SIAM J. Appl. Math.
Vol. 77, No. 1, pp. 154–180

UNCERTAINTY PRINCIPLES FOR INVERSE SOURCE PROBLEMS, FAR FIELD SPLITTING, AND DATA COMPLETION*

ROLAND GRIESMAIER† AND JOHN SYLVESTER‡

Abstract. Starting with far field data of time-harmonic acoustic or electromagnetic waves radiated by a collection of compactly supported sources in two-dimensional free space, we develop criteria and algorithms for the recovery of the far field components radiated by each of the individual sources, and the simultaneous restoration of missing data segments. Although both parts of this inverse problem are severely ill-conditioned in general, we give precise conditions relating the wavelength, the diameters of the supports of the individual source components and the distances between them, and the size of the missing data segments, which guarantee that stable recovery in the presence of noise is possible. The only additional requirement is that a priori information on the approximate location of the individual sources is available. We give analytic and numerical examples to confirm the sharpness of our results and to illustrate the performance of corresponding reconstruction algorithms, and we discuss consequences for stability and resolution in inverse source and inverse scattering problems.

Key words. inverse source problem, Helmholtz equation, uncertainty principles, far field splitting, data completion, stable recovery

AMS subject classifications. 35R30, 65N21

DOI. 10.1137/16M1086157

1. Introduction.

In signal processing, a classical uncertainty principle limits the time-bandwidth product $|T||W|$ of a signal, where $|T|$ is the measure of the essential support of the signal and $|W|$ is the measure of the essential support of its Fourier transform (cf., e.g., [7]). We may reformulate this principle as an inequality for the inner product of two functions, each of which satisfies a different support condition, i.e., *if* $\operatorname{supp}\phi \subseteq T$ *and* $\operatorname{supp}\widehat{\psi} \subseteq W$, then

$$|\langle \phi, \psi\rangle| \;\leq\; C\sqrt{|T||W|}\|\phi\|\|\psi\|\,. \tag{1.1}$$

The inequality (1.1) becomes a classical uncertainty principle when it is possible to choose $\psi = \phi$ and still satisfy the support conditions. This is the case for the N-point discrete Fourier transform, with the norm in (1.1) equal to the l^2-norm on sequences and $C = \frac{1}{N}$. The reformulation, however, is equally useful even when we cannot make this choice, as is the case for the Fourier transform on $L^2(\mathbb{R})$ with $C = 1$.

In the inverse source problem, the far field radiated by a source f is its *restricted* (to the unit sphere) *Fourier transform*, and the operator that maps the restricted Fourier transform of $f(x)$ to the restricted Fourier transform of its translate $f(x + c)$ is called the *far field translation operator*. We will prove an uncertainty principle analogous to (1.1), with the Fourier transform replaced by the far field translation operator. Combining this principle with a *regularized Picard criterion*, which characterizes the nonevanescent (i.e., detectable) far fields radiated by a (limited power) source supported in a ball, provides simple proofs and extensions of several results

---

*Received by the editors July 22, 2016; accepted for publication (in revised form) October 25, 2016; published electronically January 24, 2017.
http://www.siam.org/journals/siap/77-1/M108615.html
Funding: The work of the second author was partially supported by NSF grant DMS-1309362.

†Institut für Mathematik, Universität Würzburg, 97074 Würzburg, Germany (roland.griesmaier@uni-wuerzburg.de).
‡Department of Mathematics, University of Washington, Seattle, WA 98195 (sylvest@u.washington.edu).

about locating the support of a source and about splitting a far field radiated by well-separated sources into the far fields radiated by each source component.

We also combine the regularized Picard criterion with a more conventional uncertainty principle for the map from a far field in $L^2(S^1)$ to its Fourier coefficients. This leads to a data completion algorithm which tells us that we can deduce missing data (i.e., on part of S^1) if we know a priori that the source has small support. All of these results can be combined so that we can simultaneously complete the data and split the far fields into the components radiated by well-separated sources. We discuss both l^2 (least squares) and l^1 (basis pursuit) algorithms to accomplish this.

Perhaps the most significant point is that all of these algorithms come with bounds on their condition numbers (both the splitting and data completion problems are linear) which we show are sharp in their dependence on geometry and wavenumber. These results highlight an important difference between the inverse source problem and the inverse scattering problem. The conditioning of the linearized inverse scattering problem does not depend on wavenumber, which means that the conditioning does not deteriorate as we increase the wavenumber in order to increase resolution. The conditioning for splitting and data completion for the inverse source problem does, however, deteriorate with increased wavenumber, which means the dynamic range of the sensors must increase with wavenumber to obtain higher resolution.

We note that applications of classical uncertainty principles for the one-dimensional Fourier transform to data completion for band-limited signals have been developed in [7]. In this classical setting a problem that is somewhat similar to far field splitting is the representation of highly sparse signals in overcomplete dictionaries. Corresponding stability results for basis pursuit reconstruction algorithms have been established in [6].

The numerical algorithms for far field splitting that we are going to discuss have been developed and analyzed in [9, 10]. The novel mathematical contribution of the present work is the stability analysis for these algorithms based on new uncertainty principles, and their application to data completion. Alternative approaches to far field splitting, which do not include a rigorous analysis of stability, have been developed in [12, 19] (see also [11] for a method to separate time-dependent wave fields due to multiple sources).

This paper is organized as follows. In the next section we provide the theoretical background for the direct and inverse source problem for the two-dimensional Helmholtz equation with compactly supported sources. In section 3 we discuss the singular value decomposition of the restricted far field operator mapping sources supported in a ball to their radiated far fields, and we formulate the regularized Picard criterion to characterize nonevanescent far fields. In section 4 we discuss uncertainty principles for the far field translation operator and for the Fourier expansion of far fields, and in section 5 we utilize those to analyze the stability of least squares algorithms for far field splitting and data completion. Section 6 focuses on corresponding results for l^1 algorithms. Consequences of these stability estimates related to conditioning and resolution of reconstruction algorithms for inverse source and inverse scattering problems are considered in section 7, and in section 8–9 we provide some analytic and numerical examples.

2. Far fields radiated by compactly supported sources.

Suppose that $g \in L_0^2(\mathbb{R}^2)$ represents a compactly supported acoustic or electromagnetic source in the plane. Then the time-harmonic wave $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ radiated by g at *wavenumber*

$k > 0$ solves the *source problem* for the Helmholtz equation

$$-\Delta v - k^2 v \;=\; k^2 g \qquad \text{in } \mathbb{R}^2$$

and satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial v}{\partial r} - \mathrm{i}kv\Big) \;=\; 0\,, \qquad r = |x|\,.$$

We include the extra factor of k^2 on the right-hand side so that both v and g scale (under dilations) as functions; i.e., if $u(x) := v(kx)$ and $f(x) := g(kx)$, then

$$-\Delta u - u \;=\; f \qquad \text{in } \mathbb{R}^2 \qquad \text{and} \qquad \lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u}{\partial r} - \mathrm{i}u\Big) \;=\; 0\,. \tag{2.1}$$

With this scaling, distances are measured in wavelengths,[1] and this allows us to set $k = 1$ in our calculations and then easily restore the dependence on wavelength when we are done.

The *fundamental solution* of the Helmholtz equation (with $k = 1$) in two dimensions is

$$\Phi(x) \;:=\; \frac{\mathrm{i}}{4} H_0^{(1)}(|x|)\,, \qquad x \in \mathbb{R}^2 \setminus \{0\}\,,$$

so the solution to (2.1) can be written as a volume potential

$$u(x) \;=\; \int_{\mathbb{R}^2} \Phi(x-y) f(y)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^2\,.$$

The asymptotics of the Hankel function tell us that

$$u(x) \;=\; \frac{e^{\frac{\mathrm{i}\pi}{4}}}{\sqrt{8\pi}} \frac{e^{\mathrm{i}r}}{\sqrt{r}} \alpha(\theta_x) + O(r^{-\frac{3}{2}}) \qquad \text{as } r \to \infty\,,$$

where $x = r\theta_x$ with $\theta_x \in S^1$, and

$$\alpha(\theta_x) \;=\; \int_{\mathbb{R}^2} e^{-\mathrm{i}\theta_x \cdot y} f(y)\,\mathrm{d}y\,. \tag{2.2}$$

The function α is called the *far field* radiated by the source f, and (2.2) shows that the *far field operator* $\mathcal{F}$, which maps f to α, is a *restricted Fourier transform*, i.e.,

$$\mathcal{F} :\; L_0^2(\mathbb{R}^2) \to L^2(S^1)\,, \quad \mathcal{F}f \;:=\; \widehat{f}\big|_{S^1}\,. \tag{2.3}$$

The goal of the inverse source problem is to deduce properties of an unknown source $f \in L_0^2(\mathbb{R}^2)$ from observations of the far field. Clearly, any compactly supported source with Fourier transform that vanishes on the unit circle is in the nullspace $\mathcal{N}(\mathcal{F})$ of the far field operator. We call $f \in \mathcal{N}(\mathcal{F})$ a *nonradiating source* because a corollary of Rellich's lemma and unique continuation is that, if the far field vanishes, then the wave u vanishes on the unbounded connected component of the complement of the support of f. The nullspace of $\mathcal{F}$ is exactly

$$\mathcal{N}(\mathcal{F}) \;=\; \{g = -\Delta v - v \mid v \in H_0^2(\mathbb{R}^2)\}\,.$$

---

[1] One unit represents 2π wavelengths.

Neither the source f nor its support is uniquely determined by the far field, and, as nonradiating sources can have arbitrarily large supports, no upper bound on the support is possible. There are, however, well-defined notions of lower bounds. We say that a compact set $\Omega \subseteq \mathbb{R}^2$ *carries* α if every open neighborhood of Ω supports a source $f \in L_0^2(\mathbb{R}^2)$ that radiates α. The *convex scattering support* $\mathscr{C}(\alpha)$ of α, as defined in [16] (see also [17, 21]), is the intersection of all compact convex sets that carry α. The set $\mathscr{C}(\alpha)$ itself carries α, so that $\mathscr{C}(\alpha)$ is the smallest convex set which carries the far field α, and the convex hull of the support of the "true" source f must contain $\mathscr{C}(\alpha)$. Because two disjoint compact sets with connected complements cannot carry the same far field pattern (cf. [21, lemma 6]), it follows that $\mathscr{C}(\alpha)$ intersects any connected component of $\operatorname{supp}(f)$, as long as the corresponding source component is not nonradiating.

In [21], an analogous notion, the *UWSCS support*, was defined, showing that any far field with a compactly supported source is carried by a smallest union of well-separated convex sets (well-separated means that the distance between any two connected convex components is strictly greater than the diameter of any component). A corollary is that it makes theoretical sense to look for the support of a source with components that are small compared to the distance between them.

Here, as in previous investigations [9, 10], we study the well-posedness issues surrounding numerical algorithms to compute that support.

3. A regularized Picard criterion. If we consider the restriction of the source to far field map $\mathcal{F}$ from (2.3) to sources supported in the ball $B_R(0)$ of radius R centered at the origin, i.e.,

$$\mathcal{F}_{B_R(0)} : L^2(B_R(0)) \to L^2(S^1)\,, \quad \mathcal{F}_{B_R(0)} f := \widehat{f}\Big|_{S^1}\,, \tag{3.1}$$

we can write out a full singular value decomposition. We decompose $f \in L^2(B_R(0))$ as

$$f(x) = \left(\sum_{n=-\infty}^{\infty} f_n \, \mathrm{i}^n J_n(|x|) e^{\mathrm{i}n\varphi_x} \right) \oplus f_{\mathrm{NR}}(x)\,, \qquad x = |x|(\cos\varphi_x, \sin\varphi_x) \in B_R(0)\,,$$

where $\mathrm{i}^n J_n(|x|)e^{\mathrm{i}n\varphi_x}$, $n \in \mathbb{Z}$, span the closed subspace of *free sources*, which satisfy

$$-\Delta u - u = 0 \qquad \text{in } B_R(0)\,,$$

and f_{NR} belongs to the orthogonal complement of that subspace; i.e., f_{NR} is a nonradiating source.[2] The restricted far field operator $\mathcal{F}_{B_R(0)}$ maps

$$\mathcal{F}_{B_R(0)} : \mathrm{i}^n J_n(|x|) e^{\mathrm{i}n\varphi_x} \mapsto s_n^2(R) e^{\mathrm{i}n\theta}\,, \tag{3.2}$$

where

$$s_n^2(R) = 2\pi \int_0^R J_n^2(r) r \, \mathrm{d}r\,, \tag{3.3}$$

and the squared singular values of $\mathcal{F}_{B_R(0)}$ are given by $2\pi s_n^2(R)$, $n \in \mathbb{Z}$.

---

[2] Throughout, we identify $f \in L^2(B_R(0))$ with its continuation to $\mathbb{R}^2$ by zero whenever appropriate.

Denoting the Fourier coefficients of a far field $\alpha \in L^2(S^1)$ by

$$\alpha_n := \frac{1}{\sqrt{2\pi}} \int_{S^1} \alpha(\theta) e^{\mathrm{i}n\theta} \,\mathrm{d}\theta\,, \qquad n \in \mathbb{Z}\,, \tag{3.4}$$

so that

$$\alpha(\theta) = \sum_{n=-\infty}^{\infty} \alpha_n \frac{e^{\mathrm{i}n\theta}}{\sqrt{2\pi}}\,, \qquad \theta \in S^1\,,$$

and

$$\|\alpha\|^2_{L^2(S^1)} = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \tag{3.5}$$

by Parseval's identity, an immediate consequence of (3.2) is that

$$f^*_\alpha(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n}{s_n(R)^2}\, \mathrm{i}^n J_n(|x|) e^{\mathrm{i}n\varphi_x}\,, \qquad x \in B_R(0)\,, \tag{3.6}$$

which has L^2-norm

$$\|f^*_\alpha\|^2_{L^2(B_R(0))} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{|\alpha_n|^2}{s_n^2(R)}\,,$$

is the source with smallest L^2-norm that is supported in $B_R(0)$, and radiates the far field α. We refer to f^*_α as the *minimal power source* because, in electromagnetic applications, f^*_α is proportional to current density, so that, in a system with a constant internal resistance, $\|f^*_\alpha\|^2_{L^2(B_R(0))}$ is proportional to the input power required to radiate a far field. Similarly, $\|\alpha\|^2_{L^2(S^1)}$ measures the radiated power of the far field.

The rescaled (by a factor of $\frac{1}{2\pi}$) squared singular values $\{s_n^2(R)\}$ of the restricted Fourier transform $\mathcal{F}_{B_R(0)}$ have a number of interesting properties with immediate consequences for the inverse source problem; full proofs of the results discussed in the following can be found in the supplementary material (uncertainty_supplement.pdf [local/web 286KB]) in section SM1. The rescaled squared singular values satisfy

$$\sum_{n=-\infty}^{\infty} s_n^2(R) = \pi R^2\,, \tag{3.7}$$

and $s_n^2(R)$ decays rapidly as a function of n as soon as $|n| \geq R$,

$$s_n^2(R) \leq \frac{\pi 2^{\frac{2}{3}} n^{\frac{2}{3}}}{3^{\frac{4}{3}} (\Gamma(\frac{2}{3}))^2} \left(\frac{n+\frac{1}{2}}{n}\right)^{n+1} \left(\frac{R^2}{n^2} e^{1-\frac{R^2}{n^2}}\right)^n \frac{R^2}{n^2} \qquad \text{if } |n| \geq R\,. \tag{3.8}$$

Moreover, the odd and even rescaled squared singular values, $s_n^2(R)$, are decreasing (increasing) as functions of $n \geq 0$ ($n \leq 0$), and asymptotically

$$\lim_{R\to\infty} \frac{s^2_{\lceil \nu R \rceil}(R)}{2R} = \begin{cases} \sqrt{1-\nu^2}, & \nu \leq 1\,, \\ 0, & \nu \geq 1\,, \end{cases} \tag{3.9}$$

where $\lceil \nu R \rceil$ denotes the smallest integer that is greater than or equal to νR. This can also be seen in Figure 3.1, where we include plots of $s_n^2(R)$ (solid line) together

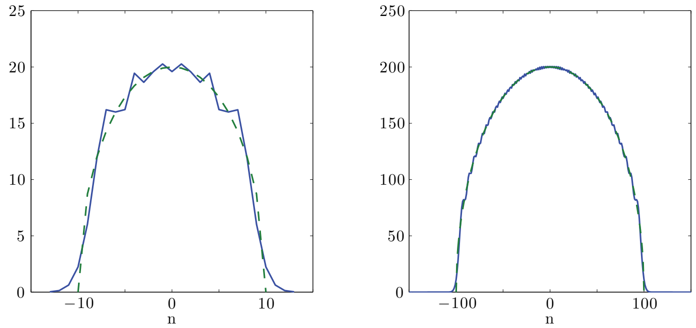

FIG. 3.1. *Rescaled squared singular values* $s_n^2(R)$ *(solid line) and asymptote* $2\sqrt{R^2-n^2}$ *(dashed line) for* $R=10$ *(left) and* $R=100$ *(right).*

with plots of the asymptote $2\sqrt{R^2-n^2}$ (dashed line) for $R=10$ (left) and $R=100$ (right). The asymptotic regime in (3.9) is already reached for moderate values of R.

The foregoing yields a very explicit understanding of the restricted Fourier transform $\mathcal{F}_{B_R(0)}$. For $|n| \lesssim R$ the rescaled singular values $s_n(R)$ are uniformly large, while for $|n| \gtrsim R$ the $s_n(R)$ are close to zero, and it is seen from (3.7)–(3.9) as well as from Figure 3.1 that as R gets large, the width of the n-interval in which $s_n(R)$ falls from uniformly large to zero decreases. Similar properties are known for the singular values of more classical restricted Fourier transforms (see [20]).

A physical source has *limited power*, which we denote by $P>0$, and a receiver has a *power threshold*, which we denote by $p>0$. If the radiated far field has power less than p, the receiver cannot detect it. Because $s_{-n}^2(R)=s_n^2(R)$ and the odd and even squared singular values, $2\pi s_n^2(R)$, are decreasing as functions of $n\geq 0$, we may define the following:

$$N(R,P,p) := \sup_{2\pi s_n^2(R)\geq \frac{p}{P}} n\,. \tag{3.10}$$

So, if $\alpha\in L^2(S^1)$ is a far field radiated by a limited power source supported in $B_R(0)$ with $\|f_\alpha^*\|^2_{L^2(B_R(0))}\leq P$, then, for $N=N(R,P,p)$,

$$P \geq \frac{1}{2\pi}\sum_{|n|>N}\frac{|\alpha_n|^2}{s_n^2(R)} \geq \frac{1}{2\pi}\frac{1}{s_{N+1}^2(R)}\sum_{|n|>N}|\alpha_n|^2 > \frac{P}{p}\sum_{|n|>N}|\alpha_n|^2\,.$$

Accordingly, $\sum_{|n|\geq N}|\alpha_n|^2 < p$ is below the power threshold. So the subspace of detectable far fields that can be radiated by a power limited source supported in $B_R(0)$ is

$$V_{\mathrm{NE}} := \left\{\alpha\in L^2(S^1)\;\middle|\;\alpha(\theta)=\sum_{n=-N}^{N}\alpha_n e^{\mathrm{i}n\theta}\right\}.$$

We refer to V_{NE} as the subspace of *nonevanescent far fields*, and to the orthogonal projection of a far field onto this subspace as the *nonevanescent* part of the far field. We use the term *nonevanescent* because it is the phenomenon of evanescence that explains why the rescaled singular values $s_n^2(R)$ decrease rapidly for $|n|\gtrsim R$, resulting in the fact that, for a wide range of p and P, $R<N(R,P,p)<1.5R$ if R is sufficiently large. This is also illustrated in Figure 3.2, where we include plots of $N(R,P,p)$ from (3.10) for $p/P=10^{-1}$, $p/P=10^{-4}$, and $p/P=10^{-8}$ and for varying R. The dotted lines in these plots correspond to $a_1(R):=R$ and $a_{1.5}(R):=1.5R$, respectively.

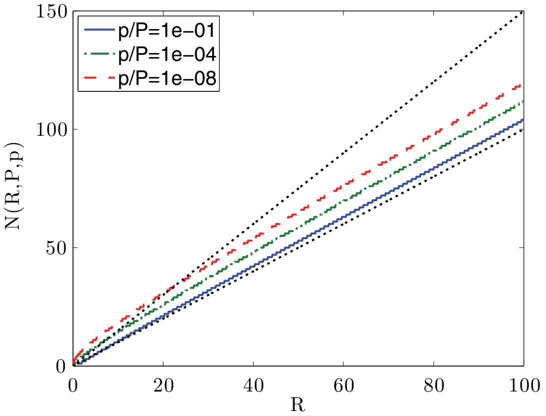

Fig. 3.2. *Threshold $N(R,P,p)$ as a function of R for different values of p/P. Dotted lines correspond to $a_1(R)=R$ and $a_{1.5}(R)=1.5R$.*

4. Uncertainty principles for far field translation.

In the inverse source problem, we seek to recover information about the size and location of the support of a source from observations of its far field. Because the far field is a restricted Fourier transform, the formula for the Fourier transform of the translation of a function,

$$\widehat{f(\cdot+c)}(\theta) = e^{\mathrm{i}c\cdot\theta}\widehat{f}(\theta)\,, \qquad \theta\in S^1\,,\ c\in\mathbb{R}^2\,,$$

plays an important role. We use T_c to denote the map from $L^2(S^1)$ to itself given by

$$T_c : \alpha\mapsto e^{\mathrm{i}c\cdot\theta}\alpha\,. \tag{4.1}$$

The mapping T_c acts on the Fourier coefficients $\{\alpha_n\}$ of α as a convolution operator, i.e., the Fourier coefficients $\{\alpha_m^c\}$ of $T_c\alpha$ satisfy

$$\alpha_m^c = \sum_{n=-\infty}^{\infty}\alpha_{m-n}(\mathrm{i}^n J_n(|c|)e^{\mathrm{i}n\varphi_c})\,, \qquad m\in\mathbb{Z}\,, \tag{4.2}$$

where $|c|$ and φ_c are the polar coordinates of c. Employing a slight abuse of notation, we also use T_c to denote the corresponding operator from l^2 to itself that maps

$$T_c : \{\alpha_n\}\mapsto\{\alpha_m^c\}\,. \tag{4.3}$$

Note that T_c is a unitary operator, i.e., $T_c^* = T_{-c}$.

The following theorem, which we call an *uncertainty principle for the translation operator*, will be the main ingredient in our analysis of far field splitting.

Theorem 4.1 (uncertainty principle for far field translation). *Let $\alpha,\beta\in L^2(S^1)$ such that the corresponding Fourier coefficients $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy* $\operatorname{supp}\{\alpha_n\}\subseteq W_1$ *and* $\operatorname{supp}\{\beta_n\}\subseteq W_2$ *with $W_1,W_2\subseteq\mathbb{Z}$, and let $c\in\mathbb{R}^2$. Then*

$$|\langle\alpha,T_c\beta\rangle_{L^2(S^1)}| \le \frac{\sqrt{|W_1||W_2|}}{|c|^{\frac{1}{3}}}\|\alpha\|_{L^2(S^1)}\|\beta\|_{L^2(S^1)}\,.$$

We will frequently be discussing properties of a far field α and those of its Fourier coefficients. The following notation will be a useful shorthand:

$$\|\alpha\|_{L^p} = \left(\int_{S^1} |\alpha(\theta)|^p \, \mathrm{d}\theta \right)^{\frac{1}{p}}, \qquad 1 \le p \le \infty, \tag{4.4}$$

$$\|\alpha\|_{l^p} = \left(\sum_{n=-\infty}^{\infty} |\alpha_n|^p \right)^{\frac{1}{p}}, \qquad 1 \le p \le \infty. \tag{4.5}$$

The notation emphasizes that we treat the representation of the function α by its values, or by the sequence of its Fourier coefficients as simply a way of inducing different norms. That is, both (4.4) and (4.5) describe different norms of the same function on S^1. Note that, because of the Plancherel equality (3.5), $\|\alpha\|_{L^2} = \|\alpha\|_{l^2}$, so we may just write $\|\alpha\|_2$, and we write $\langle \cdot, \cdot \rangle$ for the corresponding inner product.

Remark 4.2. We will extend the notation a little more and refer to the support of α in S^1 as its L^0-support and denote by $\|\alpha\|_{L^0}$ the measure of $\operatorname{supp}(\alpha) \subseteq S^1$. We will call the indices of the nonzero Fourier coefficients in its Fourier series expansion the l^0-support of α and use $\|\alpha\|_{l^0}$ to denote the number of nonzero coefficients.

With this notation, Theorem 4.1 becomes the following.

Theorem 4.3 (uncertainty principle for far field translation). *Let $\alpha, \beta \in L^2(S^1)$ and let $c \in \mathbb{R}^2$. Then*

$$|\langle \alpha, T_c \beta \rangle| \le \frac{\sqrt{\|\alpha\|_{l^0} \|\beta\|_{l^0}}}{|c|^{\frac{1}{3}}} \|\alpha\|_2 \|\beta\|_2 . \tag{4.6}$$

We refer to Theorem 4.3 as an uncertainty principle, because if we could take $\beta = T_c^* \alpha$ in (4.6), it would yield

$$1 \le \frac{\|\alpha\|_{l^0} \|T_c^* \alpha\|_{l^0}}{|c|^{\frac{2}{3}}} . \tag{4.7}$$

As stated, (4.7) is is true but not useful, because $\|\alpha\|_{l^0}$ and $\|T_c^* \alpha\|_{l^0}$ cannot simultaneously be finite.[3] We present the corollary only to illustrate the close analogy to Theorem 1 in [7], which treats the discrete Fourier transform (DFT) on sequences of length N.

Theorem 4.4 (uncertainty principle for the Fourier transform [7]). *If x represents the sequence $\{x_n\}$ for $n = 0, \ldots, N-1$ and $\widehat{x}$ its DFT, then*

$$1 \le \frac{\|x\|_{l^0} \|\widehat{x}\|_{l^0}}{N} .$$

This is a lower bound on the *time-bandwidth product.* In [7] Donoho and Stark present two important corollaries of uncertainty principles for the Fourier transform. The first is the uniqueness of sparse representations of a signal x as a superposition of vectors taken from both the standard basis and the basis of Fourier modes, and the second is the recovery of this representation by l^1 minimization.

---

[3] This would imply, using (3.6), that α could have been radiated by a source supported in an arbitrarily small ball centered at the origin, or centered at c, but Rellich's lemma and unique continuation show that no nonzero far field can have two sources with disjoint supports.

The main observation we make here is that if we phrase our uncertainty principle as in Theorem 4.3, then the far field translation operator, as well as the map from α to its Fourier coefficients, satisfies an uncertainty principle. Combining the uncertainty principle with the regularized Picard criterion from section 3 yields analogues of both results in the context of the inverse source problem. These include previous results about the splitting of far fields from [9] and [10], which can be simplified and extended by viewing them as consequences of the uncertainty principle and the regularized Picard criterion.

The proof of Theorem 4.3 is a simple corollary of the lemma below.

LEMMA 4.5. *Let* $c \in \mathbb{R}^2$*, and let* T_c *be the operator introduced in* (4.1) *and* (4.3). *Then the operator norm of* $T_c : L^p(S^1) \longrightarrow L^p(S^1)$, $1 \le p \le \infty$, *satisfies*

$$\|T_c\|_{L^p,L^p} = 1\,, \tag{4.8}$$

whereas $T_c : l^1 \longrightarrow l^\infty$ *fulfills*

$$\|T_c\|_{l^1,l^\infty} \le \frac{1}{|c|^{\frac{1}{3}}}\,. \tag{4.9}$$

Proof. Recalling (4.1), we see that T_c is multiplication by a function of modulus one, so (4.8) is immediate. On the other hand, combining (4.2) with the last inequality from page 199 of [18], more precisely,

$$|J_n(x)| < \frac{b}{|x|^{\frac{1}{3}}} \qquad \text{with } b \approx 0.6749\,,$$

shows that

$$\|T_c\|_{l^1,l^\infty} \le \sup_{n\in\mathbb{Z}} |J_n(|c|)| \le \frac{1}{|c|^{\frac{1}{3}}}\,. \qquad \square$$

Proof of Theorem 4.3. Using Hölder's inequality and (4.9), we obtain that

$$|\langle \alpha, T_c\beta\rangle| \le \|\alpha\|_{l^1}\|T_c\beta\|_{l^\infty} \le \frac{1}{|c|^{\frac{1}{3}}}\|\alpha\|_{l^1}\,\|\beta\|_{l^1} \le \frac{\sqrt{\|\alpha\|_{l^0}\|\beta\|_{l^0}}}{|c|^{\frac{1}{3}}}\|\alpha\|_{l^2}\|\beta\|_{l^2}\,. \qquad \square$$

We can improve the dependence on $|c|$ in (4.6) under hypotheses on α and β that are more restrictive but well suited to the inverse source problem.

THEOREM 4.6. *Suppose that* $\alpha \in l^2(-M,M)$, $\beta \in l^2(-N,N)$ *with* $M, N \ge 1$, *and let* $c \in \mathbb{R}^2$ *such that* $|c| > 2(M+N+1)$. *Then*

$$|\langle \alpha, T_c\beta\rangle| \le \frac{\sqrt{(2N+1)(2M+1)}}{|c|^{\frac{1}{2}}}\|\alpha\|_2\|\beta\|_2\,. \tag{4.10}$$

Proof. Because the l^0-support of β is contained in $[-N,N]$

$$\beta^c_m = \sum_{n=-N}^{N} \beta_n(\mathrm{i}^{m-n}J_{m-n}(|c|)e^{\mathrm{i}(m-n)\varphi_c}),$$

so

$$\sup_{-M\le m\le M} |\beta^c_m| \le \|\beta\|_{l^1} \sup_{-(M+N)\le n\le(M+N)} |J_n(|c|)|,$$

and it follows from Theorem 2 of [15]—using the fact that $M, N \geq 1$, together with our hypothesis, which implies that $|c| > 6$—that

$$\sup_{-(M+N)\leq n\leq(M+N)} J_n^2(|c|) \leq \frac{b}{|c|} \qquad \text{with } b \approx 0.7595 \tag{4.11}$$

(see section SM2 in the supplementary material for details; uncertainty_supplement.pdf [local/web 286KB]). We now simply repeat the proof of Theorem 4.3, replacing the estimate for $\|T_c\beta\|_{l^\infty}$ from (4.9) with the estimate we have just established in (4.11), i.e.,

$$\|T_c\|_{l^1[-N,N],l^\infty[-M,M]} \leq \frac{1}{|c|^{\frac{1}{2}}}\,. \tag{4.12}$$

□

We will also make use of another uncertainty principle. A glance at (3.4)–(3.5) reveals that the operator which maps α to its Fourier coefficients maps L^2 to l^2 with norm 1, L^1 to l^∞ with norm $1/\sqrt{2\pi}$, and its inverse maps l^1 to L^∞, also with norm $1/\sqrt{2\pi}$. The theorem below is an immediate corollary of this observation.

THEOREM 4.7. *Let $\alpha, \beta \in L^2(S^1)$ and let $c \in \mathbb{R}^2$. Then*

$$|\langle T_c\alpha, \beta\rangle| \leq \sqrt{\frac{\|\alpha\|_{l^0}\|\beta\|_{L^0}}{2\pi}}\|\alpha\|_2\|\beta\|_2\,. \tag{4.13}$$

Proof. Combining Hölder's inequality with (4.8) and using the mapping properties of the operator which maps α to its Fourier coefficients, we find that

$$\begin{aligned}|\langle T_c\alpha, \beta\rangle| &\leq \|T_c\alpha\|_{L^\infty}\|\beta\|_{L^1} \leq \|\alpha\|_{L^\infty}\|\beta\|_{L^1} \leq \frac{1}{\sqrt{2\pi}}\|\alpha\|_{l^1}\|\beta\|_{L^1}\\ &\leq \frac{1}{\sqrt{2\pi}}\sqrt{\|\alpha\|_{l^0}}\|\alpha\|_2\sqrt{\|\beta\|_{L^0}}\|\beta\|_2\,.\end{aligned}$$

□

5. l^2 corollaries of the uncertainty principles. The regularized Picard criterion tells us that, up to an L^2-small error, a far field radiated by a limited power source in $B_R(0)$ is L^2-close to an α that belongs to the subspace of nonevanescent far fields, the span of $\{e^{in\theta}\}$ with $|n| \leq N$, where $N = N(R, P, p)$ is a little bigger than the radius R. This nonevanescent α satisfies $\|\alpha\|_{l^0} \leq 2N + 1$. The uncertainty principle will show that the angle between translates of these subspaces is bounded below when the translation parameter is large enough, so that we can split the sum of the two nonevanescent far fields into the original two summands.

LEMMA 5.1. *Suppose that $\gamma, \alpha_1, \alpha_2 \in L^2(S^1)$ and $c_1, c_2 \in \mathbb{R}^2$ with*

$$\gamma = T^*_{c_1}\alpha_1 + T^*_{c_2}\alpha_2 \tag{5.1}$$

and that $\frac{\|\alpha_1\|_{l^0}\|\alpha_2\|_{l^0}}{|c_1-c_2|^{\frac{2}{3}}} < 1$. Then, for $i = 1, 2$,

$$\|\alpha_i\|_2^2 \leq \Big(1 - \frac{\|\alpha_1\|_{l^0}\|\alpha_2\|_{l^0}}{|c_1 - c_2|^{\frac{2}{3}}}\Big)^{-1}\|\gamma\|_2^2\,. \tag{5.2}$$

Proof. We first note that (5.1) and (4.1) imply

$$\begin{aligned}\|\gamma\|_2^2 &\geq \|\alpha_1\|_2^2 + \|\alpha_2\|_2^2 - 2|\langle T^*_{c_1}\alpha_1, T^*_{c_2}\alpha_2\rangle|\\ &= \|\alpha_1\|_2^2 + \|\alpha_2\|_2^2 - 2|\langle \alpha_1, T^*_{c_2-c_1}\alpha_2\rangle|\,.\end{aligned} \tag{5.3}$$

We now use (4.6),

$$
\begin{aligned}
\|\gamma\|_2^2 &\geq \|\alpha_1\|_2^2 + \|\alpha_2\|_2^2 - 2\frac{\sqrt{\|\alpha_1\|_{l^0}\|\alpha_2\|_{l^0}}}{|c_2-c_1|^{\frac{1}{3}}}\|\alpha_1\|_2\|\alpha_2\|_2 \\
&= \left(1 - \frac{\|\alpha_1\|_{l^0}\|\alpha_2\|_{l^0}}{|c_2-c_1|^{\frac{2}{3}}}\right)\|\alpha_1\|_2^2 + \left(\|\alpha_2\|_2 - \frac{\sqrt{\|\alpha_1\|_{l^0}\|\alpha_2\|_{l^0}}}{|c_2-c_1|^{\frac{1}{3}}}\|\alpha_1\|_2\right)^2 .
\end{aligned}
\tag{5.4}
$$

Dropping the second term now gives (5.2) for α_1, and we may interchange the roles α_1 and α_2 in the proof to obtain the estimate for α_2. ☐

The analogous consequence of Theorem 4.6 is as follows.

LEMMA 5.2. *Suppose that* $\gamma \in L^2(S^1)$, $\alpha_i \in l^2(-N_i, N_i)$ *for some* $N_i \in \mathbb{N}$, $i = 1, 2$, *and* $c_1, c_2 \in \mathbb{R}^2$ *with* $|c_1 - c_2| > 2(N_1 + N_2 + 1)$ *and*

$$\gamma = T^*_{c_1}\alpha_1 + T^*_{c_2}\alpha_2 ,$$

and that $\frac{(2N_1+1)(2N_2+1)}{|c_1-c_2|} < 1$. *Then, for* $i = 1, 2$,

$$\|\alpha_i\|_2^2 \leq \left(1 - \frac{(2N_1+1)(2N_2+1)}{|c_1-c_2|}\right)^{-1}\|\gamma\|_2^2 . \tag{5.5}$$

In our application to the inverse source problem, we will know that each far field is the translation of a far field α_i, radiated by a limited power source supported in a ball centered at the origin, and therefore that all but a very small amount of the radiated power is contained in the nonevanescent part, the translation of the Fourier modes $e^{in\theta}$ for $|n| < N(R, P, p)$. The estimate in the theorem below says that if the distance between the balls is large enough, we may uniquely solve for the nonevanescent parts of the individual far fields, and that this split is stable with respect to perturbations in the data.

THEOREM 5.3. *Suppose that* $\gamma^0, \gamma^1 \in L^2(S^1)$, $c_1, c_2 \in \mathbb{R}^2$, *and* $N_1, N_2 \in \mathbb{N}$ *such that* $|c_1 - c_2| > 2(N_1 + N_2 + 1)$ *and*

$$\frac{(2N_1+1)(2N_2+1)}{|c_1-c_2|} < 1 , \tag{5.6}$$

and let

$$\gamma^0 \overset{\text{LS}}{=} T^*_{c_1}\alpha_1^0 + T^*_{c_2}\alpha_2^0 , \qquad \alpha_i^0 \in l^2(-N_i, N_i) , \tag{5.7a}$$

$$\gamma^1 \overset{\text{LS}}{=} T^*_{c_1}\alpha_1^1 + T^*_{c_2}\alpha_2^1 , \qquad \alpha_i^1 \in l^2(-N_i, N_i) . \tag{5.7b}$$

Then, for $i = 1, 2$,

$$\|\alpha_i^1 - \alpha_i^0\|_2^2 \leq \left(1 - \frac{(2N_1+1)(2N_2+1)}{|c_1-c_2|}\right)^{-1}\|\gamma^1 - \gamma^0\|_2^2 . \tag{5.8}$$

The notation in (5.7) above means that the α_i^j are the (necessarily unique) least squares solutions to the equations $\gamma^j = T^*_{c_1}\alpha_1^j + T^*_{c_2}\alpha_2^j$. Recall that the far fields radiated by a limited power source from a ball have almost all, but not all, of their power (L^2-norm) concentrated in the Fourier modes with $n \leq N(R, P, p)$. Therefore the γ^i will typically not belong to the subspace that is the direct sum of

$T^*_{c_1} l^2(-N_1,N_1) \oplus T^*_{c_2} l^2(-N_2,N_2)$, and therefore α_1^j and α_2^j will usually not solve equations (5.7) exactly. The estimate in (5.8) is nevertheless always true and guarantees that the pair (α_1^j,α_2^j) is unique and that the absolute condition number of the splitting operator which maps γ to (α_1^j,α_2^j) is no larger than $\big(1-\frac{(2N_1+1)(2N_2+1)}{|c_1-c_2|}\big)^{-\frac{1}{2}}$.

Proof of Theorem 5.3. Each γ^j can be uniquely decomposed as

$$\gamma^j \;=\; w^j + w^j_\perp\,, \tag{5.9}$$

where each w^j belongs to the $2N_1+2N_2+2$-dimensional subspace

$$W \;=\; T^*_{c_1} l^2(-N_1,N_1) \oplus T^*_{c_2} l^2(-N_2,N_2),$$

and each $w^j_\perp$ is orthogonal to W. The definition of least squares solutions means that

$$w^j \;=\; T^*_{c_1}\alpha_1^j + T^*_{c_2}\alpha_2^j\,.$$

Subtracting gives

$$w^1 - w^0 \;=\; T^*_{c_1}(\alpha_1^1-\alpha_1^0) + T^*_{c_2}(\alpha_2^1-\alpha_2^0) \tag{5.10}$$

and applying the estimate (5.5) yields

$$\|\alpha_i^1-\alpha_i^0\|_2^2 \;\le\; \left(1-\frac{(2N_1+1)(2N_2+1)}{|c_1-c_2|}\right)^{-1} \|w^1-w^0\|_2^2\,. \tag{5.11}$$

Finally, we note that

$$\|\gamma^1-\gamma^0\|_2^2 \;=\; \|w^1-w^0\|_2^2 + \|w^1_\perp - w^0_\perp\|_2^2 \;\ge\; \|w^1-w^0\|_2^2\,, \tag{5.12}$$

which finishes the proof. ∎

We also have corresponding corollaries of Theorem 4.7, which tell us that if a far field is radiated from a small ball and measured on most of the circle, then it is possible to recover its nonevanescent part on the entire circle. Theorem 5.5 below describes the case where we cannot measure the far field $\alpha = T^*_c\alpha^0$ on a subset $\Omega \subseteq S^1$. We measure $\gamma = \alpha+\beta$, where $\beta = -\alpha\big|_\Omega$. The estimates (5.14) imply that we can stably recover the nonevanescent part of the far field on Ω.

Before we state the theorem, we give the corresponding analogue of Lemmas 5.1 and 5.2.

Lemma 5.4. *Suppose that* $\gamma,\alpha,\beta \in L^2(S^1)$ *and* $c\in\mathbb{R}^2$ *with*

$$\gamma \;=\; \beta + T^*_c\alpha$$

and that $\frac{\|\alpha\|_{l^0}\|\beta\|_{L^0}}{2\pi} < 1$. *Then*

$$\|\alpha\|_2^2 \;\le\; \left(1-\frac{\|\alpha\|_{l^0}\|\beta\|_{L^0}}{2\pi}\right)^{-1}\|\gamma\|_2^2 \tag{5.13a}$$

and

$$\|\beta\|_2^2 \;\le\; \left(1-\frac{\|\alpha\|_{l^0}\|\beta\|_{L^0}}{2\pi}\right)^{-1}\|\gamma\|_2^2\,. \tag{5.13b}$$

Proof. Proceeding as in (5.3)–(5.4), but replacing (4.6) by (4.13), yields the result. ∎

Theorem 5.5. *Suppose that* $\gamma^0, \gamma^1 \in L^2(S^1)$, $c \in \mathbb{R}^2$, $N \in \mathbb{N}$, *and* $\Omega \subseteq S^1$ *such that* $\frac{(2N+1)|\Omega|}{2\pi} < 1$, *and let*

$$\gamma^0 \overset{\mathrm{LS}}{=} \beta^0 + T_c\alpha^0\,, \qquad \alpha^0 \in l^2(-N,N) \text{ and } \beta^0 \in L^2(\Omega)\,,$$

$$\gamma^1 \overset{\mathrm{LS}}{=} \beta^1 + T_c\alpha^1\,, \qquad \alpha^1 \in l^2(-N,N) \text{ and } \beta^1 \in L^2(\Omega)\,.$$

Then

$$\|\alpha^1 - \alpha^0\|_2^2 \le \left(1 - \frac{(2N+1)|\Omega|}{2\pi}\right)^{-1} \|\gamma^1 - \gamma^0\|_2^2 \tag{5.14a}$$

and

$$\|\beta^1 - \beta^0\|_2^2 \le \left(1 - \frac{(2N+1)|\Omega|}{2\pi}\right)^{-1} \|\gamma^1 - \gamma^0\|_2^2\,. \tag{5.14b}$$

Proof. Just as in (5.9), we decompose each γ^j

$$\gamma^j = w^j + w^j_\perp\,,$$

where each w^j belongs to the subspace

$$W = L^2(\Omega) \oplus T_c l^2(-N,N)$$

and each $w^j_\perp$ is orthogonal to W. Proceeding as in (5.10)–(5.11), but using the estimates from (5.13), we find

$$\|\alpha^1 - \alpha^0\|_2^2 \le \left(1 - \frac{(2N+1)|\Omega|}{2\pi}\right)^{-1} \|w^1 - w^0\|_2^2$$

and

$$\|\beta^1 - \beta^0\|_2^2 \le \left(1 - \frac{(2N+1)|\Omega|}{2\pi}\right)^{-1} \|w^1 - w^0\|_2^2$$

and then note that (5.12) is true here as well to finish the proof. ∎

Remark 5.6. Theorem 5.5 tells us that as Ω gets bigger, we must choose N smaller. This is natural because as we increase the number of Fourier modes of a far field α, it becomes possible to localize α in smaller sets. If α can be localized in Ω, then we cannot stably recover it from measurements outside Ω. Because $2N+1$ is approximately the diameter in wavelengths, decreasing N means that for a source of fixed physical size, we must increase the wavelength and therefore lower the resolution.

A version of Theorem 5.3 with multiple well-separated components is also true (proofs of the following two theorems are available in the supplementary material in section SM3; uncertainty_supplement.pdf [local/web 286KB]).

Theorem 5.7. *Suppose that* $\gamma^0, \gamma^1 \in L^2(S^1)$, $c_i \in \mathbb{R}^2$, *and* $N_i \in \mathbb{N}$, $i = 1, \dots, I$, *such that* $|c_i - c_j| > 2(N_i + N_j + 1)$ *for every* $i \neq j$ *and*

$$\left(\sqrt{2N_i+1}\sum_{j\neq i}\sqrt{\frac{2N_j+1}{|c_i - c_j|}}\right) < 1 \qquad \text{for each } i\,,$$

and let

$$\gamma^0 \overset{LS}{=} \sum_{i=1}^{I} T^*_{c_i}\alpha^0_i\,, \qquad \alpha^0_i \in l^2(-N_i, N_i)\,,$$

$$\gamma^1 \overset{LS}{=} \sum_{i=1}^{I} T^*_{c_i}\alpha^1_i\,, \qquad \alpha^1_i \in l^2(-N_i, N_i)\,.$$

Then, for $i = 1, \ldots, I$,

$$\|\alpha^1_i - \alpha^0_i\|^2_2 \le \left(1 - \sqrt{2N_i+1}\sum_{j\neq i}\sqrt{\frac{2N_j+1}{|c_j - c_i|}}\right)^{-1} \|\gamma^1 - \gamma^0\|^2_2\,.$$

We may include a missing data component as well.

Theorem 5.8. *Suppose that* $\gamma^0, \gamma^1 \in L^2(S^1)$, $c_i \in \mathbb{R}^2$, $N_i \in \mathbb{N}$, $i = 1, \ldots, I$, *and* $\Omega \subseteq L^2(S^1)$ *such that* $|c_i - c_j| > 2(N_i + N_j + 1)$ *for every* $i \neq j$ *and*

$$\sqrt{\frac{|\Omega|}{2\pi}}\sum_{i=1}^{I}\sqrt{2N_i+1} \,<\, 1\,,$$

$$\sqrt{2N_i+1}\left(\sqrt{\frac{|\Omega|}{2\pi}} + \sum_{j\neq i}\sqrt{\frac{2N_i+1}{|c_i - c_j|}}\right) \,<\, 1 \qquad \textit{for each } i\,,$$

and let

$$\gamma^0 \overset{LS}{=} \beta^0 + \sum_{i=1}^{I} T^*_{c_i}\alpha^0_i\,, \qquad \alpha^0_i \in l^2(-N_i, N_i) \textit{ and } \beta^0 \in L^2(\Omega)\,, \tag{5.15a}$$

$$\gamma^1 \overset{LS}{=} \beta^1 + \sum_{i=1}^{I} T^*_{c_i}\alpha^1_i\,, \qquad \alpha^1_i \in l^2(-N_i, N_i) \textit{ and } \beta^0 \in L^2(\Omega)\,. \tag{5.15b}$$

Then

$$\|\beta^1 - \beta^0\|^2_2 \le \left(1 - \sqrt{\frac{|\Omega|}{2\pi}}\sum_{i}\sqrt{2N_i+1}\right)^{-1} \|\gamma^1 - \gamma^0\|^2_2$$

and, for $i = 1, \ldots, I$,

$$\|\alpha^1_i - \alpha^0_i\|^2_2 \le \left(1 - \sqrt{2N_i+1}\left(\sqrt{\frac{|\Omega|}{2\pi}} + \sum_{j\neq i}\sqrt{\frac{2N_i+1}{|c_i - c_j|}}\right)\right)^{-1} \|\gamma^1 - \gamma^0\|^2_2\,.$$

6. l^1 corollaries of the uncertainty principle.

The results below are analogous to those in the previous section. The main difference is that they do not require the a priori knowledge of the size of the nonevanescent subspaces (the N_i in Theorems 5.3 through 5.8).

In Theorem 6.1 below, γ^0 represents the (measured) approximate far field; the α^0_i are the nonevanescent parts of the true (unknown) far fields radiated by each of the two components, which we assume are well separated (6.1). The constant δ_0 in (6.2) accounts for both the noise and the evanescent components of the true far fields. Condition (6.3) requires that the optimization problem (6.4) be formulated with a constraint that is weak enough so that the α^0_i are feasible.

Theorem 6.1. *Suppose that* $\gamma^0, \alpha_1^0, \alpha_2^0 \in L^2(S^1)$ *and* $c_1, c_2 \in \mathbb{R}^2$ *such that*

$$\frac{4\|\alpha_i^0\|_{l^0}}{|c_1-c_2|^{\frac{1}{3}}} < 1 \qquad \text{for each } i \tag{6.1}$$

and

$$\|\gamma^0 - T_{c_1}^*\alpha_1^0 - T_{c_2}^*\alpha_2^0\|_2 \le \delta_0 \qquad \text{for some } \delta_0 \ge 0\,. \tag{6.2}$$

If $\delta \ge 0$ *and* $\gamma \in L^2(S^1)$ *with*

$$\delta \ge \delta_0 + \|\gamma - \gamma^0\|_2 \tag{6.3}$$

and

$$\begin{aligned}(\alpha_1,\alpha_2) = \operatorname{argmin} \|\alpha_1\|_{l^1} + \|\alpha_2\|_{l^1}& \\ s.t.\quad \|\gamma - T_{c_1}^*\alpha_1 - T_{c_2}^*\alpha_2\|_2 \le \delta\,,\ \alpha_1,\alpha_2 \in L^2(S^1)\,,&\end{aligned} \tag{6.4}$$

then, for $i = 1, 2$,

$$\|\alpha_i^0 - \alpha_i\|_2^2 \le \left(1 - \frac{4\|\alpha_i^0\|_{l^0}}{|c_1-c_2|^{\frac{1}{3}}}\right)^{-1} 4\delta^2\,. \tag{6.5}$$

Proof. A consequence of (6.3) is that the pair (α_1^0, α_2^0) satisfies the constraint in (6.4), which implies that

$$\|\alpha_1\|_{l^1} + \|\alpha_2\|_{l^1} \le \|\alpha_1^0\|_{l^1} + \|\alpha_2^0\|_{l^1} \tag{6.6}$$

because (α_1, α_2) is a minimizer. Additionally, with W_i representing the l^0-support of α_i^0 and W_i^c its complement,

$$\begin{aligned}\|\alpha_i\|_{l^1} &= \|\alpha_i^0 + (\alpha_i - \alpha_i^0)\|_{l^1} \\ &= \|\alpha_i^0 + (\alpha_i - \alpha_i^0)\|_{l^1(W_i)} + \|\alpha_i - \alpha_i^0\|_{l^1(W_i^c)} \\ &= \|\alpha_i^0 + (\alpha_i - \alpha_i^0)\|_{l^1(W_i)} + \|\alpha_i - \alpha_i^0\|_{l^1} - \|\alpha_i - \alpha_i^0\|_{l^1(W_i)} \\ &\ge \|\alpha_i^0\|_{l^1} + \|\alpha_i - \alpha_i^0\|_{l^1} - 2\|\alpha_i - \alpha_i^0\|_{l^1(W_i)}\,.\end{aligned} \tag{6.7}$$

Inserting (6.7) into (6.6) yields

$$\|\alpha_1 - \alpha_1^0\|_{l^1} + \|\alpha_2 - \alpha_2^0\|_{l^1} \le 2(\|\alpha_1 - \alpha_1^0\|_{l^1(W_1)} + \|\alpha_2 - \alpha_2^0\|_{l^1(W_2)})\,. \tag{6.8}$$

We now use (6.3) together with (6.2), the constraint in (6.4), and the fact that $T_{c_1-c_2}^*$ is an L^2-isometry to obtain

$$\begin{aligned}4\delta^2 &\ge (\|\gamma - \gamma^0\|_2 + \delta_0 + \delta)^2 \\ &\ge (\|\gamma - \gamma^0\|_2 + \|\gamma^0 - T_{c_1}^*\alpha_1^0 - T_{c_2}^*\alpha_2^0\|_2 + \|\gamma - T_{c_1}^*\alpha_1 - T_{c_2}^*\alpha_2\|_2)^2 \\ &\ge \|T_{c_1}^*(\alpha_1 - \alpha_1^0) + T_{c_2}^*(\alpha_2 - \alpha_2^0)\|_2^2 \\ &= \|\alpha_1 - \alpha_1^0 + T_{c_2-c_1}^*(\alpha_2 - \alpha_2^0)\|_2^2 \\ &\ge \|\alpha_1 - \alpha_1^0\|_2^2 + \|\alpha_2 - \alpha_2^0\|_2^2 - 2|\langle \alpha_1 - \alpha_1^0, T_{c_2-c_1}^*(\alpha_2 - \alpha_2^0)\rangle|\,.\end{aligned} \tag{6.9}$$

Hölder's inequality, (4.9), and (6.8) show

$$\begin{aligned} 4\delta^2 &\ge \|\alpha_1-\alpha_1^0\|_2^2+\|\alpha_2-\alpha_2^0\|_2^2-\frac{2}{|c_1-c_2|^{\frac{1}{3}}}\|\alpha_1-\alpha_1^0\|_{l^1}\|\alpha_2-\alpha_2^0\|_{l^1} \\ &\ge \|\alpha_1-\alpha_1^0\|_2^2+\|\alpha_2-\alpha_2^0\|_2^2-\frac{1}{2|c_1-c_2|^{\frac{1}{3}}}(\|\alpha_1-\alpha_1^0\|_{l^1}+\|\alpha_2-\alpha_2^0\|_{l^1})^2 \\ &\ge \|\alpha_1-\alpha_1^0\|_2^2+\|\alpha_2-\alpha_2^0\|_2^2-\frac{2}{|c_1-c_2|^{\frac{1}{3}}}(\|\alpha_1-\alpha_1^0\|_{l^1(W_1)}+\|\alpha_2-\alpha_2^0\|_{l^1(W_2)})^2\,. \end{aligned} \tag{6.10}$$

Using Hölder's inequality once more yields

$$\begin{aligned} 4\delta^2 \ge\ & \|\alpha_1-\alpha_1^0\|_2^2+\|\alpha_2-\alpha_2^0\|_2^2 \\ &-\frac{2}{|c_1-c_2|^{\frac{1}{3}}}(|W_1|^{\frac{1}{2}}\|\alpha_1-\alpha_1^0\|_2+|W_2|^{\frac{1}{2}}\|\alpha_2-\alpha_2^0\|_2)^2 \\ \ge\ & \|\alpha_1-\alpha_1^0\|_2^2+\|\alpha_2-\alpha_2^0\|_2^2 \\ &-\frac{4}{|c_1-c_2|^{\frac{1}{3}}}(|W_1|\|\alpha_1-\alpha_1^0\|_2^2+|W_2|\|\alpha_2-\alpha_2^0\|_2^2)\,, \end{aligned} \tag{6.11}$$

which implies (6.5) because $|W_i| = \|\alpha_i^0\|_{l^0}$. ∎

Assuming that some a priori information on the size of the nonevanescent subspaces is available and that the distances between the source components is large relative to their dimensions, we can improve the dependence of the stability estimates on the distances.

COROLLARY 6.2. *If we add to the hypothesis of Theorem* 6.1

$$\alpha_i^0, \alpha_i \in l^2(-N_i, N_i) \qquad \textit{and} \qquad |c_1-c_2| \,>\, 2(N_1+N_2+1)$$

for some $N_1, N_2 \in \mathbb{N}$ *and replace* (6.1) *with*

$$\frac{4\|\alpha_i^0\|_{l^0}}{|c_1-c_2|^{\frac{1}{2}}} < 1 \qquad \textit{for each } i, \tag{6.12}$$

then, for $i = 1, 2$,

$$\|\alpha_i^0-\alpha_i\|_2^2 \le \left(1-\frac{4\|\alpha_i^0\|_{l^0}}{|c_1-c_2|^{\frac{1}{2}}}\right)^{-1} 4\delta^2\,. \tag{6.13}$$

Proof. Replace (4.9) by (4.12) in (6.9)–(6.10). ∎

The analogue of Theorem 5.5 for data completion but without a priori knowledge on the size of the nonevanescent subspaces is as follows.

THEOREM 6.3. *Suppose that* $\gamma^0, \alpha^0 \in L^2(S^1)$, $\Omega \subseteq S^1$, $\beta^0 \in L^2(\Omega)$, *and* $c \in \mathbb{R}^2$ *such that*

$$\frac{2\|\alpha^0\|_{l^0}|\Omega|}{\pi} < 1$$

and

$$\|\gamma^0 - T_c^*\alpha^0 - \beta^0\|_2 \le \delta_0 \qquad \textit{for some } \delta^0 \ge 0\,.$$

If $\delta \geq 0$ *and* $\gamma \in L^2(S^1)$ *with*

$$\delta \;\geq\; \delta_0 + \|\gamma - \gamma^0\|_2$$

and

$$\alpha = \operatorname{argmin} \|\alpha\|_{l^1} \quad \textit{s.t.} \quad \|\gamma - \beta - T_c^*\alpha\|_2 \leq \delta\,,\ \alpha \in L^2(S^1)\,,\ \beta \in L^2(\Omega)\,,$$

then

$$\|\alpha^0 - \alpha\|_2^2 \;\leq\; \left(1 - \frac{2\|\alpha^0\|_{l^0}|\Omega|}{\pi}\right)^{-1} 4\delta^2 \tag{6.14a}$$

and

$$\|\beta^0 - \beta\|_2^2 \;\leq\; \left(1 - \frac{2\|\alpha^0\|_{l^0}|\Omega|}{\pi}\right)^{-1} 4\delta^2\,. \tag{6.14b}$$

Proof. Proceeding as in (6.6)–(6.8), we find that

$$\|\alpha - \alpha^0\|_{l^1} \;\leq\; 2\|\alpha - \alpha^0\|_{l^1(W)}, \tag{6.15}$$

with W representing the l^0-support of α^0. Applying arguments similar to those in (6.9) yields

$$4\delta^2 \;\geq\; \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - 2|\langle T_c^*(\alpha - \alpha^0), \beta - \beta^0\rangle|\,.$$

We now use Hölder's inequality, (4.1), the mapping properties of the operator which maps α to its Fourier coefficients, and (6.15) to obtain

(6.16)

$$\begin{aligned}
4\delta^2 \;&\geq\; \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - 2\|T_c^*(\alpha - \alpha^0)\|_{L^\infty}\|\beta - \beta^0\|_{L^1}\\
&=\; \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - 2\|\alpha - \alpha^0\|_{L^\infty}\|\beta - \beta^0\|_{L^1}\\
&\geq\; \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - \frac{2}{\sqrt{2\pi}}\|\alpha - \alpha^0\|_{l^1}\|\beta - \beta^0\|_{L^1}\\
&\geq\; \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - \frac{4}{\sqrt{2\pi}}\|\alpha - \alpha^0\|_{l^1(W)}\|\beta - \beta^0\|_{L^1}\\
&\geq\; \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - \frac{4}{\sqrt{2\pi}}\sqrt{|W|}\|\alpha - \alpha^0\|_2\sqrt{|\Omega|}\|\beta - \beta^0\|_2\\
&\geq\; \left(1 - \frac{2}{\pi}|W||\Omega|\right)\|\alpha - \alpha^0\|_2^2 + \left(\|\beta - \beta^0\|_2 - \frac{2}{\sqrt{2\pi}}\sqrt{|W||\Omega|}\|\alpha - \alpha^0\|_2\right)^2\,.
\end{aligned}$$

Dropping the second term gives (6.14) for α because $|W| = \|\alpha^0\|_{l^0}$, and we may interchange the roles of α and β when completing the square in the last line of (6.16) to obtain the estimate for β. ∎

If Ω is unknown as well, then Theorem 6.3 can be adapted as follows (for a proof of the following corollary we refer to section SM4 in the supplementary material; uncertainty_supplement.pdf [local/web 286KB]).

COROLLARY 6.4. *Suppose that* $\gamma^0, \alpha^0 \in L^2(S^1)$, $\Omega \subseteq S^1$, $\beta^0 \in L^2(\Omega)$, *and* $c \in \mathbb{R}^2$ *such that*

$$\frac{4}{\sqrt{2\pi}}\frac{1}{\tau^2}\|\alpha^0\|_{l^0} \;<\; 1 \quad \textit{and} \quad \frac{4}{\sqrt{2\pi}}\tau^2|\Omega| \;<\; 1 \qquad \textit{for some } \tau > 0 \tag{6.17}$$

and

$$\|\gamma^0 - T_c^*\alpha^0 - \beta^0\|_2 \le \delta_0 \qquad \textit{for some } \delta_0 \ge 0\,.$$

If $\delta \ge 0$ *and* $\gamma \in L^2(S^1)$ *with*

$$\delta \ge \delta_0 + \|\gamma - \gamma^0\|_2$$

and

$$(\alpha,\beta) = \operatorname{argmin} \frac{1}{\tau}\|\alpha\|_{l^1} + \tau\|\beta\|_{L^1} \quad \textit{s.t.} \quad \|\gamma - T_c^*\alpha - \beta\|_2 \le \delta\,,\ \alpha,\beta \in L^2(S^1)\,, \tag{6.18}$$

then

$$\|\alpha^0 - \alpha\|_2^2 \le \left(1 - \frac{4}{\sqrt{2\pi}}\frac{1}{\tau^2}\|\alpha^0\|_{l^0}\right)^{-1} 4\delta^2$$

and

$$\|\beta^0 - \beta\|_2^2 \le \left(1 - \frac{4}{\sqrt{2\pi}}\tau^2|\Omega|\right)^{-1} 4\delta^2\,.$$

The parameter $\tau > 0$ in Corollary 6.4 modifies the optimization problem (6.18) to improve the conditioning on the recovery of one of α or β at the cost of the other. The simplest (symmetric) choice is $\tau^2 = \sqrt{\frac{|\Omega|}{\|\alpha^0\|_{l^0}}}$, which replaces (6.17) with $\frac{8}{\pi}|\Omega|\,\|\alpha^0\|_{l^0} < 1$. A possible application of Corollary 6.4 is the problem of removing (high-amplitude) strongly localized noise from measured far field data.

Next we consider sources supported on sets with multiple disjoint components (a proof of the following theorem is available in the supplementary material in section SM4; uncertainty_supplement.pdf [local/web 286KB]).

Theorem 6.5. *Suppose that* $\gamma^0, \alpha_i^0 \in L^2(S^1)$ *and* $c_i \in \mathbb{R}^2$, $i = 1, \dots, I$, *such that*

$$\max_{j\neq k} \frac{1}{|c_k - c_j|^{\frac{1}{3}}} 4(I-1)\|\alpha_i^0\|_{l^0} < 1 \qquad \textit{for each } i \tag{6.19}$$

and

$$\|\gamma^0 - \sum_{i=1}^{I} T_{c_i}^*\alpha_i^0\|_2 \le \delta_0 \qquad \textit{for some } \delta^0 \ge 0\,.$$

If $\delta \ge 0$ *and* $\gamma \in L^2(S^1)$ *with*

$$\delta \ge \delta_0 + \|\gamma - \gamma^0\|_2$$

and

$$(\alpha_1, \dots, \alpha_I) = \operatorname{argmin} \sum_{i=1}^{I} \|\alpha_i\|_{l^1} \quad \textit{s.t.} \quad \|\gamma - \sum_{i=1}^{I} T_{c_i}^*\alpha_i\|_2 \le \delta\,,\ \alpha_i \in L^2(S^1)\,, \tag{6.20}$$

then, for $i = 1, \dots, I$,

$$\|\alpha_i^0 - \alpha_i\|_2^2 \le \left(1 - \max_{j\neq k} \frac{1}{|c_k - c_j|^{\frac{1}{3}}} 4(I-1)\|\alpha_i^0\|_{l^0}\right)^{-1} 4\delta^2\,.$$

As in Corollary 6.2 we can improve these estimates, under the assumption that some a priori knowledge of the size of the nonevanescent subspaces is available and that the individual source components are sufficiently far apart.

Corollary 6.6. *If we add to the hypothesis of Theorem* 6.5

$$\alpha_i^0, \alpha_i \in l^2(-N_i, N_i) \text{ for each } i \quad \text{and} \quad |c_i - c_j| > 2(N_i + N_j + 1) \text{ for every } i \neq j$$

for some $N_1, \dots, N_I \in \mathbb{N}$, *and replace* (6.19) *with*

$$\max_{j \neq k} \frac{1}{|c_k - c_j|^{\frac{1}{2}}} 4(I-1)\|\alpha_i^0\|_{l^0} < 1 \qquad \text{for each } i\,,$$

the conclusion becomes, for $i = 1, \dots, I$,

$$\|\alpha_i^0 - \alpha_i\|_2^2 \le \left(1 - \max_{j \neq k} \frac{1}{|c_k - c_j|^{\frac{1}{2}}} 4(I-1)\|\alpha_i^0\|_{l^0}\right)^{-1} 4\delta^2\,.$$

Next we consider multiple source components together with a missing data component (for a proof of the following theorem, see section SM4 in the supplementary material; uncertainty_supplement.pdf [local/web 286KB]).

Theorem 6.7. *Suppose that* $\gamma^0, \alpha_i^0 \in L^2(S^1)$, $c_i \in \mathbb{R}^2$, $i = 1, \dots, I$, $\Omega \subseteq S^1$, *and* $\beta^0 \in L^2(\Omega)$ *such that*

$$\frac{2}{\sqrt{2\pi}} \sum_{i=1}^{I} \sqrt{|\Omega| \|\alpha_i^0\|_{l^0}} < 1\,, \tag{6.21a}$$

$$\max_{j \neq k} \frac{1}{|c_k - c_j|^{\frac{1}{3}}} 4(I-1)\|\alpha_i^0\|_{l^0} + \frac{2}{\sqrt{2\pi}} \sqrt{|\Omega| \|\alpha_i^0\|_{l^0}} < 1 \qquad \text{for each } i\,, \tag{6.21b}$$

and

$$\|\gamma^0 - \beta^0 - \sum_{i=1}^{I} T_{c_i}^* \alpha_i^0\|_2 \le \delta_0 \qquad \text{for some } \delta_0 \ge 0\,.$$

If $\delta \ge 0$ *and* $\gamma \in L^2(S^1)$ *with*

$$\delta \ge \delta_0 + \|\gamma - \gamma^0\|_2$$

and

$$(\alpha_1, \dots, \alpha_I) = \operatorname{argmin} \sum_{i=1}^{I} \|\alpha_i\|_{l^1} \tag{6.22}$$

$$\text{s.t.} \quad \|\gamma - \beta - \sum_{i=1}^{I} T_{c_i}^* \alpha_i\|_2 \le \delta\,,\ \alpha_i \in L^2(S^1)\,,\ \beta \in L^2(\Omega)\,,$$

then

$$\|\beta^0 - \beta\|_2^2 \le \left(1 - \frac{2}{\sqrt{2\pi}} \sum_{i=1}^{I} \sqrt{|\Omega| \|\alpha_i^0\|_{l^0}}\right)^{-1} 4\delta^2 \tag{6.23a}$$

and, for $i = 1, \dots, I$,

$$\|\alpha_i^0 - \alpha_i\|_2^2 \le \left(1 - \max_{j \neq k} \frac{1}{|c_k - c_j|^{\frac{1}{3}}} 4(I-1)\|\alpha_i^0\|_{l^0} - \frac{2}{\sqrt{2\pi}} \sqrt{|\Omega| \|\alpha_i^0\|_{l^0}}\right)^{-1} 4\delta^2\,. \tag{6.23b}$$

Again, including a priori information of the size of the nonevanescent subspaces and assuming that the individual source components are well separated, the result can be improved.

Corollary 6.8. *If we add to the hypothesis of Theorem* 6.7

$$\alpha_i^0, \alpha_i \in l^2(-N_i, N_i) \text{ for each } i \quad \text{and} \quad |c_i - c_j| > 2(N_i + N_j + 1) \text{ for every } i \neq j$$

for some $N_1, \dots, N_I \in \mathbb{N}$, *and replace* (6.21b) *with*

$$\max_{j\neq k} \frac{1}{|c_k - c_j|^{\frac{1}{2}}} 4(I-1)\|\alpha_i^0\|_{l^0} + \frac{2}{\sqrt{2\pi}}\sqrt{|\Omega|\|\alpha_i^0\|_{l^0}} < 1 \qquad \text{for each } i\,,$$

the conclusion (6.23b) *becomes, for* $i = 1, \dots, I$,

$$\|\alpha_i^0 - \alpha_i\|_2^2 \leq \left(1 - \max_{j\neq k} \frac{1}{|c_k - c_j|^{\frac{1}{2}}} 4(I-1)\|\alpha_i^0\|_{l^0} + \frac{2}{\sqrt{2\pi}}\sqrt{|\Omega|\|\alpha_i^0\|_{l^0}}\right)^{-1} 4\delta^2\,.$$

Finally, we note that variants of Theorems 6.5 and 6.7, where we replace the l^1 minimization in (6.20) and (6.22) by a weighted l^1 minimization in order to obtain better estimates for certain geometric configurations of the supports of the individual source components, are established in Theorems SM4.2 and SM4.4 in section SM4 in the supplementary material (uncertainty_supplement.pdf [local/web 286KB]).

7. Conditioning, resolution, and wavelength. So far, we have suppressed the dependence on the wavenumber k. We restore it here and consider the consequences related to conditioning and resolution. We confine our discussion to Theorem 5.3, assuming that the γ^j, $j = 1, 2$, represent far fields that are radiated by superpositions of limited power sources supported in balls $B_{R_i}(c_i)$, $i = 1, 2$, and that accordingly, for $k = 1$ (following our discussion at the end of section 3), the numbers $N_i \gtrsim R_i$ are just a little bigger than the radii of these balls. This becomes $N_i \gtrsim kR_i$ when we return to conventional units, and the estimate (5.8) then depends on the quantity

$$\frac{(2N_1+1)(2N_2+1)}{k|c_1 - c_2|}\,. \tag{7.1}$$

Writing $V_i := T_{c_i}^* l^2(-N_i, N_i)$ and denoting by $P_i : l^2 \to l^2$ the orthogonal projection onto V_i, $i = 1, 2$, we have $V_1 \cap V_2 = \{0\}$ if $c_1 \neq c_2$, and the angle θ_{12} between these subspaces is given by

$$\cos\theta_{12} = \sup_{\substack{\alpha_1\in V_1\\ \alpha_2\in V_2}} \frac{|\langle\alpha_1, \alpha_2\rangle|}{\|\alpha_1\|_2\|\alpha_2\|_2} = \sup_{\alpha_1,\alpha_2\in l^2} \frac{|\langle P_1\alpha_1, P_2\alpha_2\rangle|}{\|\alpha_1\|_2\|\alpha_2\|_2} = \|P_1P_2\|_{l^2,l^2}\,.$$

A glance at the proof of Lemma 5.1 reveals that the square root of (7.1) is just an upper bound for this cosine. Furthermore, the least squares solutions to (5.7) can be constructed from simple formulas,

$$\begin{aligned} \alpha_1^j &= (I - P_1P_2)^{-1}P_1(I - P_2)\gamma^j =: P_{1|2}\gamma^j\,, \\ \alpha_2^j &= (I - P_2P_1)^{-1}P_2(I - P_1)\gamma^j =: P_{2|1}\gamma^j\,, \end{aligned}$$

where $P_{1|2}$ and $P_{2|1}$ denote the projection onto V_1 along V_2, and vice versa. These satisfy

$$\|P_{1|2}\|_{l^2,l^2} \,=\, \|P_{2|1}\|_{l^2,l^2} \,=\, \csc\theta_{12} \,=\, \Big(\frac{1}{1-\cos^2\theta_{12}}\Big)^{\frac{1}{2}}\,.$$

Consequently $\csc\theta_{12}$ is the absolute condition number for the splitting problem (5.7), and Theorem 5.3 (with our choice of N_1 and N_2) essentially says that

$$\csc(\theta_{12}) \,\le\, \frac{1}{\sqrt{1-\frac{(2N_1+1)(2N_2+1)}{k|c_1-c_2|}}} \,\lesssim\, \frac{1}{\sqrt{1-\frac{(2kR_1+1)(2kR_2+1)}{k|c_1-c_2|}}}\,. \tag{7.2}$$

We will include an example below to show that, at least for large distances, the dependence on k in estimate in (7.2) is sharp. This means that, for a fixed geometry $((c_1,R_1),(c_2,R_2))$, the condition number increases with k. Because resolution is proportional to wavelength, this means that we cannot increase resolution by simply increasing the wavenumber without increasing the dynamic range of the sensors (i.e., the number of significant figures in the measured data). Note that as k increases, the dimensions of the subspaces $V_i = T^*_{c_i} l^2(-N_i,N_i) \approx T^*_{c_i} l^2(-kR_i,kR_i)$ increase. The increase in the number of significant Fourier coefficients (nonevanescent Fourier modes) is the way we see higher resolution in this problem.

The situation changes considerably if we replace the limited power source radiated from $B_{R_1}(c_1)$ by a point source with singularity in c_1. Then we can choose for V_1 a one-dimensional subspace of l^2 (spanned by the zeroth order Fourier mode translated by $T^*_{c_1}$) and accordingly set $N_1 = R_1 = 0$. Consequently, the estimate (7.2) reduces to

$$\csc(\theta_{12}) \,\le\, \frac{1}{\sqrt{1-\frac{2N_2+1}{k|c_1-c_2|}}} \,\lesssim\, \frac{1}{\sqrt{1-\frac{2kR_2+1}{k|c_1-c_2|}}}\,. \tag{7.3}$$

Since numerator and denominator have the same units, the conditioning of the splitting operator does not depend on k in this case.

This has immediate consequences for the inverse scattering problem: Qualitative reconstruction methods like the linear sampling method [2] or the factorization method [13] determine the support of an unknown scatterer by testing pointwise whether the far field of a point source belongs to the range of a certain restricted far field operator, mapping sources supported inside the scatterer to their radiated far field. The inequality (7.3) indeed shows that (using these qualitative reconstruction algorithms for the inverse scattering problem) one can increase resolution by simply increasing the wavenumber.

Finally, if we replace both sources by point sources with singularities in c_1 and c_2, respectively, then we can choose both subspaces V_1 and V_2 to be one-dimensional and accordingly set $N_1 = N_2 = R_1 = R_2 = 0$. The estimate (7.2) reduces to

$$\csc(\theta_{12}) \,\le\, \frac{1}{\sqrt{1-\frac{1}{k|c_1-c_2|}}}\,, \tag{7.4}$$

i.e., in this case the conditioning of the splitting operator improves with increasing wavenumber k. MUSIC-type reconstruction methods [5] for inverse scattering problems with infinitesimally small scatterers recover the locations of a collection of unknown small scatterers by testing pointwise whether the far field of a point source belongs to the range of a certain restricted far field operator, mapping point sources with singularities at the positions of the small scatterers to their radiated far field. From (7.4) we conclude that (using MUSIC-type reconstruction algorithms for the inverse scattering problem with infinitesimally small scatterers) one can increase resolution by simply increasing the wavenumber, and the reconstruction becomes more stable for higher frequencies.

8. An analytic example. The example below illustrates that the estimate of the cosine of the angle between two far fields radiated by two sources supported in balls $B_{R_1}(c_1)$ and $B_{R_2}(c_2)$, respectively, cannot be better than proportional to the quantity

$$\sqrt{\frac{kR_1R_2}{|c_1-c_2|}}\,.$$

As pointed out in the previous section, we need only construct the example for $k=1$. We will let f be a single-layer source supported on a horizontal line segment of width W, and let g be the same source, translated vertically by a distance d (i.e., $c_1=(0,0)$ and $c_2=(0,d)$). Specifically, with H denoting the Heaviside or indicator function, and δ the Dirac mass, we obtain

$$\begin{aligned} f &= \frac{1}{\sqrt{W}} H_{|x|<W}\delta_{y=0}, \\ g &= \frac{1}{\sqrt{W}} H_{|x|<W}\delta_{y=d}. \end{aligned}$$

The far fields radiated by f and g are

$$\begin{aligned} \alpha_f(\theta) &= (\mathcal{F}f)(\theta) = 2\frac{\sin(W\cos t)}{\sqrt{W}\cos t}, \\ \alpha_g(\theta) &= (\mathcal{F}g)(\theta) = e^{-\mathrm{i}d\sin t}\,2\frac{\sin(W\cos t)}{\sqrt{W}\cos t} \end{aligned}$$

for $\theta=(\cos t,\sin t)\in S^1$. Accordingly

$$\begin{aligned} \|\alpha_f\|_2^2 = \|\alpha_g\|_2^2 &= 4\int_0^{2\pi}\frac{\sin^2(W\cos t)}{(W\cos t)^2}W\,\mathrm{d}t = 8\int_{-W}^{W}\frac{\sin^2(\xi)}{\xi^2}\frac{W}{\sqrt{W^2-\xi^2}}\,\mathrm{d}\xi \\ &\geq 8\int_{-W}^{W}\frac{\sin^2(\xi)}{\xi^2}\,\mathrm{d}\xi = 8\int_{-\infty}^{\infty}\frac{\sin^2(\xi)}{\xi^2}\,\mathrm{d}\xi - 16\int_{W}^{\infty}\frac{\sin^2(\xi)}{\xi^2}\,\mathrm{d}\xi\,, \end{aligned}$$

and we can evaluate the first integral on the right-hand side using the Plancherel equality as $2\frac{\sin\xi}{\xi}$ is the Fourier transform of the characteristic function of the interval $[-1,1]$, and we estimate the second, yielding

$$\|\alpha_f\|_2^2 \geq 8\Big(\pi-\frac{2}{W}\Big)\,.$$

On the other hand, for $d\gg W$, according to the principle of stationary phase (there are stationary points at $\pm\frac{\pi}{2}$),

$$\langle\alpha_f,\alpha_g\rangle = 4W\int_0^{2\pi}\frac{\sin^2(W\cos t)}{(W\cos t)^2}e^{-\mathrm{i}d\sin t}\,\mathrm{d}t = 8\sqrt{2\pi}\frac{W}{\sqrt{d}}\cos\Big(d-\frac{\pi}{4}\Big)+O(d^{-\frac{3}{2}})\,,$$

which shows that for $d \gg W \gg 1$

$$\frac{\langle \alpha_f, \alpha_g \rangle}{\|\alpha_f\|_2 \|\alpha_g\|_2} \approx \sqrt{\frac{2}{\pi}} \frac{W}{\sqrt{d}} \cos\Big(d - \frac{\pi}{4}\Big) ,$$

which decays no faster than that predicted by Theorem 5.3.

9. Numerical examples. Next we consider the numerical implementation of the l^2 approach from section 5 and the l^1 approach from section 6 for far field splitting and data completion simultaneously (cf. Theorems 5.8 and 6.7). Since both schemes are extensions of corresponding algorithms for far field splitting as described in [9] (least squares) and [10] (basis pursuit), we just briefly comment on modifications that have to be made to include data completion and refer the reader to [9, 10] for further details.

Given a far field $\alpha = \sum_{i=1}^{I} T_{c_i}^* \alpha_i$ that is a superposition of far fields $T_{c_i}^* \alpha_i$ radiated from balls $B_{R_i}(c_i)$ for some $c_i \in \mathbb{R}^2$ and $R_i > 0$, we assume in the following that we are unable to observe all of α and that a subset $\Omega \subseteq S^1$ is unobserved. The aim is to recover $\alpha|_\Omega$ from $\alpha|_{S^1 \setminus \Omega}$ and a priori information on the location of the supports of the individual source components $B_{R_i}(c_i)$, $i = 1, \ldots, I$.

We first consider the l^2 approach from section 5 and write $\gamma := \alpha|_{S^1 \setminus \Omega}$ for the observed far field data and $\beta := -\alpha|_\Omega$. Accordingly,

$$\gamma = \beta + \sum_{i=1}^{I} T_{c_i}^* \alpha_i ,$$

i.e., we are in the setting of Theorem 5.8. Using the shorthand $V_\Omega := L^2(\Omega)$ and $V_i := T_{c_i}^* l^2(-N_i, N_i)$, $i = 1, \ldots, I$, the least squares problem (5.15) is equivalent to seeking approximations $\widetilde{\beta} \in V_\Omega$ and $\widetilde{\alpha}_i \in l^2(-N_i, N_i)$, $i = 1, \ldots, I$, satisfying the Galerkin condition

$$\langle \widetilde{\beta} + T_{c_1}^* \widetilde{\alpha}_1 + \cdots + T_{c_I}^* \widetilde{\alpha}_I, \phi \rangle = \langle \gamma, \phi \rangle \qquad \text{for all } \phi \in V_\Omega \oplus V_1 \oplus \cdots \oplus V_I . \tag{9.1}$$

The size of the individual subspaces depends on the a priori information on $R_1, \ldots, R_I$. Following our discussion at the end of section 3 we choose $N_j = \frac{e}{2} k R_j$ in our numerical example below. Denoting by P_Ω and $P_1, \ldots, P_I$ the orthogonal projections onto V_Ω and $V_1, \ldots, V_I$, respectively, (9.1) is equivalent to the linear system

$$\begin{aligned} \widetilde{\beta} + P_\Omega P_1 T_{c_1}^* \widetilde{\alpha}_1 + \cdots + P_\Omega P_I T_{c_I}^* \widetilde{\alpha}_I &= 0 , \\ P_1 P_\Omega \widetilde{\beta} + T_{c_1}^* \widetilde{\alpha}_1 + \cdots + P_1 P_I T_{c_I}^* \widetilde{\alpha}_I &= P_1 \gamma , \\ &\vdots \\ P_I P_\Omega \widetilde{\beta} + P_I P_1 T_{c_1}^* \widetilde{\alpha}_1 + \cdots + T_{c_I}^* \widetilde{\alpha}_I &= P_I \gamma . \end{aligned} \tag{9.2}$$

Explicit matrix representations of the individual matrix blocks in (9.2) follow directly from (4.2)–(4.3) (see [9, Lemma 3.3] for details) for $P_1, \ldots, P_I$ and by applying a DFT to the characteristic function on $S^1 \setminus \Omega$ for P_Ω. Accordingly, the block matrix corresponding to the entire linear system can be assembled, and the linear system can be solved directly. The estimates from Theorem 5.8 give bounds on the absolute condition number of the system matrix.

The main advantage of the l^1 approach from section 6 is that no a priori information on the radii R_i of the balls $B_{R_i}(c_i)$, $i = 1, \ldots, I$, containing the individual

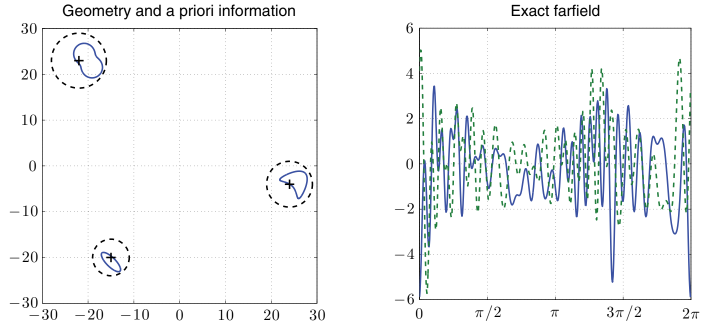

FIG. 9.1. *Left: Geometry of the scatterers (solid) and a priori information on the source locations (dashed). Right: Real part (solid) and imaginary part (dashed) of the far field α.*

source components is required. However, we still assume that a priori knowledge of the centers $c_1, \dots, c_I$ of such balls is available. Using the orthogonal projection P_Ω onto $L^2(\Omega)$, the basis pursuit formulation from Theorem 6.7 can be rewritten as

$$(\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_I) = \operatorname{argmin} \sum_{i=1}^{I} \|\alpha_i\|_{l^1} \quad \text{s.t.} \quad \left\| \gamma - P_\Omega \left(\sum_{i=1}^{I} T_{c_i}^* \alpha_i \right) \right\|_2 \le \delta \,, \ \alpha_i \in L^2(S^1) \,. \tag{9.3}$$

Accordingly, $\widetilde{\beta} := \sum_{i=1}^{I} (T_{c_i}^* \widetilde{\alpha}_i)|_\Omega$ is an approximation of the missing data segment. It is well known that the minimization problem from (9.3) is equivalent to minimizing the Tikhonov functional

$$\Psi_\mu(\alpha_1, \dots, \alpha_I) \ = \ \left\| \gamma - P_\Omega \left(\sum_{i=1}^{I} T_{c_i}^* \alpha_i \right) \right\|_2^2 + \mu \sum_{i=1}^{I} \|\alpha_i\|_{l^1} \,, \tag{9.4}$$

$[\alpha_1, \dots, \alpha_I] \in l^2 \times \cdots \times l^2$, for a suitably chosen regularization parameter $\mu > 0$ (see, e.g., [8, Proposition 2.2]). The unique minimizer of this functional can be approximated using (fast) iterative soft thresholding (cf. [1, 4]). Apart from the projection P_Ω, which can be implemented straightforwardly, our numerical implementation, as well as the convergence analysis, is analogous to the implementation of far field splitting described in [10].[4]

Example 9.1. We consider a scattering problem with three obstacles as shown in Figure 9.1 (left), which are illuminated by a plane wave $u^i(x) = e^{ikx\cdot d}$, $x \in \mathbb{R}^2$, with incident direction $d = (1, 0)$ and wavenumber $k = 1$ (i.e., the wavelength is $\lambda = 2\pi \approx 6.28$). Assuming that the ellipse is sound soft whereas the kite and the nut are sound hard, the scattered field u^s satisfies the homogeneous Helmholtz equation outside the obstacles, the Sommerfeld radiation condition at infinity, and Dirichlet (ellipse) or Neumann boundary conditions (kite and nut) on the boundaries of the obstacles. We simulate the corresponding far field α of u^s on an equidistant grid with 512 points on the unit sphere S^1 using a Nyström method (cf. [3, 14]). Figure 9.1 (right) shows the real part (solid line) and the imaginary part (dashed line) of α. Since the far field α can be written as a superposition of three far fields radiated by

---

[4] In [10] we used additional weights in the l^1 minimization problem to ensure that its solution indeed gives the exact far field split. Here we don't use these weights, but our estimates from section 6 imply that the solution of (9.3) and (9.4) is very close to the true split.

FIG. 9.2. *Reconstruction of the least squares scheme: Observed far field γ (left), reconstruction of the missing part $\alpha|_\Omega$ (middle), and the difference between exact far field and reconstructed far field (right).*

three individual smooth sources supported in arbitrarily small neighborhoods of the scattering obstacles (cf., e.g., [17, lemma 3.6]), this example fits into the framework of the previous sections.

We assume that the far field cannot be measured on the segment

$$\Omega \;=\; \{\theta = (\cos t, \sin t) \in S^1 \mid \pi/2 < t < \pi/2 + \pi/3\}\,,$$

i.e., $|\Omega| = \pi/3$. We first apply the least squares procedure and use the dashed circles shown in Figure 9.1 (left) as a priori information on the approximate source locations $B_{R_i}(c_i)$, $i = 1, 2, 3$. More precisely, $c_1 = (24, -4)$, $c_2 = (-22, 23)$, $c_3 = (-15, -20)$ and $R_1 = 5$, $R_2 = 6$, and $R_3 = 4$. Accordingly we choose $N_1 = 7$, $N_2 = 9$, and $N_3 = 6$ and solve the linear system (9.2).

Figure 9.2 shows plots of the observed data γ (left), of the reconstruction of the missing data segment obtained by the least squares algorithm (middle), and of the difference between the exact far field and the reconstructed far field (right). Again the solid line corresponds to the real part, while the dashed line corresponds to the imaginary part. The condition number of the matrix is 5.4×10^4. We note that the missing data component in this example is actually too large for the assumptions of Theorem 5.8 to be satisfied. Nevertheless the least squares approach still gives good results. The relative approximation error of the reconstruction is

$$\frac{\|\alpha|_\Omega - \widetilde{\beta}\|_{L^2(\Omega)}}{\|\alpha|_\Omega\|_{L^2(\Omega)}} \approx 0.101 \quad\text{and}\quad \frac{\|\alpha - (T^*_{c_1}\widetilde{\alpha}_1 + T^*_{c_2}\widetilde{\alpha}_2 + T^*_{c_3}\widetilde{\alpha}_3)\|_{L^2(S^1)}}{\|\alpha\|_{L^2(S^1)}} \approx 0.0248\,.$$

For comparison, we note that applying the least squares algorithm to full aperture data (i.e., $\Omega = \emptyset$) to compute just the far field split (without data completion) yields reconstructions $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, and $\widetilde{\alpha}_3$ with a relative approximation error

$$\frac{\|\alpha - (T^*_{c_1}\widetilde{\alpha}_1 + T^*_{c_2}\widetilde{\alpha}_2 + T^*_{c_3}\widetilde{\alpha}_3)\|_{L^2(S^1)}}{\|\alpha\|_{L^2(S^1)}} \approx 0.00127\,.$$

Applying the (fast) iterative soft shrinkage algorithm to this example (with regularization parameter $\mu = 10^{-3}$ in (9.4)) does not give a useful reconstruction. As indicated by the estimates in Theorem 6.7 the l^1 approach seems to be a bit less stable. Hence we halve the missing data segment; consider

$$\Omega \;=\; \{\theta = (\cos t, \sin t) \in S^1 \mid \pi/2 < t < \pi/2 + \pi/6\}\,,$$

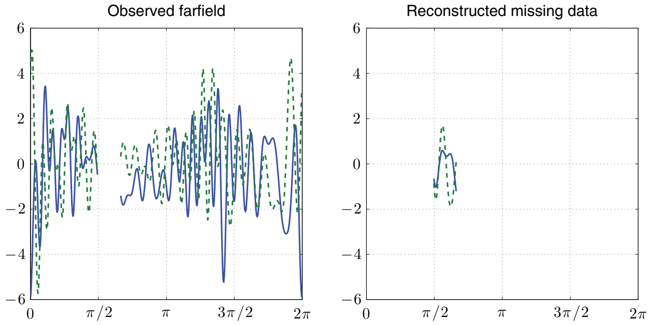

FIG. 9.3. *Reconstruction of the basis pursuit scheme: Observed far field γ (left), reconstruction of the missing part $\alpha|_\Omega$ (middle), and difference between exact far field and reconstructed far field (right).*

i.e., $|\Omega| = \pi/6$, and apply the l^1 reconstruction scheme to this data. Figure 9.3 shows plots of the observed data γ (left), of the reconstruction of the missing data segment obtained by the fast iterative soft shrinkage algorithm (with $\mu = 10^{-3}$) after 10^3 iterations (the initial guess is zero), and of the difference between the exact far field and the reconstructed far field. The relative approximation error of this reconstruction is

$$\frac{\|\alpha|_\Omega - \widetilde{\beta}\|_{L^2(\Omega)}}{\|\alpha|_\Omega\|_{L^2(\Omega)}} \approx 0.0484 \quad \text{and} \quad \frac{\|\alpha - (T^*_{c_1}\widetilde{\alpha}_1 + T^*_{c_2}\widetilde{\alpha}_2 + T^*_{c_3}\widetilde{\alpha}_3)\|_{L^2(S^1)}}{\|\alpha\|_{L^2(S^1)}} \approx 0.00865\,.$$

Again we note that applying the (fast) iterative soft shrinkage algorithm (with the same parameters as above) to full aperture data (i.e., $\Omega = \emptyset$) to compute just the far field split (without data completion) yields reconstructions $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, and $\widetilde{\alpha}_3$ with a relative approximation error

$$\frac{\|\alpha - (T^*_{c_1}\widetilde{\alpha}_1 + T^*_{c_2}\widetilde{\alpha}_2 + T^*_{c_3}\widetilde{\alpha}_3)\|_{L^2(S^1)}}{\|\alpha\|_{L^2(S^1)}} \approx 0.00642\,.$$

The behavior of both algorithms in the presence of noise in the data depends crucially on the geometrical setup of the problem (i.e., on its conditioning). The smaller the missing data segment is and the smaller the dimensions of the individual source components are relative to their distances, the more noise these algorithms can handle.

Conclusions. We have considered the source problem for the two-dimensional Helmholtz equation when the source is a superposition of finitely many well-separated compactly supported source components. We have presented stability estimates for numerical algorithms to split the far field radiated by this source into the far fields corresponding to the individual source components and to restore missing data segments. Analytic and numerical examples confirm the sharpness of these estimates and illustrate the potential and limitations of the numerical schemes.

The following are the most significant observations: (i) The conditioning of far field splitting and data completion depends on the dimensions of the source components, their relative distances with respect to wavelength, and the size of the missing data segment. The results clearly suggest combining data completion with splitting whenever possible in order to improve the conditioning of the data completion problem. (ii) The conditioning of far field splitting and data completion depends on wavelength and deteriorates with increasing wavenumber. Therefore, in order to increase resolution, one has to increase not only the wavenumber, but also the dynamic range of the sensors used to measure the far field data.

REFERENCES

[1] A. Beck and M. Teboulle, *A fast iterative shrinkage-thresholding algorithm for linear inverse problems*, SIAM J. Imaging Sci., 2 (2009), pp. 183–202, https://doi.org/10.1137/080716542.

[2] F. Cakoni and D. Colton, *A Qualitative Approach to Inverse Scattering Theory*, Springer, New York, 2014, https://doi.org/10.1007/978-1-4614-8827-9.

[3] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd ed., Springer, Berlin, 1998, https://doi.org/10.1007/978-3-662-03537-5.

[4] I. Daubechies, M. Defrise, and C. De Mol, *An iterative thresholding algorithm for linear inverse problems with a sparsity constraint*, Comm. Pure Appl. Math., 57 (2004), pp. 1413–1457, https://doi.org/10.1002/cpa.20042.

[5] A. J. Devaney, *Mathematical Foundations of Imaging, Tomography and Wavefield Inversion*, Cambridge University Press, Cambridge, UK, 2012, https://doi.org/10.1017/CBO9781139047838.

[6] D. L. Donoho, M. Elad, and V. N. Temlyakov, *Stable recovery of sparse overcomplete representations in the presence of noise*, IEEE Trans. Inform. Theory, 52 (2006), pp. 6–18, https://doi.org/10.1109/TIT.2005.860430.

[7] D. L. Donoho and P. B. Stark, *Uncertainty principles and signal recovery*, SIAM J. Appl. Math., 49 (1989), pp. 906–931, https://doi.org/10.1137/0149053.

[8] M. Grasmair, M. Haltmeier, and O. Scherzer, *Necessary and sufficient conditions for linear convergence of* ℓ^1*-regularization*, Comm. Pure Appl. Math., 64 (2011), pp. 161–182, https://doi.org/10.1002/cpa.20350.

[9] R. Griesmaier, M. Hanke, and J. Sylvester, *Far field splitting for the Helmholtz equation*, SIAM J. Numer. Anal., 52 (2014), pp. 343–362, https://doi.org/10.1137/120891381.

[10] R. Griesmaier and J. Sylvester, *Far field splitting by iteratively reweighted* ℓ^1 *minimization*, SIAM J. Appl. Math., 76 (2016), pp. 705–730, https://doi.org/10.1137/15M102839X.

[11] M. J. Grote, M. Kray, F. Nataf, and F. Assous, *Wave splitting for time-dependent scattered field separation*, C. R. Math. Acad. Sci. Paris, 353 (2015), pp. 523–527, https://doi.org/10.1016/j.crma.2015.03.008.

[12] F. ben Hassen, J. Liu, and R. Potthast, *On source analysis by wave splitting with applications in inverse scattering of multiple obstacles*, J. Comput. Math., 25 (2007), pp. 266–281.

[13] A. Kirsch and N. Grinberg, *The Factorization Method for Inverse Problems*, Oxford University Press, Oxford, 2008.

[14] R. Kress, *On the numerical solution of a hypersingular integral equation in scattering theory*, J. Comput. Appl. Math., 61 (1995), pp. 345–360, https://doi.org/10.1016/0377-0427(94)00073-7.

[15] I. Krasikov, *Uniform bounds for Bessel functions*, J. Appl. Anal., 12 (2006), pp. 83–91, https://doi.org/10.1515/JAA.2006.83.

[16] S. Kusiak and J. Sylvester, *The scattering support*, Comm. Pure Appl. Math., 56 (2003), pp. 1525–1548, https://doi.org/10.1002/cpa.3038.

[17] S. Kusiak and J. Sylvester, *The convex scattering support in a background medium*, SIAM J. Math. Anal., 36 (2005), pp. 1142–1158, https://doi.org/10.1137/S0036141003433577.

[18] L. J. Landau, *Bessel functions: Monotonicity and bounds*, J. London Math. Soc. (2), 61 (2000), pp. 197–215, https://doi.org/10.1112/S0024610799008352.

[19] R. Potthast, F. M. Fazi, and P. A. Nelson, *Source splitting via the point source method*, Inverse Problems, 26 (2010), 045002, https://doi.org/10.1088/0266-5611/26/4/045002.

[20] D. Slepian, *Some comments on Fourier analysis, uncertainty and modeling*, SIAM Rev., 25 (1983), pp. 379–393, https://doi.org/10.1137/1025078.

[21] J. Sylvester, *Notions of support for far fields*, Inverse Problems, 22 (2006), pp. 1273–1288, https://doi.org/10.1088/0266-5611/22/4/010.