SUPPLEMENTARY MATERIALS: UNCERTAINTY PRINCIPLES FOR INVERSE SOURCE PROBLEMS, FAR FIELD SPLITTING AND DATA COMPLETION*

ROLAND GRIESMAIER† AND JOHN SYLVESTER‡

SM1. Supplements to section 3. In the following we collect some interesting properties of the rescaled squared singular values $\{s_n^2(R)\}$, as introduced in (3.3), of the restricted Fourier transform $\mathcal{F}_{B_R(0)}$ from (3.1).

We first note that [SM4, 10.22.5] implies that the rescaled squared singular values from (3.3) satisfy

$$s_n^2(R) \,=\, \pi R^2\big(J_n^2(R) - J_{n-1}(R)J_{n+1}(R)\big)\,, \qquad n\in\mathbb{Z}\,,$$

and simple manipulations using recurrence relations for Bessel functions

$$\text{(SM1.1a)}\qquad \frac{nJ_n(r)}{r} \,=\, \frac{J_{n+1}(r)+J_{n-1}(r)}{2}\,, \qquad n\in\mathbb{Z}\,,$$

$$\text{(SM1.1b)}\qquad J_n'(r) \,=\, \frac{J_{n-1} - J_{n+1}(r)}{2}\,, \qquad n\in\mathbb{Z}\,,$$

(cf., e.g., [SM4, 10.6(i)]) show that, for $n\in\mathbb{Z}$

$$\begin{aligned} s_n^2(R) \,&=\, \pi\big(n^2J_n^2(R) - R^2J_{n-1}(R)J_{n+1}(R) + R^2J_n(R) - n^2J_n^2(R)\big)\\ &=\, \pi\Big(R^2\Big(\frac{J_{n-1}(R)-J_{n+1}(R)}{2}\Big)^2 + (R^2-n^2)J_n^2(R)\Big) \qquad \text{(SM1.2)}\\ &=\, \pi\big((RJ_n'(R))^2 + (R^2-n^2)J_n^2(R)\big)\,. \end{aligned}$$

LEMMA SM1.1.

$$\sum_{n=-\infty}^{\infty} s_n^2(R) \,=\, \pi R^2\,.$$

Proof. Since

$$\sum_{n=-\infty}^{\infty} J_n^2(r) \,=\, 1$$

(cf. [SM4, 10.23.3]), the definition (3.3) yields

$$\sum_{n=-\infty}^{\infty} s_n^2(R) \,=\, 2\pi\int_0^R\Big(\sum_{n=-\infty}^{\infty}J_n^2(r)\Big)r\,\mathrm{d}r \,=\, 2\pi\int_0^R r\,\mathrm{d}r \,=\, \pi R^2\,.$$

∎

The next lemma shows that odd and even rescaled squared singular values, $s_n^2(R)$, are monotonically decreasing for $n\geq 0$ and monotonically increasing for $n\leq 0$.

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†Institut für Mathematik, Universität Würzburg, 97074 Würzburg, Germany (roland.griesmaier@uni-wuerzburg.de).
‡Department of Mathematics, University of Washington, Seattle, Washington 98195, U.S.A. (sylvest@u.washington.edu). Research partially supported by NSF grant DMS-1007447.

LEMMA SM1.2.

$$s_{n-1}^2(R) - s_{n+1}^2(R) \;\geq\; 0\,, \qquad n \geq 0\,.$$

Proof. Using the recurrence relations (SM1.1) we find that

$$J_{n-1}^2(r) - J_{n+1}^2(r) \;=\; \frac{4n}{r} J_n(r) J_n'(r) \;=\; \frac{2n}{r}(J_n^2)'(r)\,.$$

Thus,

$$\begin{aligned} s_{n-1}^2(R) - s_{n+1}^2(R) \;&=\; 2\pi\Big(\int_0^R J_{n-1}^2(r) r \,\mathrm{d}r - \int_0^R J_{n+1}^2(r) r \,\mathrm{d}r\Big) \\ &=\; 2\pi \int_0^R 2n (J_n^2)'(r)\,\mathrm{d}r \;=\; 4\pi n J_n^2(R) \;\geq\; 0\,. \end{aligned}$$

∎

Integrating sharp estimates for $J_n(r)$ from [SM3], we obtain upper bounds for $s_n^2(R)$ when $|n| \geq R > 0$. Since $s_n^2(R) = s_{-n}^2(R)$, $n \in \mathbb{Z}$, it is sufficient to consider $n \geq R$.

LEMMA SM1.3. *Suppose that $n \geq R > 0$. Then*

$$s_n^2(R) \;\leq\; \frac{\pi 2^{\frac{2}{3}} n^{\frac{2}{3}}}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2} \Big(\frac{n+\frac{1}{2}}{n}\Big)^{n+1} \Big(\frac{R^2}{n^2} e^{1-\frac{R^2}{n^2}}\Big)^n \frac{R^2}{n^2}\,.$$

Proof. From theorem 2 of [SM3] we obtain for $0 < r \leq n$ that

$$J_n^2(r) \;\leq\; \frac{2^{\frac{2}{3}}}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2} \frac{r^{2n}}{n^{2n+\frac{2}{3}}} e^{\frac{n^2-r^2}{n+\frac{1}{2}}}\,.$$

Substituting this into (3.3) yields

$$\begin{aligned} s_n^2(R) \;&\leq\; 2\pi \frac{2^{\frac{2}{3}}}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2} \frac{e^{\frac{n^2}{n+\frac{1}{2}}}}{n^{2n+\frac{2}{3}}} \int_0^R r^{2n} e^{-\frac{r^2}{n+\frac{1}{2}}} r \,\mathrm{d}r \\ &=\; \pi \frac{2^{\frac{2}{3}}}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2} \frac{e^{\frac{n^2}{n+\frac{1}{2}}}}{n^{2n+\frac{2}{3}}} \Big(n+\frac{1}{2}\Big)^{n+1} \int_0^{\frac{R^2}{n+\frac{1}{2}}} t^n e^{-t}\,\mathrm{d}t \\ &\leq\; \pi \frac{2^{\frac{2}{3}}}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2} \frac{e^n}{n^{2n+\frac{2}{3}}} \Big(n+\frac{1}{2}\Big)^{n+1} \int_0^{\frac{R^2}{n}} t^n e^{-t}\,\mathrm{d}t\,. \end{aligned}$$

Since $t^n e^{-t}$ is monotonically increasing for $0 < t < \frac{R^2}{n} \leq n$, we see

$$\begin{aligned} s_n^2(R) \;&=\; \pi \frac{2^{\frac{2}{3}}}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2} \frac{(n+\frac{1}{2})^{n+1}}{n^{2n+\frac{2}{3}}} e^n \frac{R^2}{n} \frac{R^{2n}}{n^n} e^{-\frac{R^2}{n}} \\ &=\; \pi \frac{2^{\frac{2}{3}}}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2} n^{\frac{2}{3}} \Big(\frac{n+\frac{1}{2}}{n}\Big)^{n+1} \Big(\frac{R^2}{n^2} e^{1-\frac{R^2}{n^2}}\Big)^n \frac{R^2}{n^2}\,. \end{aligned}$$

∎

On the other hand, the rescaled squared singular values $s_n^2(R)$ are not small for $|n| < R$.

THEOREM SM1.4. *Suppose that* $R > n \geq 0$, *define* $\alpha \in (0, \frac{\pi}{2})$ *by* $\cos\alpha = \frac{n}{R}$, *and therefore* $\sin\alpha = \sqrt{1 - (\frac{n}{R})^2}$, *and assume* $\alpha > \varepsilon > 0$. *Then*

$$\left| J_n(R) - \sqrt{\frac{2}{\pi R \sin\alpha}} \cos\Big(R(\sin\alpha - \alpha\cos\alpha) - \frac{\pi}{4}\Big) \right| \leq \frac{C(\varepsilon)}{R}\,, \tag{SM1.3}$$

$$\left| J_n'(R) + \sqrt{\frac{2}{\pi R \sin\alpha}} \sin\alpha \sin\Big(R(\sin\alpha - \alpha\cos\alpha) - \frac{\pi}{4}\Big) \right| \leq \frac{C(\varepsilon)}{R}\,. \tag{SM1.4}$$

where the constant $C(\varepsilon)$ *depends on the lower bound* ε *but is otherwise independent of* n *and* R.

Proof. By definition,

$$\begin{aligned} J_n(R) \;&=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}(R\sin t - nt)}\,\mathrm{d}t \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}R(\sin t - t\cos\alpha)}\,\mathrm{d}t \\ &=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}R\phi(t)}\,\mathrm{d}t \end{aligned} \tag{SM1.5}$$

with

$$\phi(t) \;=\; \sin t - t\cos\alpha\,, \quad \phi'(t) \;=\; \cos t - \cos\alpha\,, \quad \phi''(t) \;=\; -\sin t\,.$$

The phase function ϕ has stationary points at $\pm\alpha$, and ϕ'' vanishes at 0 and π. We will apply stationary phase in a neighborhood of each stationary point. The neighborhood must be small enough to guarantee that $|\phi''(t)|$ is bounded from below there. Integration by parts will be used to estimate the integral in regions where ϕ' is bounded below. The hypothesis that $\alpha > \varepsilon > 0$ will guarantee that the union of these two regions covers the whole interval $(-\pi, \pi)$.

To separate the two regions, let $a_\varepsilon(\tau) = a\big(\frac{\tau}{\varepsilon}\big)$, $\tau \in \mathbb{R}$, be a C^∞ cutoff function satisfying

$$a_\varepsilon(\tau) \;=\; \begin{cases} 1 & \text{if } |\tau| > 2\varepsilon \\ 0 & \text{if } |\tau| < \varepsilon \end{cases} \qquad \text{and} \qquad |a_\varepsilon^{(j)}(\tau)| \;\leq\; \frac{C_j}{\varepsilon^j}$$

with the $C_j > 0$ independent of $\varepsilon > 0$. Define $A_\varepsilon(t) := a_\varepsilon(\phi'(t))$, $t \in (-\pi, \pi)$, then

$$A_\varepsilon(t) \;=\; \begin{cases} 1 & \text{if } |\phi'(t)| > 2\varepsilon\,, \\ 0 & \text{if } |\phi'(t)| < \varepsilon\,. \end{cases} \tag{SM1.6}$$

Theorem 7.7.1 of [SM1] tells us that for any integer $k \geq 0$

$$\left| \int_{-\pi}^{\pi} e^{\mathrm{i}R\phi(t)} A_\varepsilon(t)\,\mathrm{d}t \right| \;\leq\; \frac{C}{R^k} \sum_{j=0}^{k} \sup_{t \in \operatorname{supp} A_\varepsilon} \left| \frac{A_\varepsilon^{(j)}(t)}{(\phi'(t))^{2k-j}} \right| \;\leq\; \frac{C}{R^k \varepsilon^{2k}} \sum_{j=0}^{k} C_j \tag{SM1.7}$$

with C only depending on an upper bound for the higher order derivatives of ϕ. For the second inequality we have used (SM1.6) and the fact that all higher derivatives of ϕ are bounded by 1.

We will estimate the remainder of the integral using Theorem 7.7.5 of [SM1], which tells us that, if t_0 is the unique stationary point of ϕ in the support of a smooth function B, and $|\phi''(t)| > \delta > 0$ on the support of B, then

$$\left| \int_{-\pi}^{\pi} e^{\mathrm{i}R\phi(t)} B(t)\,\mathrm{d}t - \sqrt{\frac{2\pi\mathrm{i}}{R\phi''(t_0)}} e^{\mathrm{i}R\phi(t_0)} B(t_0) \right| \le \frac{C}{R} \sum_{j=0}^{2} \sup_{t\in\operatorname{supp} B} |B^{(j)}(t)| \tag{SM1.8}$$

with $C > 0$ depending only on the lower bound δ for $|\phi''(t)|$ and an upper bound for higher derivatives of ϕ on the support of B. We will set $B = 1 - A_\varepsilon(t)$, which is supported in two intervals, one containing α and the other containing $-\alpha$, so (SM1.8), becomes

$$\begin{aligned} \left| \int_{-\pi}^{\pi} e^{\mathrm{i}R\phi(t)} (1 - A_\varepsilon(t))\,\mathrm{d}t - \sqrt{\frac{2\pi}{R\sin\alpha}} 2\cos\Big(R(\sin\alpha - \alpha\cos\alpha) - \frac{\pi}{4}\Big) \right| \\ \le \frac{C}{R} \sum_{j=0}^{2} \frac{C_j}{\varepsilon^j} \end{aligned} \tag{SM1.9}$$

as long as ε is chosen so that $|\phi''(t)| = |\sin(t)| \ge \delta$ on the support of $1 - A_\varepsilon(t)$.

The following lemma suggests a proper choice of ε.

Lemma SM1.5. *Let t and α belong to $\left[0, \frac{\pi}{2}\right]$ then*

$$|\cos t - \cos\alpha| < \varepsilon$$

implies that

$$\sin t > \sin\alpha - \frac{2\varepsilon}{\sin\alpha}.$$

Proof. Since

$$\sin^2 t - \sin^2\alpha = \cos^2\alpha - \cos^2 t$$

we deduce

$$\sin t - \sin\alpha = \frac{\cos\alpha + \cos t}{\sin t + \sin\alpha} (\cos\alpha - \cos t).$$

Consequently

$$|\sin t - \sin\alpha| \le \frac{2}{\sin\alpha}\varepsilon.$$

∎

End of proof of theorem SM1.4. We choose $\varepsilon = \frac{\sin^2\alpha}{4}$ and assume that $|\phi'(t)| = |\cos t - \cos\alpha| < \varepsilon$, then lemma SM1.5 gives $\sin t > \frac{\sin\alpha}{2}$. We use this value of ε in (SM1.6), i.e.

$$|\phi'(t)| > 2\varepsilon \quad \text{on } \operatorname{supp} A_\varepsilon \qquad \text{and} \qquad |\phi'(t)| < \varepsilon \quad \text{on } \operatorname{supp}(1 - A_\varepsilon).$$

Accordingly,

$$|\phi''(t)| = |\sin t| > \frac{\sin\alpha}{2} = \sqrt{\varepsilon} =: \delta \qquad \text{on } \operatorname{supp}(1 - A_\varepsilon).$$

Now, adding (SM1.7) and (SM1.9) establishes (SM1.3).

The calculation for (SM1.4) is analogous with (SM1.5) replaced by

$$J_n'(R) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{i}\sin(t) e^{\mathrm{i}(R\sin t - nt)}\, \mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{i}\sin(t) e^{\mathrm{i}R\phi(t)}\, \mathrm{d}t\,,$$

which has the same phase and hence the same stationary points. The only difference is that the term $B(t_0)$ in (SM1.8) at $t_0 = \pm\alpha$ will be $\pm\mathrm{i}\sin\alpha$ rather than 1. ∎

We now combine (SM1.3) and (SM1.4) with the equality (SM1.2) to obtain, for $\cos\alpha = \frac{n}{R} < 1-\varepsilon$

$$\left|s_n^2(R) - 2R\sin\alpha\right| = \left|s_n^2(R) - 2\sqrt{R^2-n^2}\right| \le C(\varepsilon)\sqrt{R}\,.$$

Since equation (SM1.5) is only a valid definition of the Bessel function $J_n(R)$ when n is an integer[1], we denote in the following by $\lceil \nu R\rceil$ is smallest integer that is greater than or equal to νR, so that we can state a convergence result.

Corollary SM1.6.

$$\lim_{R\to\infty} \frac{s^2_{\lceil \nu R\rceil}(R)}{2R} = \begin{cases} \sqrt{1-\nu^2} & \nu \le 1 \\ 0 & \nu \ge 1 \end{cases}$$

SM2. Supplements to section 4. Here we give a proof of estimate (4.11).

Let $n \in \mathbb{Z}$ and $\mu := (2n+1)(2n+3)$. Theorem 2 of [SM2] establishes that for $r > \frac{\sqrt{\mu+\mu^{\frac{3}{2}}}}{2}$,

$$J_n^2(r) \le \frac{4(4r^2-(2n+1)(2n+5))}{\pi((4r^2-\mu)^{\frac{3}{2}}-\mu)}\,. \tag{SM2.1}$$

The following lemma shows that, under the assumptions of theorem 4.6, the estimate (SM2.1) implies (4.11).

Lemma SM2.1. *Let $M, N \ge 1$ and $r > 2(M+N+1)$, then*

$$\sup_{-(M+N)<n<(M+N)} J_n^2(r) \le \frac{b}{r} \qquad \text{with } b \approx 0.7595\,.$$

Proof. Since $J_{-n}^2(r) = J_n^2(r)$ we may assume w.l.o.g. that $n \ge 0$. Let $\eta := \sqrt{(n+\frac{1}{2})(n+\frac{3}{2})}$. Then $\frac{\mu}{4} = \eta^2 = n^2+2n+\frac{3}{4}$, i.e.

$$\frac{3}{4} \le \eta^2 \le (n+1)^2 \tag{SM2.2}$$

and therefore our assumption $r > 2(M+N+1)$ implies for $0 \le n < M+N$ that

$$r > 2(n+1) \ge 2\eta\,. \tag{SM2.3}$$

Accordingly,

$$\frac{1}{2}\sqrt{\mu+\mu^{\frac{3}{2}}} = \frac{1}{2}\sqrt{4\eta^2+(4\eta^2)^{\frac{3}{2}}} = \eta\sqrt{1+\frac{1}{(4\eta^2)^{\frac{1}{3}}}} \le \eta\sqrt{1+\frac{1}{(4(\frac{3}{4})^2)^{\frac{1}{3}}}}$$
$$\le \sqrt{2}\eta \le \frac{r}{\sqrt{2}} \le r\,.$$

[1]The definition requires a contour integral when ν is not an integer.

This shows that the assumptions of theorem 2 of [SM2] are satisfied.

Next we consider (SM2.1) and further estimate its right hand side:

$$\begin{aligned} J_n^2(r) \;&\leq\; \frac{4(4r^2-(2n+1)(2n+5))}{\pi((4r^2-\mu)^{3/2}-\mu)} \;\leq\; \frac{4(4r^2-4\eta^2)}{\pi(8(r^2-\eta^2)^{\frac32}-4\eta^2)}\\ &=\; \frac{2}{\pi}\frac{1}{(r^2-\eta^2)^{\frac12}}\frac{1}{1-\frac12\frac{\eta^2}{(r^2-\eta^2)^{\frac32}}} \;=\; \frac{2}{\pi}\frac{1}{r}\frac{1}{\left(1-\left(\frac{\eta}{r}\right)^2\right)^{\frac12}}\frac{1}{1-\frac12\frac{\eta^2}{(r^2-\eta^2)^{\frac32}}}\,. \end{aligned}$$

Since $r > 2(M+N+1) \geq 6$, applying (SM2.2) and (SM2.3) yields

$$\frac{\eta^2}{(r^2-\eta^2)^{\frac32}} \;=\; \frac{1}{r}\frac{\left(\frac{\eta}{r}\right)^2}{\left(1-\left(\frac{\eta}{r}\right)^2\right)^{\frac32}} \;\leq\; \frac{1}{r}\frac{\frac14}{\left(\frac34\right)^{\frac32}} \;=\; \frac{1}{3\sqrt{27}}\,,$$

whence

$$J_n^2(r) \;\leq\; \frac{2}{\pi}\frac{1}{r}\left(\frac43\right)^{\frac12}\frac{1}{1-\frac12\frac{1}{3\sqrt{27}}} \;=\; \frac{b}{r} \qquad \text{with } b \approx 0.7595\,.$$

∎

SM3. Supplements to section 5.

Proof of theorem 5.7. As in (5.9), we decompose each γ^j

$$\gamma^j \;=\; w^j + w_\perp^j\,,$$

where each w^j belongs to the subspace

$$W \;=\; \bigoplus_{i=1}^{I} T_{c_i}^* l^2(-N_i, N_i)$$

and each $w_\perp^j$ is orthogonal to W. Subtracting gives

$$w^1 - w^0 \;=\; \sum_{i=1}^{I} T_{c_i}^*(\alpha_i^1 - \alpha_i^0)$$

and applying (4.10) shows that

$$\begin{aligned} \|w^1-w^0\|_2^2 \;&\geq\; \sum_{i=1}^{I}\|\alpha_i^1-\alpha_i^0\|_2^2 - \sum_{i=1}^{I}\sum_{j\neq i}|\langle\alpha_i^1-\alpha_i^0, T_{c_j-c_i}^*(\alpha_j^1-\alpha_j^0)\rangle|\\ &\geq\; \sum_{i=1}^{I}\|\alpha_i^1-\alpha_i^0\|_2^2 - \sum_{i=1}^{I}\sum_{j\neq i}\left(\left(\frac{r_ir_j}{|c_{ij}|}\right)^{\frac14}\|\alpha_i^1-\alpha_i^0\|_2\left(\frac{r_ir_j}{|c_{ij}|}\right)^{\frac14}\|\alpha_j^1-\alpha_j^0\|_2\right)\\ &\geq\; \sum_{i=1}^{I}\|\alpha_i^1-\alpha_i^0\|_2^2 \qquad \text{(SM3.1)}\\ &\quad - \frac12\left(\sum_{i=1}^{I}\sum_{j\neq i}\left(\left(\frac{r_ir_j}{|c_{ij}|}\right)^{\frac12}\|\alpha_i^1-\alpha_i^0\|_2^2 + \left(\frac{r_ir_j}{|c_{ij}|}\right)^{\frac12}\|\alpha_j^1-\alpha_j^0\|_2^2\right)\right.\\ &=\; \sum_{i=1}^{I}\|\alpha_i^1-\alpha_i^0\|_2^2\left(1-\sum_{j\neq i}\left(\frac{r_ir_j}{|c_{ij}|}\right)^{\frac12}\right)\,, \end{aligned}$$

where $r_i = 2N_i + 1$, $r_j = 2N_j + 1$ and $c_{ij} = c_i - c_j$. Since (5.12) is true here as well this ends the proof. ∎

Proof of theorem 5.8. As in (5.9), we decompose each γ^j

$$\gamma^j \;=\; w^j + w^j_\perp\,,$$

where each w^j belongs to the subspace

$$W \;=\; L^2(\Omega) \oplus \bigoplus_{i=1}^{J} T^*_{c_i} l^2(-N_i, N_i)$$

and each $w^j_\perp$ is orthogonal to W. Subtracting gives

$$w^1 - w^0 \;=\; \beta^1 - \beta^0 + \sum_{i=1}^{I} T^*_{c_i}(\alpha^1_i - \alpha^0_i)$$

and thus

$$\begin{aligned}\|w^1 - w^0\|_2^2 \;\geq\; &\|\beta^1 - \beta^0\|_2^2 - 2\sum_{i=1}^{I} |\langle T^*_{c_i}(\alpha^1_i - \alpha^0_i), \beta^1 - \beta^0\rangle| \\ &+ \sum_{i=1}^{I} \|\alpha^1_i - \alpha^0_i\|_2^2 - \sum_{i=1}^{I}\sum_{j\neq i} |\langle \alpha^1_i - \alpha^0_i, T^*_{c_j - c_i}(\alpha^1_j - \alpha^0_j)\rangle|\,.\end{aligned}$$

Proceeding as in (SM3.1), using (4.10) and (4.13), and applying (5.12) finishes the proof. ∎

SM4. Supplements to section 6. The constraint (6.12) in corollary 6.2 is much more restrictive than the corresponding assumption (5.6) in theorem 5.3, and also the estimate (6.13) is weaker than (5.8). However, if we add to the hypothesis of theorem 6.1 all *a priori* assumptions on the non-evanescent subspaces used in theorem 5.3, the result improves as follows.

COROLLARY SM4.1. *If we add to the hypothesis of theorem 6.1:*

$$\alpha^0_i, \alpha_i \in l^2(-N_i, N_i) \qquad \textit{and} \qquad |c_1 - c_2| \;>\; 2(N_1 + N_2 + 1)$$

for some $N_1, N_2 \in \mathbb{N}$ and replace (6.1) with

$$\frac{(2N_1+1)(2N_2+1)}{|c_1 - c_2|} \;<\; 1 \tag{SM4.1}$$

the conclusion becomes

$$\|\alpha^0_i - \alpha^1_i\|_2^2 \;\leq\; \Big(1 - \frac{(2N_1+1)(2N_2+1)}{|c_1 - c_2|}\Big)^{-1} 4\delta^2\,. \tag{SM4.2}$$

The constraint (SM4.1) in corollary SM4.1 is now the same as the corresponding assumption (5.6) in theorem 5.3, but the estimate (SM4.2) still differs from (5.8) (after taking the square root on both sides of these inequalities) by a factor of two. However, the main advantage of the l^1 approach is clearly that no a priori knowledge of the size of the non-evanescent subspaces is required. If such a priori information is available, we recommend using least squares.

Proof of corollary SM4.1. Proceeding as in (6.9) and applying (4.10), we find that

$$\begin{aligned} 4\delta^2 &\geq \|\alpha_1 - \alpha_1^0\|_2^2 + \|\alpha_2 - \alpha_2^0\|_2^2 - 2|\langle \alpha_1 - \alpha_1^0, T^*_{c_2-c_1}(\alpha_2 - \alpha_2^0)\rangle| \\ &\geq \|\alpha_1 - \alpha_1^0\|_2^2 + \|\alpha_2 - \alpha_2^0\|_2^2 - 2\Big(\frac{r_1 r_2}{|c_{12}|}\Big)^{\frac{1}{2}} \|\alpha_1 - \alpha_1^0\|_2 \|\alpha_2 - \alpha_2^0\|_2 \\ &\geq \Big(1 - \frac{r_1 r_2}{|c_{12}|}\Big)\|\alpha_1 - \alpha_1^0\|_2^2 + \Big(\|\alpha_2 - \alpha_2^0\|_2 - \Big(\frac{r_1 r_2}{|c_{12}|}\Big)^{\frac{1}{2}} \|\alpha_1 - \alpha_1^0\|_2\Big)^2 , \end{aligned}$$

where $r_1 = 2N_1 + 1$, $r_2 = 2N_2 + 1$ and $c_{12} = c_1 - c_2$. Dropping the second term gives (6.13) for α_1, and we may interchange the roles of α_1 and α_2 in the proof to obtain the estimate for α_2. ◻

Proof of corollary 6.4. Proceeding as in (6.6)–(6.8) we find that

$$\frac{1}{\tau}\|\alpha - \alpha^0\|_{l^1} + \tau\|\beta - \beta^0\|_{L^1} \leq 2\big(\frac{1}{\tau}\|\alpha - \alpha^0\|_{l^1(W)} + \tau\|\beta - \beta^0\|_{L^1(\Omega)}\big) \tag{SM4.3}$$

with W representing the l^0-support of α^0. Applying similar arguments as in (6.9)–(6.11) and (6.16) together with (SM4.3) yields

$$\begin{aligned} 4\delta^2 &\geq \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - \frac{2}{\sqrt{2\pi}}\|\alpha - \alpha^0\|_{l^1}\|\beta - \beta^0\|_{L^1} \\ &\geq \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - \frac{1}{2\sqrt{2\pi}}\Big(\frac{1}{\tau}\|\alpha - \alpha^0\|_{l^1} + \tau\|\beta - \beta^0\|_{L^1}\Big)^2 \\ &\geq \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - \frac{2}{\sqrt{2\pi}}\Big(\frac{1}{\tau}\|\alpha - \alpha^0\|_{l^1(W)} + \tau\|\beta - \beta^0\|_{L^1(\Omega)}\Big)^2 \\ &\geq \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - \frac{2}{\sqrt{2\pi}}\Big(\frac{1}{\tau}\sqrt{|W|}\|\alpha - \alpha^0\|_2 + \tau\sqrt{|\Omega|}\|\beta - \beta^0\|_2\Big)^2 \\ &\geq \|\alpha - \alpha^0\|_2^2 + \|\beta - \beta^0\|_2^2 - \frac{4}{\sqrt{2\pi}}\Big(\frac{1}{\tau^2}|W|\|\alpha - \alpha^0\|_2^2 + \tau^2|\Omega|\|\beta - \beta^0\|_2^2\Big) . \end{aligned}$$

This ends the proof because $|W| = \|\alpha^0\|_{l^0}$. ◻

Proof of theorem 6.5. Proceeding as in (6.6)–(6.8) we find that

$$\sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_{l^1} \leq 2\sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_{l^1(W_i)} \tag{SM4.4}$$

with W_i representing the l^0-support of α_i^0. Applying similar arguments as in (6.9)–

(6.10) and using the inequality (SM5.3) and (SM4.4) we obtain

$$
\begin{aligned}
4\delta^2 &\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - \sum_{i=1}^{I} \sum_{j\neq i} |\langle \alpha_i - \alpha_i^0, T^*_{c_j - c_i}(\alpha_j - \alpha_j^0)\rangle| \\
&\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - \sum_{i=1}^{I} \sum_{j\neq i} \frac{1}{|c_i - c_j|^{\frac{1}{3}}} \|\alpha_i - \alpha_i^0\|_{l^1} \|\alpha_j - \alpha_j^0\|_{l^1} \\
&\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - \max_{j\neq k} \frac{1}{|c_j - c_k|^{\frac{1}{3}}} \sum_{i=1}^{I} \sum_{j\neq i} \|\alpha_i - \alpha_i^0\|_{l^1} \|\alpha_j - \alpha_j^0\|_{l^1} \\
&\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - \max_{j\neq k} \frac{1}{|c_j - c_k|^{\frac{1}{3}}} \frac{I-1}{I} \Big(\sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_{l^1}\Big)^2 \\
&\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - \max_{j\neq k} \frac{1}{|c_j - c_k|^{\frac{1}{3}}} \frac{I-1}{I} 4\Big(\sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_{l^1(W_i)}\Big)^2 .
\end{aligned} \tag{SM4.5}
$$

Applying Hölder's inequality and (SM5.2) yields

$$
\begin{aligned}
4\delta^2 &\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - \max_{j\neq k} \frac{1}{|c_j - c_k|^{\frac{1}{3}}} \frac{I-1}{I} 4\Big(\sum_{i=1}^{I} |W_i|^{\frac{1}{2}} \|\alpha_i - \alpha_i^0\|_2\Big)^2 \\
&\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - \max_{j\neq k} \frac{1}{|c_j - c_k|^{\frac{1}{3}}} 4(I-1) \sum_{i=1}^{I} |W_i| \|\alpha_i - \alpha_i^0\|_2^2 ,
\end{aligned} \tag{SM4.6}
$$

where $|W_i| = \|\alpha_i^0\|_{l^0}$. □

Proof of theorem 6.7. Proceeding as in (6.6)–(6.8) we find that

$$
\sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_{l^1} \leq 2 \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_{l^1(W_i)} \tag{SM4.7}
$$

with W_i representing the l^0-support of α_i^0. Applying similar arguments as in (6.9) we obtain

$$
\begin{aligned}
4\delta^2 \geq &\sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 + \|\beta - \beta^0\|_2^2 - \sum_{i=1}^{I} \sum_{j\neq i} |\langle \alpha_i - \alpha_i^0, T^*_{c_j - c_i}(\alpha_j - \alpha_j^0)\rangle| \\
&- 2\sum_{i=1}^{I} |\langle T^*_{c_i}(\alpha_i - \alpha_i^0), \beta - \beta^0\rangle| .
\end{aligned} \tag{SM4.8}
$$

The third term on the right hand side of (SM4.8) can be estimated as in (SM4.5)–(SM4.6), while for the last term we find using Hölder's inequality, the mapping properties of the operator which maps α to its Fourier coefficients and (SM4.7) that

$$\begin{aligned} 2\sum_{i=1}^{I} |\langle T_{c_i}^*(\alpha_i - \alpha_i^0), \beta - \beta^0\rangle| \;&\leq\; 2\Big(\sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_{L^\infty}\Big)\|\beta - \beta^0\|_{L^1} \\ &\leq\; \frac{2}{\sqrt{2\pi}}\Big(\sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_{l^1}\Big)\|\beta - \beta^0\|_{L^1} \\ &\leq\; \frac{4}{\sqrt{2\pi}}\Big(\sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_{l^1(W_i)}\Big)\|\beta - \beta^0\|_{L^1} \\ &\leq\; \frac{4}{\sqrt{2\pi}}\Big(\sum_{i=1}^{I} \sqrt{|W_i|}\|\alpha_i - \alpha_i^0\|_2\Big)\sqrt{|\Omega|}\|\beta - \beta^0\|_2 \\ &\leq\; \frac{2}{\sqrt{2\pi}}\Big(\sum_{i=1}^{I} \sqrt{|\Omega||W_i|}\|\alpha_i - \alpha_i^0\|_2^2 + \sum_{i=1}^{I} \sqrt{|\Omega||W_i|}\|\beta - \beta^0\|_2^2\Big)\,, \end{aligned}$$

where $|W_i| = \|\alpha_i^0\|_{l^0}$. Combining these estimates ends the proof. ∎

Finally, we establish variants of theorem 6.5 and theorem 6.7, where we replace the l^1 minimization in (6.20) and (6.22) by a weighted l^1 minimization in order to obtain better estimates for certain geometric configurations of the supports of the individual source components. In contrast to theorem 6.5 the constant in the stability estimate (SM4.10) in theorem SM4.2 below only depends on the distances of source components relative to the source component corresponding to the far field component appearing on the left hand side of the estimate.

THEOREM SM4.2. *Suppose that $\gamma^0, \alpha_i^0 \in L^2(S^1)$ and $c_i \in \mathbb{R}^2$, and set*

$$a_i^2 \;=\; \max_{j\neq i}\Big(\frac{2}{|c_i - c_j|}\Big)^{\frac{1}{3}} \qquad \text{or} \qquad a_i^2 \;=\; \max_{\substack{j\neq i\\ k\neq i,j}}\Big(\frac{1}{|c_i - c_j|} + \frac{1}{|c_i - c_k|}\Big)^{\frac{1}{3}}\,, \tag{SM4.9}$$

$i = 1, \ldots, I$. Assume that

$$4(I-1)a_i^2\|\alpha_i^0\|_{l^0} \;<\; 1 \qquad \text{for each } i\,,$$

and

$$\|\gamma^0 - \sum_{i=1}^{I} T_{c_i}^*\alpha_i^0\|_2 \;\leq\; \delta_0 \qquad \text{for some } \delta_0 \geq 0\,.$$

If $\delta \geq 0$ and $\gamma \in L^2(S^1)$ with

$$\delta \;\geq\; \delta_0 + \|\gamma - \gamma^0\|_2$$

and

$$(\alpha_1, \ldots, \alpha_I) \;=\; \operatorname{argmin} \sum_{i=1}^{I} a_i\|\alpha_i\|_{l^1} \quad \text{s.t.} \quad \|\gamma - \sum_{i=1}^{I} T_{c_i}^*\alpha_i\|_2 \leq \delta\,,\; \alpha_i \in L^2(S^1)\,,$$

then, for $i = 1, \ldots, I$

$$\|\alpha_i^0 - \alpha_i\|_2^2 \;\leq\; \big(1 - 4(I-1)a_i^2\|\alpha_i^0\|_{l^0}\big)^{-1} 4\delta^2\,. \tag{SM4.10}$$

Proof. Proceeding as in (6.6)–(6.8) we find

$$\sum_{i=1}^{I} a_i \|\alpha_i - \alpha_i^0\|_{l^1} \;\le\; 2\sum_{i=1}^{I} a_i \|\alpha_i - \alpha_i^0\|_{l^1(W_i)} \tag{SM4.11}$$

with W_i representing the l^0-support of α_i^0.

Abbreviating $c_{ij} = c_i - c_j$, the triangle inequality shows

$$|c_{ij}| + |c_{ik}| \;=\; |c_i - c_j| + |c_i - c_k| \;\ge\; |c_j - c_k| \;=\; |c_{jk}|\,,$$

i.e.,

$$\frac{1}{|c_{ij}|} + \frac{1}{|c_{ik}|} \;\ge\; \frac{|c_{jk}|}{|c_{ij}||c_{ik}|}\,.$$

Thus our assumption (SM4.9) implies

$$a_i^2 \;\ge\; \Big(\frac{|c_{jk}|}{|c_{ij}||c_{ik}|}\Big)^{\frac{1}{3}} \qquad \text{for every } j \ne k\,,$$

and therefore, using (SM5.1)

$$\begin{aligned}
\Big(\sum_{i=1}^{I} a_i\|\alpha_i\|_{l^1}\Big)^2 \;&=\; \sum_{i=1}^{I} a_i^2\|\alpha_i\|_{l^1}^2 + \sum_{i=1}^{I}\sum_{j\ne i} a_i a_j \|\alpha_i\|_{l^1}\|\alpha_j\|_{l^1} \\
&\ge\; \frac{I}{I-1}\sum_{i=1}^{I}\sum_{j\ne i} a_i a_j \|\alpha_i\|_{l^1}\|\alpha_j\|_{l^1} \\
&\ge\; \frac{I}{I-1}\sum_{i=1}^{I}\sum_{j\ne i} \Big(\frac{|c_{jk}|}{|c_{ij}||c_{ik}|}\Big)^{\frac{1}{6}}\Big(\frac{|c_{ik}|}{|c_{ij}||c_{jk}|}\Big)^{\frac{1}{6}} \|\alpha_i\|_{l^1}\|\alpha_j\|_{l^1} \\
&=\; \frac{I}{I-1}\sum_{i=1}^{I}\sum_{j\ne i} \frac{1}{|c_{ij}|^{\frac{1}{3}}} \|\alpha_i\|_{l^1}\|\alpha_j\|_{l^1}\,.
\end{aligned} \tag{SM4.12}$$

Now, applying similar arguments as in (6.9)–(6.10), using (SM4.11) and (SM4.12), we obtain

$$\begin{aligned}
4\delta^2 \;&\ge\; \sum_{i=1}^{I}\|\alpha_i - \alpha_i^0\|_2^2 - \sum_{i=1}^{I}\sum_{j\ne i}|\langle \alpha_i - \alpha_i^0, T^*_{c_j - c_i}(\alpha_j - \alpha_j^0)\rangle| \\
&\ge\; \sum_{i=1}^{I}\|\alpha_i - \alpha_i^0\|_2^2 - \sum_{i=1}^{I}\sum_{j\ne i}\frac{1}{|c_{ij}|^{\frac{1}{3}}}\|\alpha_i - \alpha_i^0\|_{l^1}\|\alpha_j - \alpha_j^0\|_{l^1} \\
&\ge\; \sum_{i=1}^{I}\|\alpha_i - \alpha_i^0\|_2^2 - \frac{I-1}{I}\Big(\sum_{i=1}^{I} a_i\|\alpha_i - \alpha_i^0\|_{l^1}\Big)^2
\end{aligned} \tag{SM4.13}$$

Using Hölder's inequality and (SM5.2) we deduce

$$\begin{aligned}4\delta^2 &\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - 4\frac{I-1}{I}\Big(\sum_{i=1}^{I} a_i \|\alpha_i - \alpha_i^0\|_{l^1(W_i)}\Big)^2 \\ &\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - 4\frac{I-1}{I}\Big(\sum_{i=1}^{I} a_i |W_i|^{\frac{1}{2}} \|\alpha_i - \alpha_i^0\|_2\Big)^2 \\ &\geq \sum_{i=1}^{I} \|\alpha_i - \alpha_i^0\|_2^2 - 4(I-1)\sum_{i=1}^{I} a_i^2 |W_i| \|\alpha_i - \alpha_i^0\|_2^2\,,\end{aligned} \tag{SM4.14}$$

where $|W_i| = \|\alpha_i^0\|_{l^0}$. This ends the proof. ∎

COROLLARY SM4.3. *If we add to the hypothesis of theorem SM4.2:*

$$\alpha_i^0, \alpha_i \in l^2(-N_i, N_i); \text{ for each } i \quad \text{and} \quad |c_i - c_j| > 2(N_i + N_j + 1) \text{ for every } j \neq i$$

for some $N_1, \ldots, N_I \in \mathbb{N}$, *and replace* (SM4.9) *with*

$$a_i^2 = \max_{j\neq i}\Big(\frac{2}{|c_i - c_j|}\Big)^{\frac{1}{2}} \qquad \text{or} \qquad a_i^2 = \max_{\substack{j\neq i\\ k\neq i,j}}\Big(\frac{1}{|c_i - c_j|} + \frac{1}{|c_i - c_k|}\Big)^{\frac{1}{2}} \qquad \text{for each } i\,,$$

the conclusion remains true.

THEOREM SM4.4. *Suppose that* $\gamma^0, \alpha_i^0 \in L^2(S^1)$, $c_i \in \mathbb{R}^2$, $\Omega \subseteq S^1$ *and* $\beta^0 \in L^2(\Omega)$, *and set*

$$a_i^2 = \max_{j\neq i}\Big(\frac{2}{|c_i - c_j|}\Big)^{\frac{1}{3}} \qquad \text{or} \qquad a_i^2 = \max_{\substack{j\neq i\\ k\neq i,j}}\Big(\frac{1}{|c_i - c_j|} + \frac{1}{|c_i - c_k|}\Big)^{\frac{1}{3}}, \tag{SM4.15}$$

$i = 1, \ldots, I$. *Assume that*

$$\frac{2}{\sqrt{2\pi}}\Big(\max_j \frac{1}{a_j}\Big)\sum_{i=1}^{I} a_i \sqrt{|\Omega| \|\alpha_i^0\|_{l^0}} < 1\,,$$

$$4(I-1)a_i^2 \|\alpha_i^0\|_{l^0} + \frac{2}{\sqrt{2\pi}}\Big(\max_j \frac{1}{a_j}\Big) a_i \sqrt{|\Omega| \|\alpha_i^0\|_{l^0}} < 1 \qquad \text{for each } i\,,$$

and

$$\|\gamma^0 - \beta^0 - \sum_{i=1}^{I} T_{c_i}^* \alpha_i^0\|_2 \leq \delta_0 \qquad \text{for some } \delta_0 \geq 0\,.$$

If $\delta \geq 0$ *and* $\gamma \in L^2(S^1)$ *with*

$$\delta \geq \delta_0 + \|\gamma - \gamma^0\|_2$$

and

$$(\alpha_1, \ldots, \alpha_I) = \operatorname{argmin} \sum_{i=1}^{I} a_i \|\alpha_i\|_{l^1}$$

$$\text{s.t.} \quad \|\gamma - \beta - \sum_{i=1}^{I} T_{c_i}^* \alpha_i\|_2 \leq \delta\,,\ \alpha_i \in L^2(S^1)\,,\ \beta \in L^2(\Omega)\,,$$

then

$$\|\beta^0-\beta\|_2^2 \le \Big(1-\frac{2}{\sqrt{2\pi}}\Big(\max_j\frac{1}{a_j}\Big)\sum_{i=1}^I a_i\sqrt{|\Omega|\|\alpha_i^0\|_{l^0}}\Big)^{-1}4\delta^2\,,$$

and, for $i=1,\dots,I$

$$\|\alpha_i^0-\alpha_i\|_2^2 \le \Big(1-4(I-1)a_i^2\|\alpha_i^0\|_{l^0}+\frac{2}{\sqrt{2\pi}}\Big(\max_j\frac{1}{a_j}\Big)a_i\sqrt{|\Omega|\|\alpha_i^0\|_{l^0}}\Big)^{-1}4\delta^2\,.$$

Proof. Proceeding as in (6.6)–(6.8) we find that

$$\sum_{i=1}^I a_i\|\alpha_i-\alpha_i^0\|_{l^1} \le 2\sum_{i=1}^I a_i\|\alpha_i-\alpha_i^0\|_{l^1(W_i)} \tag{SM4.16}$$

with W_i representing the l^0-support of α_i^0. Applying similar arguments as in (6.9) we obtain

$$\begin{aligned}4\delta^2 \ge \sum_{i=1}^I\|\alpha_i-\alpha_i^0\|_2^2+\|\beta-\beta^0\|_2^2-\sum_{i=1}^I\sum_{j\neq i}|\langle\alpha_i-\alpha_i^0,T^*_{c_j-c_i}(\alpha_j-\alpha_j^0)\rangle|\\ -2\sum_{i=1}^I|\langle T^*_{c_i}(\alpha_i-\alpha_i^0),\beta-\beta^0\rangle|\,.\end{aligned} \tag{SM4.17}$$

The third term on the right hand side of (SM4.17) can be estimated as in (SM4.13)–(SM4.14), while for the last term we find using Hölder's inequality, the mapping properties of the operator which maps α to its Fourier coefficients, and (SM4.16) that

$$\begin{aligned}2\sum_{i=1}^I|\langle T^*_{c_i}(\alpha_i-\alpha_i^0),\beta-\beta^0\rangle| &\le 2\Big(\sum_{i=1}^I\|\alpha_i-\alpha_i^0\|_{L^\infty}\Big)\|\beta-\beta^0\|_{L^1}\\ &\le \frac{2}{\sqrt{2\pi}}\max_j\frac{1}{a_j}\Big(\sum_{i=1}^I a_i\|\alpha_i-\alpha_i^0\|_{l^1}\Big)\|\beta-\beta^0\|_{L^1}\\ &\le \frac{4}{\sqrt{2\pi}}\max_j\frac{1}{a_j}\Big(\sum_{i=1}^I a_i\|\alpha_i-\alpha_i^0\|_{l^1(W_i)}\Big)\|\beta-\beta^0\|_{L^1}\,.\end{aligned}$$

Applying Hölder's inequality once more yields

$$\begin{aligned}&2\sum_{i=1}^I|\langle T^*_{c_i}(\alpha_i-\alpha_i^0),\beta-\beta^0\rangle|\\ &\le \frac{4}{\sqrt{2\pi}}\max_j\frac{1}{a_j}\Big(\sum_{i=1}^I a_i\sqrt{|W_i|}\|\alpha_i-\alpha_i^0\|_2\Big)\sqrt{|\Omega|}\|\beta-\beta^0\|_2\\ &= \frac{4}{\sqrt{2\pi}}\max_j\frac{1}{a_j}\sum_{i=1}^I\Big(\sqrt{a_i}(|\Omega||W_i|)^{\frac14}\|\alpha_i-\alpha_i^0\|_2\sqrt{a_i}(|\Omega||W_i|)^{\frac14}\|\beta-\beta^0\|_2\Big)\\ &\le \frac{2}{\sqrt{2\pi}}\max_j\frac{1}{a_j}\sum_{i=1}^I\Big(a_i(|\Omega||W_i|)^{\frac12}\|\alpha_i-\alpha_i^0\|_2^2+a_i(|\Omega||W_i|)^{\frac12}\|\beta-\beta^0\|_2^2\Big)\,,\end{aligned}$$

where $|W_i|=\|\alpha_i^0\|_{l^0}$. Combining these estimates ends the proof. ∎

COROLLARY SM4.5. *If we add to the hypothesis of theorem SM4.4:*

$$\alpha_i^0, \alpha_i \in l^2(-N_i, N_i) \text{ for each } i \quad \text{and} \quad |c_i - c_j| > 2(N_i + N_j + 1) \text{ for every } j \neq i$$

for some $N_1, \dots, N_I \in \mathbb{N}$, *and replace* (SM4.15) *with*

$$a_i^2 = \max_{j\neq i}\Big(\frac{2}{|c_i - c_j|}\Big)^{\frac12} \qquad \text{or} \qquad a_i^2 = \max_{\substack{j\neq i\\ k\neq i,j}}\Big(\frac{1}{|c_i - c_j|} + \frac{1}{|c_i - c_k|}\Big)^{\frac12} \qquad \text{for each } i\,,$$

the conclusion remains true.

SM5. Some elementary inequalities. Here we prove some elementary inequalities that we haven't been able to find in the literature.

LEMMA SM5.1. *Let* $a_1, \dots, a_I \in \mathbb{R}$. *Then*

(a)

$$\sum_{i=1}^I \sum_{j\neq i} a_i a_j \le (I-1) \sum_{i=1}^I a_i^2\,, \tag{SM5.1}$$

(b)

$$\Big(\sum_{i=1}^I a_i\Big)^2 \le I \sum_{i=1}^I a_i^2\,, \tag{SM5.2}$$

(c)

$$\sum_{i=1}^I \sum_{j\neq i} a_i a_j \le \frac{I-1}{I} \Big(\sum_{i=1}^I a_i\Big)^2\,. \tag{SM5.3}$$

Proof. (a)

$$0 \le \sum_{i=1}^I \sum_{j\neq i} (a_j - a_i)^2 = 2(I-1)\sum_{i=1}^I a_i^2 - 2\sum_{i=1}^I\sum_{j\neq i} a_i a_j\,.$$

(b) Using (SM5.1) we find that

$$\Big(\sum_{i=1}^I a_i\Big)^2 = \sum_{i=1}^I a_i^2 + \sum_{i=1}^I\sum_{j\neq i} a_i a_j \le I\sum_{i=1}^I a_i^2\,.$$

(c) Proceeding as in (b) but applying (SM5.1) the other way round yields

$$\Big(\sum_{i=1}^I a_i\Big)^2 = \sum_{i=1}^I a_i^2 + \sum_{i=1}^I\sum_{j\neq i} a_i a_j \ge \frac{I}{I-1}\sum_{i=1}^I\sum_{j\neq i} a_i a_j\,.$$

∎

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