_{L^2(S^{d-1})}\big) \qquad \text{for all } g \in \mathcal{V}_1^\perp \text{ with } \|g\|_{L^2(S^{d-1})} = \widetilde{\rho} \tag{5.3}$$

for some $0 < \widetilde{\rho} \leq \rho$ and a finite dimensional subspace $\mathcal{V}_1 \subseteq L^2(S^{d-1})$. Using (2.25) we find that

$$\begin{aligned}
\mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{F(g)}\,\mathrm{d}s \Big) &= \mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{F_0 g}\,\mathrm{d}s \Big) + \mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{(F - F_0)(g)}\,\mathrm{d}s \Big) \\
&\leq \mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{F_0 g}\,\mathrm{d}s \Big) + C \|Hg\|_{L^\infty(D)}^\alpha \|Hg\|_{L^2(D)}^2 .
\end{aligned}$$

Applying the monotonicity relation (3.3) in [13, Cor. 3.4] with $q_1 = 0$ and $q_2 = q_0$, shows that there exists a finite dimensional subspace $\mathcal{V}_2 \subseteq L^2(S^{d-1})$ such that

$$\mathrm{Re}\Big(\int_{S^{d-1}} g\,\overline{F_0 g}\,\mathrm{d}s\Big) \,\leq\, k^2 \int_D q_0 |V_0 H g|^2 \,\mathrm{d}x \qquad \text{for all } g \in \mathcal{V}_2^\perp\,. \tag{5.4}$$

Combining (5.3)–(5.4), we obtain for $\widetilde{\mathcal{V}} := \mathcal{V}_1 + \mathcal{V}_2$ that, for all $g \in \widetilde{\mathcal{V}}^\perp$ with $\|g\|_{L^2(S^{d-1})} = \widetilde{\rho}$,

$$\begin{aligned} k^2\beta\|Hg\|^2_{L^2(B)} \,&\leq\, k^2 \int_D q_0|V_0Hg|^2 \,\mathrm{d}x + C\|Hg\|^\alpha_{L^\infty(D)}\|Hg\|^2_{L^2(D)}\\ &\leq\, k^2 q_{0,\max}\int_D |V_0Hg|^2\,\mathrm{d}x + C\|Hg\|^\alpha_{L^\infty(D)}\|Hg\|^2_{L^2(D)}\,. \end{aligned}$$

Applying [13, Thm. 4.5] with $q_1 = 0$ and $q_2 = q_0$, this implies that there exists a constant $\widetilde{C} > 0$ such that, for all $g \in \widetilde{\mathcal{V}}^\perp$ with $\|g\|_{L^2(S^{d-1})} = \widetilde{\rho}$,

$$\begin{aligned} k^2\beta\|Hg\|^2_{L^2(B)} \,&\leq\, \big(\widetilde{C}k^2 q_{0,\max} + C\|Hg\|^\alpha_{L^\infty(D)}\big)\|Hg\|^2_{L^2(D)}\\ &\leq\, \big(\widetilde{C}k^2 q_{0,\max} + C\omega_{d-1}^{\alpha/2}\|g\|^\alpha_{L^2(D)}\big)\|Hg\|^2_{L^2(D)}\,. \end{aligned} \tag{5.5}$$

In the following we denote by $\mathcal{P}_\mathcal{V} : L^2(S^{d-1}) \to L^2(S^{d-1})$ the orthogonal projection onto $\mathcal{V}$. Using [13, Lmm. 4.4] we obtain as in the proof of [13, Thm. 4.1] a sequence $(\widetilde{g}_m)_{m\in\mathbb{N}} \subseteq L^2(S^{d-1})$ such that $\|\widetilde{g}_m\|_{L^2(S^{d-1})} = \rho/2$, and

$$\|H\widetilde{g}_m\|_{L^2(B)} \,\geq\, m\big(\|H\widetilde{g}_m\|_{L^2(D)} + \|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})}\big)\,, \qquad m \in \mathbb{N}\,.$$

Therefore, $g_m := \widetilde{g}_m - \mathcal{P}_\mathcal{V}\widetilde{g}_m \in \mathcal{V}^\perp$ and by rescaling $\widetilde{g}_m$ we can assume without loss of generality that $\|g_m\|_{L^2(S^{d-1})} = \widetilde{\rho} \leq \rho$. Accordingly, if $\|H\| < 1$, then

$$\begin{aligned} \|Hg_m\|_{L^2(B)} \,&\geq\, \|H\widetilde{g}_m\|_{L^2(B)} - \|H_B\|\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})}\\ &\geq\, m\|H\widetilde{g}_m\|_{L^2(D)} + m\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})} - \|H_B\|\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})}\\ &\geq\, m\|Hg_m\|_{L^2(D)} + \big(m(1-\|H\|) - \|H_B\|\big)\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})}\\ &\geq\, m\|Hg_m\|_{L^2(D)} \end{aligned}$$

for all $m \in \mathbb{N}$ such that $m \geq \|H_B\|/(1-\|H\|)$. On the other hand, if $\|H\| \geq 1$, then

$$\begin{aligned} \|Hg_m\|_{L^2(B)} \,&\geq\, \|H\widetilde{g}_m\|_{L^2(B)} - \|H_B\|\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})}\\ &\geq\, m\|H\widetilde{g}_m\|_{L^2(D)} + m\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})} - \|H_B\|\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})}\\ &\geq\, \frac{m}{2\|H\|}\|H\widetilde{g}_m\|_{L^2(D)} + m\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})} - \|H_B\|\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})}\\ &\geq\, \frac{m}{2\|H\|}\|Hg_m\|_{L^2(D)} + \Big(\frac{m}{2} - \|H_B\|\Big)\|\mathcal{P}_\mathcal{V}\widetilde{g}_m\|_{L^2(S^{d-1})}\\ &\geq\, \frac{m}{2\|H\|}\|Hg_m\|_{L^2(D)}\,. \end{aligned}$$

for all $m \in \mathbb{N}$ with $m \geq 2\|H_B\|$. This contradicts (5.5), and we have shown part (b). □

Remark 5.3 (Numerical implementation of Theorem 5.2). Considering for any $z \in \mathbb{R}^d$ a probing domain $B = B_\varepsilon(z)$ that is a ball of radius $\varepsilon > 0$ around z, the identity (5.1) gives

$$\begin{aligned} \langle P_B g, g\rangle_{L^2(S^{d-1})} \,&=\, k^2\int_{B_\varepsilon(z)}\Big|\int_{S^{d-1}} e^{\mathrm{i}k\theta\cdot z}e^{\mathrm{i}k\theta\cdot(x-z)}g(\theta)\,\mathrm{d}s(\theta)\Big|^2\,\mathrm{d}x\\ &=\, k^2|B_\varepsilon(z)|\,\Big|\int_{S^{d-1}} e^{\mathrm{i}k\theta\cdot z}g(\theta)\,\mathrm{d}s(\theta)\Big|^2 + O\big(k^3\varepsilon|B_\varepsilon(z)|\|g\|^2_{L^2(S^{d-1})}\big)\\ &=\, k^2|B_\varepsilon(z)|\,|\langle g, \phi_z\rangle_{L^2(S^{d-1})}|^2 + O\big(k^3\varepsilon|B_\varepsilon(z)|\|g\|^2_{L^2(S^{d-1})}\big)\,, \end{aligned}$$

uniformly with respect to $z \in \mathbb{R}^d$. Here we used that $|e^{it} - 1| \le |t|$ for $t \in \mathbb{R}$.

If $z \in D$, then part (a) of Theorem 5.2 implies that there is a finite dimensional subspace $\mathcal{V} \subseteq L^2(S^{d-1})$ such that for all $\beta \le \frac{q_{0,\min}}{2}$ and for all $g \in \mathcal{V}^\perp$ with $\|g\|_{L^2(S^{d-1})} = \widetilde{\rho}$,

$$\frac{\mathrm{Re}\big(\langle F(g), g\rangle_{L^2(S^{d-1})}\big)}{\beta\langle P_B g, g\rangle_{L^2(S^{d-1})}} = \frac{\mathrm{Re}\big(\langle F(g), g\rangle_{L^2(S^{d-1})}\big)}{\beta k^2 |B_\varepsilon(z)| |\langle \phi_z, g\rangle_{L^2(S^{d-1})}|^2 + O\big(k^3\varepsilon |B_\varepsilon(z)| \|g\|^2_{L^2(S^{d-1})}\big)} \ge 1\,, \tag{5.6}$$

i.e.,

$$\frac{\mathrm{Re}\big(\langle F(g), g\rangle_{L^2(S^{d-1})}\big)}{|\langle \phi_z, g\rangle_{L^2(S^{d-1})}|^2 + O\big(k\varepsilon \|g\|^2_{L^2(S^{d-1})}\big)} \ge k^2\beta|B_\varepsilon|\,, \tag{5.7}$$

as $\varepsilon \to 0$. This shows that for any fixed $g \in \mathcal{V}^\perp$ with $\|g\|_{L^2(S^{d-1})} = \widetilde{\rho}$ and $\langle g, \phi_z\rangle_{L^2(S^{d-1})} \neq 0$ we can choose $\varepsilon > 0$ sufficiently small such that

$$\frac{\mathrm{Re}\big(\langle F(g), g\rangle_{L^2(S^{d-1})}\big)}{|\langle \phi_z, g\rangle_{L^2(S^{d-1})}|^2} \ge \frac{k^2\beta|B_\varepsilon|}{2}\,.$$

Similarly, if $z \notin D$, then part (b) of Theorem 5.2 says that there is no finite dimensional subspace $\mathcal{W} \subseteq L^2(S^{d-1})$ and no $\beta > 0$ such that (5.6)–(5.7) hold for all $g \in \mathcal{W}^\perp$ with $\|g\|_{L^2(S^{d-1})} = \widetilde{\rho}$ as $\varepsilon \to 0$.

Assuming that $\phi_z \notin \mathcal{V}$, this says that

$$z \in D \iff \inf\left\{ \frac{\mathrm{Re}\big(\langle F(g), g\rangle_{L^2(S^{d-1})}\big)}{|\langle \phi_z, g\rangle_{L^2(S^{d-1})}|^2} \;\middle|\; g \in \mathcal{V}^\perp\,,\ \|g\|_{L^2(S^{d-1})} = \widetilde{\rho},\ \langle g, \phi_z\rangle_{L^2(S^{d-1})} \neq 0 \right\} > 0\,. \tag{5.8}$$

This is closely related to the inf-criterion from the nonlinear factorization method in (4.5). For the monotonicity criterion we have to exclude the finite dimensional subspace $\mathcal{V}^\perp$, and we assumed that $\phi_z \notin \mathcal{V}$ in the derivation of (5.8), while for the factorization method we had to assume that k^2 is such that the homogeneous linear transmission eigenvalue problem has no nontrivial solution. $\diamondsuit$

In Section 6, we will use (5.8) to implement the nonlinear monotonicity based reconstruction method. However, since the finite dimensional subspace $\mathcal{V}^\perp$ that has to be excluded is a priori unknown, we will neglect this constraint. For a numerical implementation in the linear case we refer to [13].

6 Numerical examples

In this section we comment on a numerical implementation of the shape characterizations in Theorems 4.6 and 5.2. We consider the two-dimensional case only, i.e., $d = 2$.

Let $D \subseteq \mathbb{R}^2$ be open and Lipschitz bounded such that $D \subseteq B_R(0)$ for some $R > 0$ sufficiently large and $\mathbb{R}^2 \setminus \overline{D}$ is connected. We consider at third-order Kerr-type nonlinear material law that is given by

$$q(x, |z|) := q_0(x) + q_1(x)|z|^2\,, \qquad x \in \mathbb{R}^2\,,\ z \in \mathbb{C}\,, \tag{6.1}$$

where $q_0, q_1 \in L^\infty(\mathbb{R}^2)$ with support in $\overline{D}$ and $\operatorname{ess\,inf} q_0 > -1$. Accordingly, the scattering problem (2.4) consists in determining $u = u^i + u^s$ such that

$$\Delta u + k^2\big(1 + q_0 + q_1|u|^2\big)u = 0 \qquad \text{in } \mathbb{R}^2\,,$$

and u^s satisfies the Sommerfeld radiation condition. This fits into the framework of the previous sections.

We evaluate approximate solutions of this nonlinear scattering problem using a fixed point iteration for the nonlinear Lippmann-Schwinger equation

$$u^s(x) \;=\; k^2 \int_D \Phi_k(x-y) q(x, |u^i(y) + u^s(y)|)(u^i(y) + u^s(y)) \,\mathrm{d}y\,, \qquad x \in [-R, R]^2\,,$$

as in the proof of Proposition 2.6. Denoting the solution to the linear problem by

$$u_0^s \;:=\; \bigl(I - k^2\Phi_k * (q_0\,\cdot\,)\bigr)^{-1}\bigl(k^2\Phi_k * (q_0 u^i)\bigr) \qquad \text{on } [-R, R]^2\,, \tag{6.2}$$

the fixed point iteration determines the difference $w := u^s - u_0^s$. Starting with the initial guess $w_0 = 0$ on $[-R, R]^2$ we evaluate, for $\ell = 0, 1, 2, \dots,$

$$w_{\ell+1} \;:=\; \bigl(I - k^2\Phi_k * (q_0\,\cdot\,)\bigr)^{-1}\Bigl(k^2\Phi_k * \bigl(q_1|w_\ell + u_0^s + u^i|^2(w_\ell + u_0^s + u^i)\bigr)\Bigr) \qquad \text{on } [-R, R]^2\,. \tag{6.3}$$

We have seen in the proof of Proposition 2.6 that this fixed point iteration converges whenever the product $\|q_1\|_{L^\infty(D)}\|u^i\|_{L^\infty(D)}$ is sufficiently small (see Remark 2.7). In our numerical example below we stop the fixed point iteration when

$$\frac{\|w_{\ell+1} - w_\ell\|_{L^\infty([-R,R]^2)}}{\|w_{\ell+1}\|_{L^\infty([-R,R]^2)}} \;<\; \varepsilon \tag{6.4}$$

for some tolerance $\varepsilon > 0$, and we denote the final iterate by $w_\varepsilon \approx w$. Accordingly, an approximation for the far field pattern u^∞ can be evaluated using Proposition 2.8 by

$$u_\varepsilon^\infty(\widehat{x}) \;=\; k^2 \int_D \bigl(q_0(y) + q_1(y)|w_\varepsilon(y) + u_0^s(y) + u^i(y)|^2\bigr)\bigl(w_\varepsilon(y) + u_0^s(y) + u^i(y)\bigr)e^{-\mathrm{i}k\widehat{x}\cdot y} \,\mathrm{d}y\,, \quad \widehat{x} \in S^1\,.$$

In (6.2) and in each step of the fixed point iteration (6.3) we have to solve a linear Lippmann-Schwinger integral equation. For this purpose we use the simple cubature method from [45, Sec. 2].

Next we turn to the inverse scattering problem. We consider an equidistant grid of points

$$\triangle \;=\; \{z_{ij} = (ih, jh) \mid -J \le i, j \le J\} \;\subseteq\; [-R, R]^2 \tag{6.5}$$

with step size $h = R/J$ in the region of interest $[-R, R]^2$. For each $z_{ij} \in \triangle$ we approximate a solution to the minimization problem

$$\text{Minimize} \quad \frac{|\langle F(g), g\rangle_{L^2(S^1)}|}{|\langle g, \phi_{z_{ij}}\rangle_{L^2(S^1)}|^2} \quad \text{subject to} \quad \|g\|_{L^2(S^1)} = \widetilde{\rho} \quad \text{and} \quad \langle g, \phi_{z_{ij}}\rangle_{L^2(S^1)} \neq 0 \tag{6.6}$$

for the nonlinear factorization method (see Theorem 4.6), and

$$\text{Minimize} \quad \frac{\mathrm{Re}\bigl(\langle F(g), g\rangle_{L^2(S^1)}\bigr)}{|\langle \phi_{z_{ij}}, g\rangle_{L^2(S^1)}|^2} \quad \text{subject to} \quad \|g\|_{L^2(S^1)} = \widetilde{\rho} \quad \text{and} \quad \langle g, \phi_{z_{ij}}\rangle_{L^2(S^1)} \neq 0 \tag{6.7}$$

for the nonlinear monotonicity method (see Theorems 5.2 and Remark 5.3).

We use a composite trapezoid rule on an equidistant grid of points

$$\{(\cos\phi_m, \sin\phi_m) \mid \phi_m = 2\pi m/M\,,\; m = 0, \dots, M-1\} \subseteq S^1\,, \qquad M \in \mathbb{N}\,, \tag{6.8}$$

to approximate the inner products in (6.6) and (6.7), and we discretize the densities $g \in L^2(S^1)$ using a truncated Fourier series expansion

$$g(\cos(t), \sin(t)) \;=\; \sum_{n=-N/2}^{N/2-1} \widehat{g}_n \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}nt}\,, \qquad t \in [0, 2\pi)\,,\; N/2 \in \mathbb{N}\,. \tag{6.9}$$

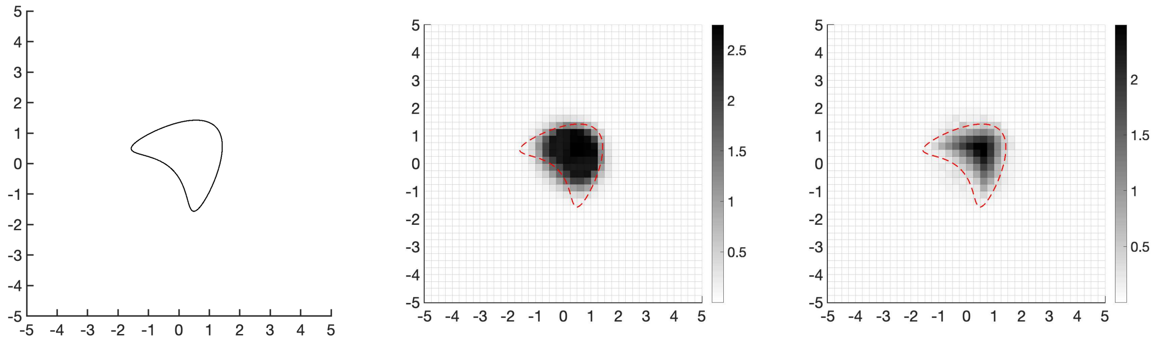

Figure 6.1: Nonlinear factorization method: Exact shape of the scattering object (left), initial guess $I_{\mathrm{fac}}^{(0)}$ (center), final result I_{fac} (right).

Accordingly, we minimize (6.6) and (6.7) with respect to the finite dimensional vector of Fourier coefficients $[\widehat{g}_{-N/2}, \dots, \widehat{g}_{N/2-1}]^\top \in \mathbb{C}^N$. From our theoretical results in Theorems 4.6 and 5.2 (see also Remark 5.3), we expect the values of the minima in (6.6) and (6.7) to be close to zero when $z \in \mathbb{R}^2 \setminus D$, and significantly larger than zero when $z \in D$.

In each grid point $z_{ij} \in \triangle$ we approximate solutions of (6.6) and (6.7) using the interior point method provided by Matlab's `fmincon`. To find an appropriate initial guess $g_{ij}^{(0)}$ at each sampling point $z_{ij} \in \triangle$, we first perform a preliminary global search and evaluate

$$g_{ij}^{(0)} := \operatorname{argmin}_{p,\ell,z} \frac{|\langle F(g_{p,\ell,z}), g_{p,\ell,z}\rangle_{L^2(S^1)}|}{|\langle g_{p,\ell,z}, \phi_{z_{ij}}\rangle_{L^2(S^1)}|^2} \tag{6.10}$$

for the optimization problem (6.6) and

$$g_{ij}^{(0)} := \operatorname{argmin}_{p,\ell,z} \frac{\operatorname{Re}\big(\langle F(g_{p,\ell,z}), g_{p,\ell,z}\rangle_{L^2(S^1)}\big)}{|\langle \phi_{z_{ij}}, g_{p,\ell,z}\rangle_{L^2(S^1)}|^2} \tag{6.11}$$

for the optimization problem (6.7). Here, $g_{p,\ell,z} \in L^2(S^1)$ is given by

$$g_{p,\ell,z}(\cos(t), \sin(t)) = \widetilde{\rho}\, \mathrm{i}^p \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}\ell t} e^{-\mathrm{i}k(z_1\cos(t)+z_2\sin(t))},$$

and the minimization in (6.10) and (6.11) is over $p = 0, 1$, $\ell = -N/2, \dots, N/2-1$, and $z = (z_1, z_2) \in \triangle$. The densities $g_{p,\ell,z}$ generate shifted Herglotz incident fields $(Hg_\ell)(x-z)$, where g_ℓ has just one active Fourier mode.

For each $z_{ij} \in \triangle$ we denote the values of the final result of the optimization by $I_{\mathrm{fac}}(z_{ij})$ for (6.6) and $I_{\mathrm{mon}}(z_{ij})$ for (6.7). Color coded plots of these indicator functions should give a reconstruction of the support D of the scattering object.

Example 6.1. We consider a kite shaped scattering object D as shown in the left plot in Figure 6.1 and in the left plot in Figure 6.2. The coefficients in the Kerr-type nonlinear material law in (6.1) are determined to be

$$q_0 = \begin{cases} 1.16 & \text{in } D\,, \\ 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}\,, \end{cases} \qquad \text{and} \qquad q_1 = \begin{cases} 2.5 \cdot 10^{-22} & \text{in } D\,, \\ 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}\,. \end{cases}$$

These coefficients correspond to fused silica (see table 4.1.2 on p. 212 in [3] with $q_0 = n_0^2 - 1$ and $q_1 = \chi^{(3)}$). For the wave number in the exterior we choose $k = 1$, and the norm constraint in (6.6) and (6.7) is set to $\widetilde{\rho} = 3.0 \times 10^{10}$.

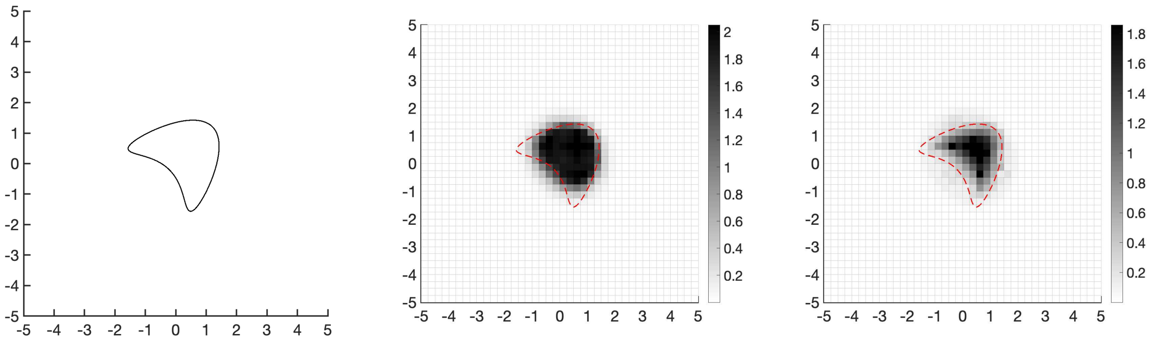

Figure 6.2: Nonlinear monotonicity method: Exact shape of the scattering object (left), initial guess $I_{\mathrm{mon}}^{(0)}$ (center), final result I_{mon} (right).

A simple rescaling argument shows that we can equivalently work with

$$u_{\mathrm{resc}} := u/\tau\,, \quad q_{1,\mathrm{resc}} := \tau^2 q_1\,, \quad \text{and} \quad \widetilde{\rho}_{\mathrm{resc}} := \widetilde{\rho}/\tau \qquad \text{for any } \tau > 0\,.$$

In the numerical implementation we choose $\tau = 3.0 \times 10^{10}$, i.e., $q_{1,\mathrm{resc}} = 0.26\,\mathbf{1}_D$ and $\widetilde{\rho}_{\mathrm{resc}} = 1$. We use a sampling grid $\triangle$ as in (6.5) with $R = 5$ and $J = 20$, i.e., the step size in each direction is $h = 0.25$. Furthermore, we choose $M = 256$ quadrature nodes in (6.8), $N = 16$ Fourier modes in (6.9), and for the tolerance in (6.4) we choose $\varepsilon = 10^{-5}$. We compute the starting guess for the optimization (6.6) and (6.7) for each sampling point $z_{ij} \in \triangle$ as in (6.10) or (6.11). The corresponding values of the cost functional in (6.6) and (6.7) for each grid point $z_{ij} \in \triangle$ are denoted by $I_{\mathrm{fac}}^{(0)}(z_{ij})$ and $I_{\mathrm{fac}}^{(0)}(z_{ij})$, respectively. Color coded plots of $I_{\mathrm{fac}}^{(0)}$ and $I_{\mathrm{mon}}^{(0)}$ are shown in Figure 6.1 (center) and Figure 6.2 (center), respectively. These give already a reasonable reconstruction of the location of the nonlinear scattering object. The dashed lines indicate the exact geometry of the scatterer.

Then we approximate solutions to the optimization problems (6.6) and (6.7) for each sampling point $z_{ij} \in \triangle$ using Matlab's `fmincon` algorithm. These approximations are denoted by $I_{\mathrm{fac}}(z_{ij})$ and $I_{\mathrm{mon}}(z_{ij})$, respectively. Color coded plots of the indicator functions I_{fac} and I_{mon} for the nonlinear factorization method and for the nonlinear monotonicity method are shown in Figure 6.1 (right) and Figure 6.2 (right), respectively. Again the dashed lines indicate the exact geometry of the scatterer. The results obtained by the two methods are of similar quality. A significant improvement of the reconstruction is observed when compared to the initial guesses. The shape of the support of the scattering object is nicely recovered. $\Diamond$

Conclusions

We have discussed a direct and inverse scattering problem for a class of nonlinear Helmholtz equations in unbounded free space. Assuming that the intensities of the incident waves are sufficiently small relative to the size of the nonlinearity, we have established existence and uniqueness of solutions to the direct and inverse scattering problem. Our analysis relies on linearization techniques and estimates for the linearization error. We have also considered extensions of two shape reconstruction techniques for the inverse scattering problem, and we have provided numerical examples.

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A Appendix: A useful estimate

In Lemma A.1 below we show a simple estimate that is used in the proof of Theorem 3.1, but that we have not been able to find in the literature.

Lemma A.1. *Let $a, b \in \mathbb{C}$ and $\alpha > 0$. Then,*

$$\big||a|^\alpha a - |b|^\alpha b\big| \,\leq\, 2(|a| + |b|)^\alpha |a - b| \,.$$

Proof. Without loss of generality we can assume that $|a| > |b| > 0$. Then $t := b/a \in \mathbb{C}$ satisfies $0 < |t| < 1$, and we are left to show that

$$\big|1 - |t|^\alpha t\big| \,\leq\, 2(1 + |t|)^\alpha |1 - t| \,. \tag{A.1}$$

Defining

$$f(\theta) \,:=\, \frac{1 - 2|t|^\alpha \theta + |t|^{2\alpha+2}}{1 - 2\theta + |t|^2} \,, \qquad \theta \in \mathbb{R} \,,\; \theta \neq \frac{1}{2}(1 + |t|^2) \,,$$

we have

$$f'(\theta) \,:=\, 2\frac{(1 - |t|^{\alpha+2})(1 - |t|^\alpha)}{(1 - 2\theta + |t|^2)^2} \,>\, 0 \,, \qquad \theta \in \mathbb{R} \,,\; \theta \neq \frac{1}{2}(1 + |t|^2) \,.$$

Accordingly, f is monotonically increasing on $(-\infty, \frac{1}{2}(1+|t|^2))$. Since $0 < (1-|t|)^2 = 1-2|t|+|t|^2$ implies that

$$\mathrm{Re}(t) \,\leq\, |t| \,<\, \frac{1}{2}(1 + |t|^2) \,,$$

we find that $f(\mathrm{Re}(t)) \leq f(|t|)$. This yields

$$\frac{\big|1 - |t|^\alpha t\big|^2}{|1 - t|^2} \,=\, \frac{1 - 2|t|^\alpha \,\mathrm{Re}(t) + |t|^{2\alpha+2}}{1 - 2\,\mathrm{Re}(t) + |t|^2} \,\leq\, \frac{1 - 2|t|^{\alpha+1} + |t|^{2\alpha+2}}{1 - 2|t| + |t|^2} \,=\, \frac{\big(1 - |t|^{\alpha+1}\big)^2}{(1 - |t|)^2} \,.$$

Therefore, it suffices to show that

$$\frac{1 - |t|^{1+\alpha}}{1 - |t|} \,\leq\, 2(1 + |t|)^\alpha \,.$$

Let $n := \lfloor \alpha \rfloor$ and $\beta := \alpha - n$. Then,

$$(1 + |t|)^n \,=\, \sum_{\ell=0}^{n} \binom{n}{\ell} |t|^\ell \,\geq\, \sum_{\ell=0}^{n} |t|^\ell \,=\, \frac{1 - |t|^{n+1}}{1 - |t|} \,,$$

and $2(1 + |t|)^\beta \geq 1 + |t|^\beta$. Accordingly,

$$2(1 + |t|)^\alpha \,\geq\, \frac{1 + |t|^\beta - |t|^{n+1} - |t|^{n+\beta+1}}{1 - |t|} \,\geq\, \frac{1 - |t|^{\alpha+1}}{1 - |t|} \,.$$

∎

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