Corrigendum concerning Theorems 5.12 to 5.15

List of changes:

- *Theorem 5.12:* Part (d) has been corrected, part (e) has been added.
- *Theorem 5.13:* The assumption on ε_r has been modified.
- *Theorem 5.14:* In the proof the definition of $\mathcal{T}_0$ has been changed according to the definition in Theorem 5.12, part (d).
- *Theorem 5.15:* We changed the condition on ε_r according to the new assumptions on q in Theorem 5.12, part (e).

Theorem 5.12 *Let the conditions of Assumption 5.9 hold and let $\mathcal{T} : L^2(D,\mathbb{C}^3) \to L^2(D,\mathbb{C}^3)$ be defined as in Theorem 5.10, i.e.,*

$$\mathcal{T}f \;=\; (\operatorname{sign} q)\left[f + \sqrt{|q|}\,\operatorname{curl} v\right]$$

and $v \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ solves (5.54), i.e. is a radiating variational solution of

$$\operatorname{curl}\left[\frac{1}{\varepsilon_r}\operatorname{curl} v\right] - k^2 v \;=\; \operatorname{curl}\left[\frac{q}{\sqrt{|q|}}\,f\right] \quad \text{in } \mathbb{R}^3 . \tag{5.58}$$

Here, again, $q = 1 - 1/\varepsilon_r$ denotes the contrast. Then we have:

(a)

$$\operatorname{Im}(\mathcal{T}f, f)_{L^2(D)} \;\geq\; 0 \quad \text{for all } f \in L^2(D,\mathbb{C}^3) .$$

(b) Assume that k^2 is not an eigenvalue of the interior transmission eigenvalue problem of Definition 5.8. Then

$$\operatorname{Im}(\mathcal{T}f, f)_{L^2(D)} \;>\; 0$$

for all $f \in \overline{\mathcal{R}(\mathcal{H})} \subset L^2(D,\mathbb{C}^3)$ with $f \neq 0$. Here, again $\mathcal{H} : L^2_t(S^2) \to L^2(D,\mathbb{C}^3)$ is defined in (5.53), and $\overline{\mathcal{R}(\mathcal{H})}$ denotes the closure of $\mathcal{R}(\mathcal{H})$ in $L^2(D,\mathbb{C}^3)$.

(c) Assume that there exists a constant $\gamma_0 > 0$ such that $\operatorname{Im} q \geq \gamma_0 |q|$ almost everywhere in D. Then there exists $\gamma_1 > 0$ such that

$$\operatorname{Im}(\mathcal{T}f, f)_{L^2(D)} \;\geq\; \gamma_1\,\|f\|^2_{L^2(D)} \tag{5.59}$$

for all $f \in L^2(D,\mathbb{C}^3)$.

(d) Define the operator $\mathcal{T}_0$ from $L^2(D,\mathbb{C}^3)$ into itself by $\mathcal{T}_0 f = (\operatorname{sign} q)\left[f + \sqrt{|q|}\,\operatorname{curl}\hat{v}\right]$ and $\hat{v} \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ solves (5.58) for $k = i$ and $f \in L^2(D,\mathbb{C}^3)$. Then $\mathcal{T} - \mathcal{T}_0$ is compact in $L^2(D,\mathbb{C}^3)$.

(e) Let $\mathcal{T}_0$ be defined as in part (d). If $\operatorname{Re} q \leq -\gamma_2 |q|$ on D for some $\gamma_2 > 0$ then $-\operatorname{Re}\mathcal{T}_0$ is coercive on $L^2(D,\mathbb{C}^3)$. If q is real valued and $q > 0$ on D then $\mathcal{T}_0$ is coercive on $L^2(D,\mathbb{C}^3)$.

Proof: For $f \in L^2(D,\mathbb{C}^3)$ we have $\mathcal{T}f = (\operatorname{sign} q)\,\big[f + \sqrt{|q|}\,\operatorname{curl} v\big] = (\operatorname{sign} q)\,\tilde{w}$ with $\tilde{w} := f + \sqrt{|q|}\,\operatorname{curl} v$ and where v solves (5.58). The variational form of (5.58) can be written as

$$\iint\limits_{\mathbb{R}^3} \big[\operatorname{curl} v \cdot \operatorname{curl}\psi - k^2\, v\cdot\psi\big]\,dx \;=\; \iint\limits_{D} \frac{q}{\sqrt{|q|}}\,\tilde{w}\cdot\operatorname{curl}\psi\,dx \tag{5.60}$$

for all $\psi \in H(\operatorname{curl},\mathbb{R}^3)$ with compact support. Then, since $f = \tilde{w} - \sqrt{|q|}\,\operatorname{curl} v$,

$$\begin{aligned}(\mathcal{T}f,f)_{L^2(D)} &= \iint\limits_D (\operatorname{sign} q)\,|\tilde{w}|^2\,dx \;-\; \iint\limits_D (\operatorname{sign} q)\,\tilde{w}\cdot\operatorname{curl}\overline{v}\,\sqrt{|q|}\,dx \\ &= \iint\limits_D (\operatorname{sign} q)\,|\tilde{w}|^2\,dx \;-\; \iint\limits_D \frac{q}{\sqrt{|q|}}\,\tilde{w}\cdot\operatorname{curl}\overline{v}\,dx\,.\end{aligned}$$

Now we choose a mollifier $\phi \in C^\infty(\mathbb{R}^3)$ with $\phi \equiv 1$ for $|x| \le R$ and $\phi \equiv 0$ for $|x| \ge 2R$ and set $\psi = \phi\,\overline{v}$ in (5.60). This yields

$$\begin{aligned}(\mathcal{T}f,f)_{L^2(D)} &= \iint\limits_D (\operatorname{sign} q)\,|\tilde{w}|^2\,dx \;-\; \iint\limits_{|x|<R} \big[|\operatorname{curl} v|^2 - k^2|v|^2\big]\,dx \\ &\quad - \iint\limits_{R<|x|<2R} \big[\operatorname{curl} v\cdot\operatorname{curl}(\overline{v}\phi) - k^2|v|^2\phi\big]\,dx \\ &= \iint\limits_D (\operatorname{sign} q)\,|\tilde{w}|^2\,dx \;-\; \iint\limits_{|x|<R} \big[|\operatorname{curl} v|^2 - k^2|v|^2\big]\,dx \;- \\ &\quad - \int\limits_{|x|=R} (\hat{x}\times\operatorname{curl} v)\cdot\overline{v}ds\,.\end{aligned} \tag{5.61}$$

Here we have used Green's theorem in the angular region $\{x : R < |x| < 2R\}$ and the fact that v solves $\operatorname{curl}^2 v - k^2 v = 0$ in this region.

(a) Taking the imaginary part of (5.61) yields

$$\operatorname{Im}(\mathcal{T}f,f)_{L^2(D)} \;=\; \iint\limits_D \frac{\operatorname{Im} q}{|q|}\,|\tilde{w}|^2\,dx \;-\; \operatorname{Im}\int\limits_{|x|=R} (\hat{x}\times\operatorname{curl} v)\cdot\overline{v}\,ds\,.$$

From the radiation condition

$$\lim_{|x|\to\infty} |x|\,\big(\operatorname{curl} v(x)\times\hat{x} - ikv(x)\big) \;=\; 0$$

we conclude that for $R \to \infty$

$$\operatorname{Im}(\mathcal{T}f,f)_{L^2(D)} \;=\; \iint\limits_D \frac{\operatorname{Im} q}{|q|}\,|\tilde{w}|^2\,dx \;+\; \operatorname{Im}\left[ik\int\limits_{S^2} |v^\infty|^2 ds\right] \;\ge\; 0\,. \tag{5.62}$$

(b) Assume now that, for some $f \in \overline{\mathcal{R}(\mathcal{H})}$, the term $\operatorname{Im}(\mathcal{T}f,f)_{L^2(D)}$ vanishes. Then v^∞ vanishes on S^2 by (5.62) and thus $v \equiv 0$ outside of D. We recall that $v \in H_{loc}(\operatorname{curl},\mathbb{R}^3)$

is the radiating solution of (5.58), i.e. in variational form (using the fact that v vanishes outside of D)

$$\iint_D \left[\frac{1}{\varepsilon_r} \operatorname{curl} v \cdot \operatorname{curl} \psi - k^2\, v \cdot \psi\right] dx \;=\; \iint_D \frac{q}{\sqrt{|q|}}\, f \cdot \operatorname{curl} \psi \, dx \quad \text{for all } \psi \in H(\operatorname{curl}, D)\,.$$

Furthermore, $v|_D \in H_0(\operatorname{curl}, D)$. Setting $w = f/\sqrt{|q|}$ we observe that (v, w) satisfies the condition (5.51) from Definition 5.8. Since $f \in \overline{\mathcal{R}(\mathcal{H})}$ there exist $\tilde{w}_j \in \mathcal{R}(\mathcal{H})$ with $\tilde{w}_j \to f$ in $L^2(D, \mathbb{C}^3)$. From the form of $\mathcal{H}$ we note that $\tilde{w}_j = \sqrt{|q|}\, w_j$ for some smooth solutions w_j of

$$\operatorname{curl}^2 w_j \;-\; k^2\, w_j \;=\; 0 \quad \text{in } D\,.$$

Therefore, w_j tends to $w = f/\sqrt{|q|}$ in $L^2(D, \mathbb{C}^3, |q| dx)$. We have thus shown that (v, w) satisfies all conditions from Definition 5.8. Since k^2 is not an eigenvalue we conclude that v and $w = f/\sqrt{|q|}$ vanish which proves this part of the theorem.

(c) This follows from (5.62) by standard arguments. Indeed, if there exists no such constant γ_1 we can find a sequence $\{f_j\}$ such that $\|f_j\|_{L^2(D)} = 1$ and $\operatorname{Im}(Tf_j, f_j)_{L^2(D)} \to 0$. From (5.62) and the definition of $\tilde{w}$ we conclude that $f_j + \sqrt{|q|}\operatorname{curl} v_j \to 0$ in $L^2(D)$ where v_j denotes the solution of (5.58) for f replaced by f_j. Writing this equation (5.58) as

$$\operatorname{curl}^2 v - k^2 v \;=\; \operatorname{curl}\left[\frac{q}{\sqrt{|q|}}\,(f + \sqrt{|q|}\operatorname{curl} v)\right]$$

we note from Theorem 5.5 that $\{v_j\}$ converges to zero in $H(\operatorname{curl}, D)$ and therefore $f_j \to 0$ in $L^2(D)$. This contradicts $\|f_j\|_{L^2(D)} = 1$.

(d) From the definitions of $\mathcal{T}$ and $\mathcal{T}_0$ we note that $\mathcal{T} f - \mathcal{T}_0 f = \frac{q}{\sqrt{|q|}} \operatorname{curl}(v - \hat{v})$ where $v, \hat{v} \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ are the radiating solutions of (5.58) for k and $k = i$, respectively, i.e in variational form

$$\iint_{\mathbb{R}^3} \left[\frac{1}{\varepsilon_r} \operatorname{curl} v \cdot \operatorname{curl} \psi - k^2\, v \cdot \psi\right] dx \;=\; \iint_D \frac{q}{\sqrt{|q|}}\, f \cdot \operatorname{curl} \psi \, dx\,,$$

$$\iint_{\mathbb{R}^3} \left[\frac{1}{\varepsilon_r} \operatorname{curl} \hat{v} \cdot \operatorname{curl} \psi + \hat{v} \cdot \psi\right] dx \;=\; \iint_D \frac{q}{\sqrt{|q|}}\, f \cdot \operatorname{curl} \psi \, dx$$

for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support. By substituting $\psi = \nabla\varphi$ for scalar functions $\varphi \in H^1(\mathbb{R}^3)$ we note that $\iint_{\mathbb{R}^3} v \cdot \nabla\varphi \, dx = 0$ for all $\varphi \in H^1(\mathbb{R}^3)$ with compact support, i.e. $\operatorname{div} v = 0$ and, analogously, $\operatorname{div} \hat{v} = 0$ in $\mathbb{R}^3$. The difference $w = v - \hat{v}$ solves

$$\iint_{\mathbb{R}^3} \left[\frac{1}{\varepsilon_r} \operatorname{curl} w \cdot \operatorname{curl} \psi - k^2\, w \cdot \psi\right] dx \;=\; (k^2 + 1) \iint_{\mathbb{R}^3} \hat{v} \cdot \psi \, dx$$

for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support.

Let now the sequence $\{f_j\}$ converge weakly to zero $L^2(D, \mathbb{C}^3)$ and denote by $v_j, \hat{v}_j \in H(\operatorname{curl}, \mathbb{R}^3)$ the corresponding radiating solutions of (5.58) for k and $k = i$, respectively. Define $w_j \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ again by the difference $w_j = v_j - \hat{v}_j$. Let $B \subset \mathbb{R}^3$ be a ball which contains D in its interior. By the boundedness of the solution operator we conclude that $\{v_j\}$ and $\{\hat{v}_j\}$ converge weakly to zero in $H(\mathrm{curl}, B)$. Furthermore, from Remark 5.6 we conclude that $\hat{v}_j$ and v_j are smooth outside of $\overline{D}$ and converges uniformly (with all of its derivatives) to zero on ∂B. We determine $p_j \in H^1_\diamond(B)$ as the solution of

$$\iint\limits_B \nabla p_j \cdot \nabla \overline{\varphi}\, dx \;=\; \int\limits_{\partial B} (\nu \cdot w_j)\, \overline{\varphi}\, ds \tag{5.63}$$

for all $\varphi \in H^1_\diamond(B)$. Here, the subspace $H^1_\diamond(B)$ of $H^1(B)$ is defined as $H^1_\diamond(B) = \big\{\varphi \in H^1(B) : \iint_B \varphi\, dx = 0\big\}$. The solution of (5.63) exists and is unique since the form

$$(p, \varphi) \;\mapsto\; \iint\limits_B \nabla p \cdot \nabla \overline{\varphi}\, dx$$

is bounded and coercive on $H^1_\diamond(B)$ by the inequality of Poincaré (cf. [3]). The latter states that there exists a constant $c > 0$ with

$$\iint\limits_B |\nabla \varphi|^2\, dx \;\geq\; c\, \|\varphi\|^2_{H^1(B)} \quad \text{for all } \varphi \in H^1_\diamond(B)\,. \tag{5.64}$$

Problem (5.63) is the variational form of the Neumann boundary value problem

$$\Delta p_j = \mathrm{div}\; w_j = 0 \text{ in } B\,, \quad \frac{\partial p_j}{\partial \nu} = \nu \cdot w_j \text{ on } \partial B\,.$$

We observe that (5.63) holds even for all $\varphi \in H^1(B)$ since $\int_{\partial B} \nu \cdot w_j\, ds$ vanishes by the divergence theorem and the fact that $\mathrm{div}\, w_j = 0$. Substituting $\varphi = p_j$ into (5.63) yields, using (5.64) and the trace theorem,

$$c\, \|p_j\|^2_{H^1(B)} \;\leq\; \iint\limits_B |\nabla p_j|^2\, dx \;=\; \int\limits_{\partial B} (\nu \cdot w_j)\, p_j\, ds \;\leq\; \tilde{c}\, \|w_j\|_{C(\partial B)}\, \|p_j\|_{H^1(B)}\,,$$

i.e. $\|p_j\|_{H^1(B)} \leq (\tilde{c}/c)\, \|w_j\|_{C(\partial B)}$ which converges to zero.
Therefore, the functions $\tilde{w}_j := w_j - \nabla p_j \in H(\mathrm{curl}, B)$ satisfy:

- $\tilde{w}_j \in H_{\mathrm{div}}(\mathrm{curl}, B) := \big\{u \in H(\mathrm{curl}, B) : \iint_B \nabla \varphi \cdot u\, dx = 0 \text{ for all } \varphi \in H^1(B)\big\}$,
- $\tilde{w}_j \rightharpoonup 0$ weakly in $L^2(B, \mathbb{C}^3)$,
- $\mathrm{curl}\, \tilde{w}_j = \mathrm{curl}\, w_j \rightharpoonup 0$ weakly in $L^2(B, \mathbb{C}^3)$.

These three conditions assure that $\tilde{w}_j$ converges to zero in the norm of $L^2(B, \mathbb{C}^3)$ since the closed subspace $H_{\mathrm{div}}(\mathrm{curl}, B)$ of $H(\mathrm{curl}, B)$ is compactly imbedded in $L^2(B, \mathbb{C}^3)$. We refer to Weber [4], see also [2], Theorem 4.7, or [1], Corollary 5.33. Since also $\|\nabla p_j\|_{L^2(B)} \to 0$ this yields $\|w_j\|_{L^2(B)} \to 0$ as j tends to infinity. Now we return to the variational equation for w_j and substitute $\psi = \phi\, \overline{w_j}$ where $\phi \in C^\infty(\mathbb{R}^3)$ is some function with compact support such that $\phi = 1$ on B. This yields

$$\iint\limits_B \left[\frac{1}{\varepsilon_r}\, |\,\mathrm{curl}\, w_j|^2 - k^2\, |w_j|^2\right] dx \;=\; -\iint\limits_{\mathbb{R}^3 \setminus B} \big[\mathrm{curl}\, w_j \cdot \mathrm{curl}(\phi \overline{w_j}) - k^2\, \phi\, |w_j|^2\big]\, dx$$
$$+\; (k^2 + 1) \iint\limits_{\mathbb{R}^3} \phi\, \hat{v}_j \cdot \overline{w_j}\, dx\,.$$

We note that w_j is smooth outside of B. Green's theorem in $K(0,R)\setminus B$ (for a sufficiently large value of R) and application of $\operatorname{curl}^2 w_j - k^2 w_j = (k^2+1)\hat{v}_j$ in this region yields

$$\iiint_B \left[\frac{1}{\varepsilon_r}\,|\operatorname{curl} w_j|^2 - k^2\,|w_j|^2\right] dx \;=\; -\int_{\partial B} (\nu \times \overline{w}_j)\cdot \operatorname{curl} w_j\, ds$$

which tends to zero as j tends to infinity. Therefore, also $\operatorname{curl} w_j$ tends to zero in $L^2(B,\mathbb{C}^3)$ and ends the proof of part (d).

(e) We return to (5.61) for $\mathcal{T}_0$ instead of $\mathcal{T}$. Since $\hat{v}$ decays exponentially to zero as $|x|$ tends to infinity we conclude, by letting R tend to infinity,

$$(\mathcal{T}_0 f, f)_{L^2(D)} \;=\; \iiint_D (\operatorname{sign} q)\,|\tilde{w}|^2\,dx \;-\; \iiint_{\mathbb{R}^3} \left[|\operatorname{curl}\hat{v}|^2 + |\hat{v}|^2\right] dx$$

where again $\tilde{w} =:= f + \sqrt{|q|}\,\operatorname{curl}\hat{v}$. Therefore, in the case where Re $q \le -\gamma_2|q|$ on D we conclude that

$$-\operatorname{Re}(\mathcal{T}_0 f, f)_{L^2(D)} \;\ge\; \gamma_2\|\tilde{w}\|^2_{L^2(D)}$$

which proves coercivity of $-\mathcal{T}_0$ by the same arguments as in part (c).

Finally, let q be real valued and $q > 0$ on D. By similar arguments as in the derivation of (5.61) – note that $\operatorname{sign} q = 1$ – we calculate

$$\begin{aligned}(\mathcal{T}_0 f, f)_{L^2(D)} &= \iiint_D |f|^2\,dx \;+\; \iiint_D \sqrt{q}\,\operatorname{curl}\hat{v}\cdot\overline{f}\,dx \\ &= \iiint_D |f|^2\,dx \;+\; \iiint_{\mathbb{R}^3}\left[\frac{1}{\varepsilon_r}\,|\operatorname{curl}\hat{v}|^2 + |\hat{v}|^2\right] dx \\ &\ge \|f\|^2_{L^2(D)}\,.\end{aligned}$$

□

After these preparations we are able to treat the following three cases of contrasts:

(1) All of D is absorbing, i.e. $\operatorname{Im}\varepsilon_r > 0$ on D.

(2) No part of D is absorbing, i.e. ε_r is real valued and positive on D.

(3) Parts of D may be absorbing, i.e. we allow quite general values of ε.

We will apply the abstract characterizations of Corollary 1.22, Theorem 1.23, and Theorem 2.15 in combination with Theorem 5.11 which characterizes D by the range of H^*.

We formulate and prove the first main result of this section in which we treat the *absorbing medium*. We recall the definition of the self-adjoint operator $\operatorname{Im} F := \frac{1}{2i}(F - F^*)$. For $\gamma_0 > 0$ define the disks $C_+(\gamma_0)$ and $C_-(\gamma_0)$ as

$$\begin{aligned}C_-(\gamma_0) &= \left\{ z \in \mathbb{C} : \left| z - \frac{1}{2}\left(1 - \frac{\sqrt{1-\gamma_0^2}}{\gamma_0}\,i\right)\right| \le \frac{1}{2\gamma_0}\right\}, \\ C_+(\gamma_0) &= \left\{ z \in \mathbb{C} : \left| z - \frac{1}{2}\left(1 + \frac{\sqrt{1-\gamma_0^2}}{\gamma_0}\,i\right)\right| \le \frac{1}{2\gamma_0}\right\}.\end{aligned}$$

Theorem 5.13 *In addition to Assumption 5.9 we assume that there exists $\gamma_0 > 0$ with $\varepsilon_r(x) \in C_+(\gamma_0) \backslash C_-(\gamma_0)$ for almost all $x \in D$.*[1] *Furthermore, let ϕ_z be defined in (5.57) for $z \in \mathbb{R}^3$ (where again $p \in \mathbb{C}^3$ is kept fixed). Then $z \in D$ if, and only if, $\phi_z \in \mathcal{R}\big((\operatorname{Im} F)^{1/2}\big)$.*

Proof: Analogously to $\operatorname{Im} F$ we define $\operatorname{Im} \mathcal{T}$ as $\operatorname{Im} \mathcal{T} := \frac{1}{2i}(\mathcal{T} - \mathcal{T}^*)$ where $\mathcal{T}^*$ is the adjoint of $\mathcal{T}$ in $L^2(D, \mathbb{C}^3)$. By our assumption on ε_r we conclude ba a simple calculation that $\operatorname{Im} q \geq \gamma_0 |q|$ on D, i.e. the additional assumption of part (c) of Theorem 5.12 is satisfied. Noting that

$$\operatorname{Im}(\mathcal{T} f, f)_{L^2(D)} \;=\; ((\operatorname{Im} \mathcal{T}) f, f)_{L^2(D)}$$

we conclude that the self-adjoint operator $(\operatorname{Im} \mathcal{T})$ is coercive on $L^2(D, \mathbb{C}^3)$. Furthermore, the factorization (5.55) yields the factorization of $(\operatorname{Im} F)$ in the form

$$\operatorname{Im} F \;=\; \mathcal{H}^* (\operatorname{Im} \mathcal{T})\, \mathcal{H}\,. \tag{5.65}$$

This is exactly the situation where we can apply Corollary 1.22 of Chapter 1 which yields that the ranges of $\mathcal{H}^*$ and $(\operatorname{Im} F)^{1/2}$ coincide. The combination with the characterization of D by Theorem 5.11 yields the assertion. □

In the second situation we consider *non-absorbing media.*

Theorem 5.14 *In addition to Assumption 5.9 we assume that ε_r is real valued and either $\varepsilon_r > 1$ on D or $\varepsilon_r < 1$ on D (and equal to one on $\mathbb{R}^3 \setminus D$). Furthermore, we assume that k^2 is not an interior transmission eigenvalue in the sense of Definition 5.8. Let ϕ_z be again defined in (5.57) for any $z \in \mathbb{R}^3$ (where again $p \in \mathbb{C}^3$ is kept fixed). Then $z \in D$ if, and only if, $\phi_z \in \mathcal{R}\big((F^*F)^{1/4}\big)$.*

Proof: This time, we have to check the assumptions of Theorem 1.23 of Chapter 1. By Theorem 5.7 F is normal and the scattering operator $\mathcal{S} := I + \frac{ik}{8\pi^2} F$ is unitary. Furthermore, the factorization

$$F \;=\; \mathcal{H}^* \mathcal{T} \mathcal{H}$$

holds and $\operatorname{Im} \mathcal{T}$ is positive on $\overline{\mathcal{R}(\mathcal{H})} \setminus \{0\}$. By Theorem 5.12, parts (d) and (e), the operator $\mathcal{T}$ has the decomposition into $\mathcal{T} = \mathcal{T}_0 + (\mathcal{T} - \mathcal{T}_0)$ where $\mathcal{T} - \mathcal{T}_0$ is compact and $+\mathcal{T}_0$ or $-\mathcal{T}_0$ is coercive. Therefore, Theorem 1.23 is applicable and yields that the ranges of $\mathcal{H}^*$ and $(F^*F)^{1/4}$ coincide. The combination with Theorem 5.11 yields the assertion. □

In the third situation we consider more **general electric permittivities** ε_r.

Theorem 5.15 *In addition to Assumptions 5.9 we assume that either $\varepsilon_r > 1$ on D or there exists a closed set $A \subset \{z \in \mathbb{C} : \operatorname{Im} z \geq 0,\ |z - 1/2| < 1/2\}$ with $\varepsilon_r(x) \in A$ for almost all $x \in D$. Furthermore, we assume that k^2 is not an interior transmission eigenvalue in the sense of Definition 5.8. Let ϕ_z be again defined in (5.57) for $z \in \mathbb{R}^3$ (where again $p \in \mathbb{C}^3$ is kept fixed). Then $z \in D$ if, and only if, $\phi_z \in \mathcal{R}\big(F_\#^{1/2}\big)$ where $F_\# : L^2_t(S^2) \to L^2_t(S^2)$ is defined by*

$$F_\# \;=\; \big|\operatorname{Re} F\big| \;+\; \operatorname{Im} F\,. \tag{5.66}$$

[1]Recall that $\overline{D}$ is the support of q, i.e. $\varepsilon_r = 1$ outside of (the open and bounded set) D.

Proof: If ε_r is real valued and $\varepsilon_r > 1$ then q is real valued and $q > 0$ on D. Therefore, by part (e) of Theorem 2.15 the operator $\operatorname{Re}\mathcal{T}$ is coercive. If $\varepsilon_r(x) \in A$ on D then it is not difficult to show the existence of $\gamma_2 > 0$ such that

$$\left|\varepsilon_r(x) - \frac{1}{2}\left(1 - \frac{\gamma_2}{\sqrt{1+\gamma_2^2}}\, i\right)\right| \leq \frac{1}{2\sqrt{1-\gamma_2^2}}.$$

In terms of $q = 1 - 1/\varepsilon_r$ this is written as $\operatorname{Re} q \leq -\gamma_2 |q|$. Again, by part (e) of Theorem 2.15 the operator $-\operatorname{Re}\mathcal{T}$ is coercive. In both cases we write $\operatorname{Re}\mathcal{T}$ as $\operatorname{Re}\mathcal{T} = \operatorname{Re}\mathcal{T}_0 + \operatorname{Re}(\mathcal{T} - \mathcal{T}_0)$. By Theorem 5.12 the operator $\operatorname{Re}(\mathcal{T} - \mathcal{T}_0)$ is compact. Finally, also assumptions (A3) and (A4) of Theorem 2.15 are satisfied because $\operatorname{Im}\mathcal{T}$ is positive on $\overline{\mathcal{R}(\mathcal{H})} \setminus \{0\}$ by Theorem 5.12, part (b). $\square$

Bibliography

[1] A. Kirsch and F. Hettlich. *The Mathematical Theory of Maxwell's Equations: Expansion-, Integral-, and Variational methods*, volume 190 of *Applied Mathematical Sciences*. Springer, 2015.

[2] P. Monk. *Finite Element Methods for Maxwell's Equations*. Oxford Science Publications, Oxford, 2003.

[3] H. Triebel. *Höhere Analysis*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1972.

[4] C. Weber. A local compactness theorem for Maxwell's equations. *Math. Methods Appl. Sci.*, 2:12–25, 1980.