TIME-HARMONIC SCATTERING BY LOCALLY PERTURBED PERIODIC STRUCTURES WITH DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

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Abstract. The paper is concerned with well-posedness of TE and TM polarizations of time-harmonic electromagnetic scattering by perfectly conducting periodic surfaces and periodically arrayed obstacles with local perturbations. The classical Rayleigh Expansion radiation condition does not always lead to well-posedness of the Helmholtz equation even in unperturbed periodic structures. We propose two equivalent radiation conditions to characterize the radiating behavior of time-harmonic wave fields incited by a source term in an open waveguide under impenetrable boundary conditions. With these open waveguide radiation conditions, uniqueness and existence of time-harmonic scattering by incoming point source waves, plane waves and surface waves from locally perturbed periodic structures are established under either the Dirichlet or Neumann boundary condition. A Dirichlet-to-Neumann operator without using the Green's function is constructed for proving well-posedness of perturbed scattering problems.

Keywords: Helmholtz equation, periodic structures, radiation condition, uniqueness, existence, Dirichlet boundary condition, Neumann boundary condition.

1. Introduction

The electromagnetic scattering theory in periodic structures has many applications in micro-optics, radar imaging and non-destructive testing. We refer to [23] for historical remarks and details of these applications. As a standard model, we consider a time-harmonic electromagnetic plane wave incident onto a perfectly reflecting periodic surface or periodically arrayed conducting obstacles which remain invariant in the x_3-direction. Without loss of generality the direction of periodicity is supposed to be x_1 and the arrayed obstacles lie in a layer of finite height in the x_2-direction. We consider both the TE polarization case where the electric field is transversal to the ox_1x_2-plane by assuming $E(x) = (0, 0, u(x_1, x_2))$ and the TM polarization case where the magnetic field is transversal to the ox_1x_2-plane by assuming $H(x) = (0, 0, u(x_1, x_2))$. The background medium above the periodic surface or in the exterior of the periodically arrayed obstacles is supposed to be homogeneous and isotropic. The time-harmonic Maxwell's equations for $(E(x), H(x))$ will be reduced to the scalar Helmholtz equation for $u(x_1, x_2)$ over the ox_1x_2-plane together with the Dirichlet/Neumann boundary condition in TE/TM case and with proper radiation conditions as $|x_2| \to \infty$; see Figure 1 (a) and (b) for illustration of the scattering problems.

In periodic structures, a frequently used radiation condition is the so-called quasi-periodic Rayleigh expansion (see (2b)), which was firstly used by Lord Rayleigh in 1907 [21] for plane wave incidence. The Rayleigh expansion consists of a finite number of plane waves

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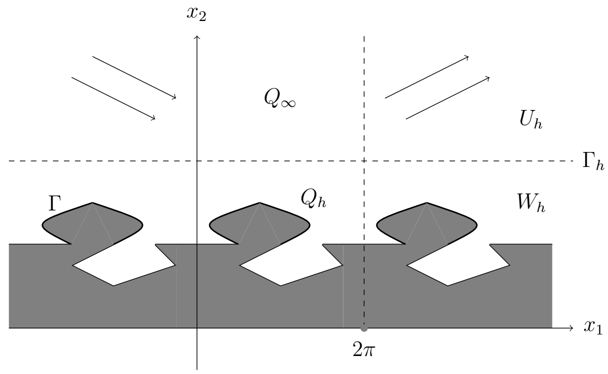
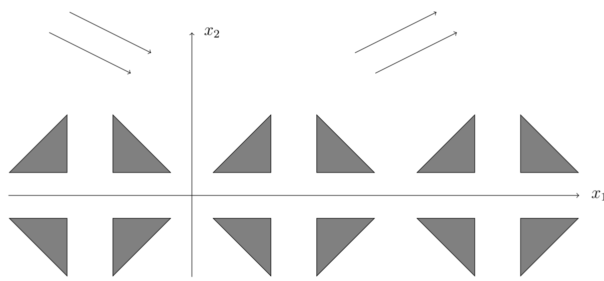


(a) A Lipschitz periodic curve



(b) Periodically arrayed obstacles

FIGURE 1. Illustration of wave scattering from (a) a perfectly reflecting periodic curve and (b) perfectly conducting obstacles. Guided waves might exist in (a)-(b), leading to difficulties in establishing well-posedness of the scattering problem with the classical Rayleigh Expansion radiation condition (2b).

and infinitely many evanescent waves. However, such a radiation condition does not always lead to uniqueness of solutions for all frequencies due to the presence of evanescent/surface waves propagating along the unbounded periodic curve, or due to the existence of guided waves propagating between the arrayed obstacles, both of them decaying exponentially in x_2. Examples of surface waves for unbounded periodic curves of Dirichlet kind were constructed in [24] where the reflecting curve is not a graph and in [14] under the Neumann boundary condition. We also refer to [1] for non-uniqueness examples of solutions incited by periodically arrayed obstacles immersed in a dielectric layer. On the other hand, it is well known that surface waves do not exist if a Dirichlet periodic curve is given by the graph of some function or satisfies the geometrical condition (20); see [4, 6, 15] for different regularity and geometry assumptions made on the reflecting curve. We also mention that the Rayleigh expansion condition does not apply to incoming source waves given by the fundamental solution of the Helmholtz equation and does not hold for scattering by compactly supported source terms. In these cases the incident waves lose the quasi-periodicity in x_1. It was firstly discussed in [2] that the radiated field should satisfy a Sommerfeld-type radiation condition and was recently proved in [11] for Dirichlet rough surfaces given by graphs and in [19] for periodic inhomogeneous layers. Hence, the radiating behavior of wave fields in periodic structures also depends on the type of incident waves. To sum up, precise and sharp radiation conditions are still needed in order to mathematically interpret the radiating behavior of time-harmonic wave fields in periodic structures, in particular for non-quasiperiodic incoming waves or when guided waves exist.

In recent years, a new radiation condition has been derived from the limiting absorption principle for scattering by layered periodic media in $\mathbb{R}^2$ and by periodic tubes in $\mathbb{R}^3$; see [9, 16, 17, 18, 19, 20]. Such a radiation condition turns out to be equivalent to the radiation condition based on dispersion curves for closed periodic wave guides (see e.g., [8] and [19, Remark 2.4]). By this new radiation condition, the diffracted fields caused by a compactly supported source term or a local defect can be decomposed into the sum of a radiating part and a propagating (guided) part. The former decays as $|x_1|^{-3/2}$ in the horizontal direction x_1 and decays as $|x|^{-1/2}$ in the radical direction, whereas the latter is a finite number of quasi-periodic left-going and right-going evanescent modes which decay exponentially in the vertical direction x_2 ([19]). Moreover, this new radiation condition is stronger than the angular spectrum representation [4] and the upward propagating radiation condition [3] for rough surface scattering problems. It can also be used for proving well-posedness of scattering by locally perturbed inhomogeneous layers in the presence of guided waves; see [9, 18, 19].

The aim of this paper is to investigate well-posedness of time-harmonic scattering by locally perturbed periodic curves and periodically arrayed obstacles of Dirichlet and Neumann kinds. The main results of this paper are summarized as follows.

(i) Propose two equivalent radiation conditions to prove uniqueness of weak solutions for periodic Lipschitz interfaces with local perturbations. The first radiation condition was adapted from [18, 19] for characterizing left-going and right-going evanescent waves of the propagating part of wave fields. It is referred to as the open waveguide radiation condition, in comparision with the closed waveguide radiation condition of [8]. The second radiation condition, which modifies the asymptotic behavior of radiating part of the first one, was motivated by the Sommerfeld radiation condition justified in [11] and [19, Section 6] for point source waves. The second radiation condition extends the well-posedness result of [11] to general periodic Lipschitz curves of Dirichlet or Neumann kind, in particular when guides waves are present. Since the decaying condition of Sommerfeld type contains more information on the radiating part, the second radiation condition yields a simplified proof of the uniqueness; see Theorem 2.15.

(ii) Existence of solutions for incoming plane waves, surfaces waves and point source waves in a locally perturbed periodic structure under a priori assumptions (Sections 4). Unlike the scattering by inhomogeneous periodic layers with local perturbations [18, 19, 9], there is no analogue of the Lippmann-Schwinger integral equation under the Dirichlet and Neumann boundary conditions. This leads to difficulties in the analysis of wave scattering from perfectly reflecting periodic curves with local perturbations. Our idea is to reduce the scattering problem to a bounded domain enclosing the perturbed part by constructing the DtN operator. For this purpose, we construct a Dirichlet-to-Neumann operator without using the Green's function for proving well-posedness of the perturbed scattering problem.

The remaining part of the paper is organized as follows. We first consider the perturbed/unperturbed scattering problem due to a compact source term. In Section 2, we describe an open waveguide radiation condition and its equivalent version, and use them to prove the uniqueness results. In comparison with the results for layered media [18, 19], a more general transmission problem and the scattering by exponentially decaying source terms without a compact support will be investigated in the unperturbed periodic domain (see Theorems 3.4 and 3.5). In Section 4, we prove well-posedness results for incoming point source waves, plane waves as well as surface waves in the perturbed setting. Finally, concluding remarks will made in Section 5 on how to carry out the analysis for unbounded periodic Dirichlet curves to Neumann curves and to periodically arrayed obstacles with boundary conditions.

2. Scattering by Dirichlet periodic curves with local perturbations: radiation condition and uniqueness

2.1. **Notations.** Let $D \subset \mathbb{R}^2$ be a 2π-periodic domain with respect to the x_1-direction. The boundary $\Gamma := \partial D$ is supposed to be given by a non-self-intersecting Lipschitz curve which is bounded in x_2-direction and 2π-periodic with respect to x_1. Therefore, in this paper we exclude the case of Figure 1 (b) but refer to Section 5. Let $\tilde{D}$ be a local perturbation of D in the way that $\Gamma \setminus \tilde{\Gamma}$ and $\tilde{\Gamma} \setminus \Gamma$ are bounded where $\tilde{\Gamma} = \partial \tilde{D}$ is the perturbed boundary which is also assumed to be a non-self-intersecting curve. Suppose that $\tilde{D}$ is filled by a homogeneous and isotropic medium and that $\tilde{\Gamma}$ is a perfectly reflecting curve of Dirichlet kind. Denote by $f \in L^2(\tilde{D})$ a source term of compact support which radiates wave fields at the wavenumber $k > 0$.

We consider the problem of determining the radiated wave $u \in H^1_{loc}(\tilde{D}) := \{w|_{\tilde{D}} : w \in H^1_{loc}(\mathbb{R}^2)\}$ such that

$$\Delta u + k^2 u = -f \text{ in } \tilde{D}, \quad u = 0 \text{ on } \tilde{\Gamma}, \tag{1}$$

and complemented by the open waveguide radiation condition explained in the next section. Without loss of generality (changing the period of the periodic structure if otherwise) we can assume that the perturbations $\Gamma \setminus \tilde{\Gamma}$ and $\tilde{\Gamma} \setminus \Gamma$ and also the support of f are contained in the disc $\{x \in \mathbb{R}^2 : (x_1 - \pi)^2 + x_2^2 < \pi^2\}$. We fix $R > \pi$ and $h_0 > \pi$ throughout this paper and use the following notations for $h > \pi$ (see Figure 1 (a) and Figure 2).

$$\begin{aligned}
Q_h &:= \{x \in D : 0 < x_1 < 2\pi,\ x_2 < h\}, \\
Q_\infty &:= \{x \in D : 0 < x_1 < 2\pi\}, \\
\Gamma_h &:= (0, 2\pi) \times \{h\}, \\
W_h &:= \{x \in D : x_2 < h\}, \\
U_h &:= \{x \in D : x_2 > h\}, \\
C_R &:= \{x \in D : (x_1 - \pi)^2 + x_2^2 = R^2\}, \\
\Sigma_R &:= \{x \in D : (x_1 - \pi)^2 + x_2^2 > R^2\}, \\
D_R &:= \{x \in D : (x_1 - \pi)^2 + x_2^2 < R^2\}, \\
\tilde{D}_R &:= \{x \in \tilde{D} : (x_1 - \pi)^2 + x_2^2 < R^2\}.
\end{aligned}$$

In the unperturbed setting we introduce the following function spaces [1].

$$\begin{aligned}
H^1_{loc,0}(\tilde D) &:= \left\{u \in H^1_{loc}(\tilde D) : u = 0 \text{ on } \partial\tilde D\right\},\\
H^1_*(\tilde D) &:= \left\{u \in H^1_{loc}(\tilde D) : u|_{W_h\cap\tilde D} \in H^1(W_h \cap \tilde D) \text{ for all } h > h_0, \right\},\\
H^1_*(\Sigma_R) &:= \left\{u \in H^1_{loc}(\Sigma_R) : \begin{array}{l} u|_{W_h\cap\Sigma_R} \in H^1(W_h \cap \Sigma_R) \text{ for all } h > h_0, \\ u = 0 \text{ on } \partial\Sigma_R \cap \partial D \end{array}\right\},\\
H^1_{\alpha,loc}(D) &:= \left\{u \in H^1_{loc}(D) : u(\cdot, x_2) \text{ is } \alpha\text{-quasi-periodic}\right\},\\
H^1_{\alpha,loc,0}(D) &:= \left\{u \in H^1_{\alpha,loc}(D) : u = 0 \text{ on } \partial D\right\}.
\end{aligned}$$

2.2. **The Open Waveguide Radiation Condition And An Energy Formula.** As mentioned in the introduction part the diffracted field will have a decomposition into a (guided) propagating part and a radiating part. The loss of exponential decay of the radiating part is a consequence of the existence of cut-off values while the propagative wave numbers determine the behavior of the guided part along the waveguide. We first recall that a function $\phi \in L^2_{loc}(\mathbb{R})$ is called α-quasi-periodic if $\phi(x_1 + 2\pi) = e^{2\pi\alpha i}\phi(x_1)$ for all $x_1 \in \mathbb{R}$.

Definition 2.1. *(i)* $\alpha \in [-1/2, 1/2]$ *is called a* cut-off value *if there exists* $\ell \in \mathbb{Z}$ *such that* $|\alpha + \ell| = k$.
(ii) $\alpha \in [-1/2, 1/2]$ *is called a* propagative wave number *if there exists a non-trivial* $\phi \in H^1_{\alpha,loc,0}(D)$ *such that*

$$\Delta\phi + k^2\phi = 0 \text{ in } D, \tag{2a}$$

and ϕ *satisfies the upward Rayleigh expansion*

$$\phi(x) = \sum_{\ell\in\mathbb{Z}} \phi_\ell\, e^{i(\ell+\alpha)x_1}\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)} \quad \text{for } x_2 > h_0 \tag{2b}$$

for some $\phi_\ell \in \mathbb{C}$ *where the convergence is uniform for* $x_2 \geq h_0 + \varepsilon$ *for every* $\varepsilon > 0$. *The functions* ϕ *are called guided (or propagating or Floquet) modes.*

In all of the paper, we choose the square root function to be holomorphic in the cutted plane $\mathbb{C} \setminus (i\mathbb{R}_{\leq 0})$. In particular, $\sqrt{t} = i\sqrt{|t|}$ for $t \in \mathbb{R}_{<0}$. In Definition 2.1 we restrict the quasi-periodic parameter α to the interval $[-1/2, 1/2]$, because an α-quasi-periodic function must be also $(\alpha + j)$-quasi-periodic for any $j \in \mathbb{N}$. Throughout this paper we make the following assumptions.

Assumption 2.2. *Let* $|\ell + \alpha| \neq k$ *for every propagative wave number* $\alpha \in [-1/2, 1/2]$ *and every* $\ell \in \mathbb{Z}$*; that is, no cut-off value is a propagative wave number.*

Under Assumption 2.2 it can be shown (see, e.g. [18] for the case of a flat curve $\Gamma = \Gamma_0$ and an additional index of refraction) that at most a finite number of propagative wave numbers exists in the interval $[-1/2, 1/2]$. Furthermore, if α is a propagative wave number with mode ϕ then $-\alpha$ is a propagative wave number with mode $\overline{\phi}$. Therefore, we can number the propagative wave numbers in $[-1/2, 1/2]$ such that they are given by $\{\hat\alpha_j : j \in J\}$ where $J \subset \mathbb{Z}$ is finite and symmetric with respect to 0 and $\hat\alpha_{-j} = -\hat\alpha_j$ for $j \in J$. Furthermore, it is known that (under Assumption 2.2) every mode ϕ is evanescent; that

[1] The definitions hold also for D instead of $\tilde D$

is, exponentially decaying as x_2 tends to infinity in D; that is, satisfies $|\phi(x)| \leq c\, e^{-\delta|x_2|}$ for $x_2 \geq h_0$ and some $c, \delta > 0$ which are independent of x. The corresponding space

$$X_j \;:=\; \left\{\phi \in H^1_{\hat\alpha_j,loc,0}(D) : u \text{ satisfies (2a) and (2b) for } \alpha = \hat\alpha_j\right\} \tag{3}$$

of modes is finite dimensional with some dimension $m_j > 0$. On X_j we define the sesqui-linear form $B : X_j \times X_j \to \mathbb{C}$ by

$$B(\phi,\psi) \;:=\; -2i \int\limits_{Q_\infty} \frac{\partial\phi}{\partial x_1}\, \overline{\psi}\, dx\,, \quad \phi,\psi \in X_j\,. \tag{4}$$

Note that B is hermitian. We make the assumption that B is non-degenerated on every X_j; that is,

Assumption 2.3. *For every $j \in J$ and $\psi \in X_j$, $\psi \neq 0$, the linear form $B(\cdot,\psi) : X_j \to \mathbb{C}$ is non-trivial on X_j; that is, there exists $\phi \in X_j$ with $B(\phi,\psi) \neq 0$.*

The hermitian sesqui-linear form B defines the cones $\{\psi \in X_j : B(\psi,\psi) \gtrless 0\}$ of propagating waves traveling to the right and left, respectively. We construct a basis of X_j with elements in these cones by taking any inner product $(\cdot,\cdot)_{X_j}$ and consider the following eigenvalue problem in X_j for every fixed $j \in J$. Determine $\lambda_{\ell,j} \in \mathbb{R}$ and non-trivial $\hat\phi_{\ell,j} \in X_j$ with

$$B(\hat\phi_{\ell,j},\psi) \;=\; -2i \int\limits_{Q_\infty} \frac{\partial\hat\phi_{\ell,j}}{\partial x_1}\, \overline{\psi}\, dx \;=\; \lambda_{\ell,j}\left(\hat\phi_{\ell,j},\psi\right)_{X_j} \quad \text{for all } \psi \in X_j \tag{5}$$

and $\ell = 1,\ldots,m_j$. We normalize the basis such that $\left(\hat\phi_{\ell,j},\hat\phi_{\ell',j}\right)_{X_j} = \delta_{\ell,\ell'}$ for $\ell,\ell' = 1,\ldots,m_j$. Then $\lambda_{\ell,j} = B(\hat\phi_{\ell,j},\hat\phi_{\ell,j})$ and the function $\psi \in X_j$ in Assumption 2.3 must take the form $\psi = \sum_{\ell=1}^{m_j} c_\ell \hat\phi_{\ell,j}$ with $c_\ell \neq 0$ for some $\ell = 1,2,\cdots,m_j$. Choosing $\phi = \hat\phi_{\ell,j}$, one deduces $B(\phi,\psi) = c_\ell\lambda_{\ell,j}$. Hence, the Assumption 2.3 is equivalent to $\lambda_{\ell,j} \neq 0$ for all $\ell = 1,\ldots,m_j$ and $j \in J$.

Remark 2.4. (i) *The set of propagative wave numbers obviously depends on $k \in \mathbb{R}_+$. Analogously, one may define $k_\ell = k_\ell(\alpha)$ for $\alpha \in [-1/2,1/2]$ as the wave number if the problem* (2a) *and* (2b) *admits a non-trivial solution. Since the solutions are in $H^1_{\alpha,loc,0}(D)$ the values $k_\ell(\alpha)$ are just eigenvalues of $-\Delta$ with respect to α-quasi-periodic boundary conditions on the vertical boundary of Q_∞ and homogeneous Dirichlet boundary condition on Γ. The functions $\alpha \to k_\ell(\alpha)$ are well known as the dispersion relations/curves. Throughout our paper the wavenumber k is fixed. Under the Assumption 2.2, the set $\{(\hat\alpha_j(k_0),k_0)\}_{j\in J}$ constitutes the intersection points of the dispersion curves with the line $k = k_0$ in the (α,k)-plane. Assumption 2.2 implies the absence of flat dispersion curves.*

(ii) *The eigenvalue problem* (5) *originates from the limiting absorption principle (LAP) by applying an abstract functional theorem that goes back to* [18]*. We refer to* [8, 20] *for detailed discussions in justifying the radiation conditions for closed full and half-waveguide problems. Note that the choice of the inner product in X_j relies on the way how to perturb the original scattering problem by applying the LAP. For example, if the LAP is applied to the wavenumber k then $(\phi,\psi)_{X_j} = \int_{Q_\infty} \phi\overline{\psi}dx$.*

In all of the paper we make Assumptions 2.2 and 2.3 without mentioning this always. The one-dimensional Fourier transform is defined as

$$(\mathcal{F}\phi)(\omega) \;:=\; \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \phi(s)\, e^{-is\omega}\, ds\,, \quad \omega \in \mathbb{R}\,.$$

It can be considered as an unitary operator from $L^2(\mathbb{R})$ onto itself. Now we are able to formulate the radiation condition caused by compactly supported source terms, which will also serve as the radiation condition of the Green's function to perturbed and unperturbed scattering problems (see Theorem 4.1 and Remark 4.2).

Definition 2.5. *Let $\psi_+, \psi_- \in C^\infty(\mathbb{R})$ be any functions with $\psi_\pm(x_1) = 1$ for $\pm x_1 \geq \sigma_0$ (for some $\sigma_0 > \max\{R, 2\pi\} + 1$) and $\psi_\pm(x_1) = 0$ for $\pm x_1 \leq \sigma_0 - 1$.*
A solution $u \in H^1_{loc}(\Sigma_R)$ of (1) satisfies the open waveguide radiation condition with respect to an inner product $(\cdot,\cdot)_{X_j}$ in X_j if u has in Σ_R a decomposition into $u = u_{rad} + u_{prop}$ which satisfy the following conditions.

(a) *The propagating part u_{prop} has the form*

$$u_{prop}(x) \;=\; \sum_{j\in J}\Bigg[\psi_+(x_1) \sum_{\ell:\lambda_{\ell,j}>0} a_{\ell,j}\, \hat{\phi}_{\ell,j}(x) \;+\; \psi_-(x_1) \sum_{\ell:\lambda_{\ell,j}<0} a_{\ell,j}\, \hat{\phi}_{\ell,j}(x)\Bigg] \tag{6}$$

for $x \in \Sigma_R$ and some $a_{\ell,j} \in \mathbb{C}$. Here, for every $j \in J$ the scalars $\lambda_{\ell,j} \in \mathbb{R}$ and $\hat{\phi}_{\ell,j} \in \hat{X}_j$ for $\ell = 1, \dots, m_j$ are given by the eigenvalues and corresponding eigenfunctions, respectively, of the self adjoint eigenvalue problem (5). Note that by the choice of $\psi_\pm$ the propagating part vanishes for $|x_1| < \sigma_0 - 1$ and is therefore well defined in Σ_R.

(b) *The radiating part $u_{rad} \in H^1_*(\Sigma_R)$ satisfies the generalized angular spectrum radiation condition*

$$\int\limits_{-\infty}^{\infty} \left| \frac{\partial(\mathcal{F}u_{rad})(\omega, x_2)}{\partial x_2} - i\sqrt{k^2 - \omega^2}\,(\mathcal{F}u_{rad})(\omega, x_2)\right|^2 d\omega \;\longrightarrow\; 0\,, \quad x_2 \to \infty\,. \tag{7}$$

The radiation condition (7) can be used to prove well-posedness of the Helmholtz equation with a source term which is supported in x_1-direction and exponentially decays in x_2 (see (9a)). It has been shown in [18] for the case of a half plane problem with an inhomogeneous period layer that the radiation condition of Definition 2.5 for the inner product $(\phi,\psi)_{X_j} = 2k\int_{Q_\infty} n\,\phi\,\overline{\psi}\,dx$ is a consequence of the limiting absorption principle by replacing k with $k + i\epsilon$, $\epsilon > 0$. In this paper we will not justify this radiation condition, although we are sure that this can be done in the same way as [18, 20]. A second motivation of our radiation condition is the following result on the direction of the energy flow which will play a central role in the proof of uniqueness.

Lemma 2.6. *Let $u = \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\hat{\phi}_{\ell,j}$ for some $a_{\ell,j} \in \mathbb{C}$ and write $q + Q_\infty := \{x \in D : q < x_1 < q + 2\pi\}$ for $q \in \mathbb{R}$. Then we have*

$$2\,\mathrm{Im} \int\limits_{q+Q_\infty} \overline{u}\, \frac{\partial u}{\partial x_1}\, dx \;=\; \sum_{j\in J}\sum_{\ell=1}^{m_j} \lambda_{\ell,j}\, |a_{\ell,j}|^2\,.$$

By Lemma 2.6, the propagating part u_{prop} satisfies the energy formula

$$2\,\mathrm{Im}\int\limits_{q+Q_\infty}\overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,dx\;=\;\begin{cases}\displaystyle\sum_{j\in J}\sum_{\lambda_{\ell,j}>0}\lambda_{\ell,j}\,|a_{\ell,j}|^2\,, & q>\sigma_0\,,\\ \displaystyle\sum_{j\in J}\sum_{\lambda_{\ell,j}<0}\lambda_{\ell,j}\,|a_{\ell,j}|^2\,, & q<-\sigma_0\,,\end{cases}$$

where $\sigma_0 > 2\pi + 1$ is the number specified in Definition 2.5. To prove Lemma 2.6, we have to modify the arguments of [19] for inhomogeneous layered media, because solutions of the Dirichlet and Neumann boundary value problems are in $H^1_{loc}(\tilde{D})$ but fail to be in $H^2_{loc}(\tilde{D})$ if $\tilde{\Gamma}$ is Lipschitz. For C^2-smooth boundaries, the quantity in Lemma 2.6 also equals to $4\pi\,\mathrm{Im}\int_{D\cap\{x_1=q\}}\overline{u}\,\frac{\partial u}{\partial x_1}\,ds$; see [18, Lemma 6.3] and [19, Lemma 2.6].

Proof of Lemma 2.6. We recall the following form of Green's formula valid in any Lipschitz domain Ω: For $u\in H^1(\Omega)$ with $\Delta u\in L^2(\Omega)$ we have

$$\int\limits_\Omega[\nabla u\cdot\nabla\psi+\psi\,\Delta u]\,dx\;=\;0\quad\text{for all }\psi\in H^1_0(\Omega)\,.$$

Let $j\in\{1,2\}$. First we show for α_j-quasi-periodic solutions $u_j\in H^1_{loc}(D)$ of $\Delta u_j+k^2u_j=0$ in D with $u_j=0$ on Γ and $\alpha_j\in(-1/2,1/2]$ with $\alpha_1\neq\alpha_2$ that

$$\int\limits_{q+Q_\infty}\left[\overline{u_2}\,\frac{\partial u_1}{\partial x_1}-u_1\,\frac{\partial\overline{u_2}}{\partial x_1}\right]dx\;=\;0\,. \tag{8}$$

Indeed, defining $\psi(x_1):=1-|x_1-q|/(2\pi)$ and applying Green's theorem in $\Omega:=\{x\in D: q-2\pi<x_1<q+2\pi\}$ yields (note that u_j decay exponentially as x_2 tends to infinity)

$$\begin{aligned}0\;&=\;\int\limits_\Omega[\nabla u_1\cdot\nabla(\psi\overline{u_2})-k^2(\psi\overline{u_2})\,u_1]\,dx\\ &=\;\int\limits_\Omega\psi\,[\nabla u_1\cdot\nabla\overline{u_2}-k^2\overline{u_2}\,u_1]\,dx\;+\;\int\limits_\Omega\psi'\,\overline{u_2}\,\frac{\partial u_1}{\partial x_1}\,dx\,.\end{aligned}$$

Interchanging the roles of u_1 and $\overline{u_2}$ and subtraction yields

$$\begin{aligned}0\;&=\;\int\limits_\Omega\psi'\left[\overline{u_2}\,\frac{\partial u_1}{\partial x_1}-u_1\,\frac{\partial\overline{u_2}}{\partial x_1}\right]dx\\ &=\;\int\limits_{q+Q_\infty}\psi'\left[\overline{u_2}\,\frac{\partial u_1}{\partial x_1}-u_1\,\frac{\partial\overline{u_2}}{\partial x_1}\right]dx+\int\limits_{q-2\pi+Q_\infty}\psi'\left[\overline{u_2}\,\frac{\partial u_1}{\partial x_1}-u_1\,\frac{\partial\overline{u_2}}{\partial x_1}\right]dx\\ &=\;\frac{1}{2\pi}\int\limits_{q+Q_\infty}\left[\overline{u_2}\,\frac{\partial u_1}{\partial x_1}-u_1\,\frac{\partial\overline{u_2}}{\partial x_1}\right]dx-\frac{1}{2\pi}\int\limits_{q-2\pi+Q_\infty}\left[\overline{u_2}\,\frac{\partial u_1}{\partial x_1}-u_1\,\frac{\partial\overline{u_2}}{\partial x_1}\right]dx\\ &=\;\left(1-e^{2\pi i(\alpha_2-\alpha_1)}\right)\frac{1}{2\pi}\int\limits_{q+Q_\infty}\left[\overline{u_2}\,\frac{\partial u_1}{\partial x_1}-u_1\,\frac{\partial\overline{u_2}}{\partial x_1}\right]dx\end{aligned}$$

where we used the quasi-periodicity of u_j. This yields (8).

Now we rewrite u as

$$u \;=\; \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\,\hat\phi_{\ell,j} \;=\; \sum_{j\in J} u_j \quad\text{with}\quad u_j \;:=\; \sum_{\ell=1}^{m_j} a_{\ell,j}\,\hat\phi_{\ell,j}.$$

Then u_j is $\hat\alpha_j$-quasi-periodic. Using (8) and the orthonormalization of $\hat\phi_{\ell,j}$, we arrive at

$$\begin{aligned}
&2i\,\mathrm{Im}\int\limits_{q+Q_\infty} \overline{u}\,\frac{\partial u}{\partial x_1}\,dx \;=\; \int\limits_{q+Q_\infty}\left[\overline{u}\,\frac{\partial u}{\partial x_1} - u\,\frac{\partial \overline{u}}{\partial x_1}\right]dx\\
=\;& \sum_{j\in J}\sum_{j'\in J}\int\limits_{q+Q_\infty}\left[\overline{u_j}\,\frac{\partial u_{j'}}{\partial x_1} - u_{j'}\,\frac{\partial \overline{u_j}}{\partial x_1}\right]dx \;=\; \sum_{j\in J}\int\limits_{q+Q_\infty}\left[\overline{u_j}\,\frac{\partial u_j}{\partial x_1} - u_j\,\frac{\partial \overline{u_j}}{\partial x_1}\right]dx\\
=\;& 2i\,\mathrm{Im}\sum_{j\in J}\int\limits_{q+Q_\infty}\overline{u_j}\,\frac{\partial u_j}{\partial x_1}\,dx \;=\; i\,\mathrm{Re}\Biggl[-2i\sum_{j\in J}\int\limits_{Q_\infty}\overline{u_j}\,\frac{\partial u_j}{\partial x_1}\,dx\Biggr] \;=\; i\sum_{j\in J}\sum_{\ell=1}^{m_j}\lambda_{\ell,j}\,|a_{\ell,j}|^2\,,
\end{aligned}$$

which proves the lemma. □

Below we review a result on the asymptotic behavior of u_{rad} which will be needed in the proof of uniqueness. By (1) and (6), the radiating part u_{rad} to the scattering problem satisfies

$$\Delta u_{rad} \;+\; k^2\,u_{rad} \;=\; -f \;-\; \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\varphi_{\ell,j} \quad\text{in } \tilde D\,,\quad u_{rad}=0 \text{ on } \tilde\Gamma\,, \tag{9a}$$

where

$$\varphi_{\ell,j}(x) \;=\; \begin{cases} 2\,\psi_+'(x_1)\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1} + \psi_+''(x_1)\,\hat\phi_{\ell,j}(x) & \text{if } \lambda_{\ell,j}>0\,,\\ 2\,\psi_-'(x_1)\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1} + \psi_-''(x_1)\,\hat\phi_{\ell,j}(x) & \text{if } \lambda_{\ell,j}<0\,.\end{cases} \tag{9b}$$

We note that f has compact support in Q_{h_0} and $\varphi_{\ell,j}$ vanish for $|x_1|\le\sigma_0-1$ and $|x_1|\ge\sigma_0$, and are evanescent; that is, there exist $\hat c,\delta>0$ with $|\varphi_{\ell,j}(x)|\le\hat c\exp(-\delta x_2)$ for all $x_2\ge h_0$. Furthermore, u_{rad} satisfies the generalized angular spectrum radiation condition (7). In [19] the following result has been shown.[2]

Lemma 2.7. *Let Assumptions 2.2 and 2.3 hold, and let $u\in H^1_{loc}(D)$ be a solution of (1) satisfying the radiation condition of Definition 2.5. Then the radiating part u_{rad} satisfies a stronger form of the radiation condition (7), namely,*

$$\left|\frac{\partial(\mathcal{F}u_{rad})(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}u_{rad})(\omega,x_2)\right| \le \frac{c}{\delta+\sqrt{|\omega^2-k^2|}}\,e^{-\delta x_2} \tag{10}$$

for almost all $\omega\in\mathbb{R}$ and $x_2>h_0$ where $c>0$ is independent of ω and x.
Furthermore, there exists $c>0$ with

$$\bigl|u_{rad}(x)\bigr| \;+\; \bigl|\nabla u_{rad}(x)\bigr| \;\le\; c\,(1+|x_2|)\,\rho(x_1) \tag{11}$$

[2]These properties are consequences of the differential equation and radiation condition above the line $x_2=h_0$ solely and are therefore independent of the differential equation or boundary condition below this line.

for all $x \in \tilde{D}$ with $x_2 \geq h_0 + 1$, where $\rho \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is given by

$$(12) \qquad \rho(x_1) \; := \; \int\limits_{\mathbb{R}} \frac{|u_{rad}(y_1, h_0)|}{(1 + |x_1 - y_1|)^{3/2}} \, dy_1 \; + \; \frac{1}{1 + |x_1|^{3/2}}, \quad x_1 \in \mathbb{R}.$$

2.3. **A Modified Open Waveguide Radiation Condition.** In this subsection we propose another open waveguide radiation condition that is equivalent to the Def. 2.5. We first define the half-plane Sommerfeld radiation condition used in [11, 19]. Introduce the weighted Sobolev space $H^1_\rho(\Omega)$ by

$$H^1_\rho(\Omega) \; := \; \left\{ u : (|1 + |x_1|^2)^{\rho/2} u \in H^1(\Omega) \right\}, \quad \rho \in \mathbb{R}.$$

Definition 2.8. *A function $v \in C^\infty(U_{h_0} \cap \Sigma_R)$ satisfies the Sommerfeld radiation condition in $U_{h_0} \cap \Sigma_R$ if $v \in H^1_\rho(W_h \cap \Sigma_R)$ for all $h > h_0$ and all $\rho < 0$ and*

$$(13) \quad \sup_{x \in C_a \cap U_h} |x|^{1/2} \Big| \frac{\partial v(x)}{\partial r} - ikv(x) \Big| \; \to \; 0, \quad a \to \infty, \qquad \sup_{x \in U_h} |x|^{1/2} |v(x)| \; < \; \infty$$

for all $h > h_0$ where $r = |x|$.

Remark 2.9. *Since $\Phi(x, y) = \mathcal{O}(|x|^{-1/2})$ and $\frac{\partial \Phi(x,y)}{\partial r} - ik\Phi(x, y) = \mathcal{O}(|x|^{-3/2})$ as $r = |x| \to \infty$, it holds that $\Phi(\cdot, y) \in H^1_\rho(W_h \cap \Sigma_R)$ for all $\rho < 0$ if $R > |y_1 - \pi|$. Hence, the above Sommerfeld radiation condition covers two-dimensional point source waves, but excludes plane waves and surface (evanescent) waves, which do not decay along the horizontal direction.*

If Γ is a Lipschitz function, it was shown in [11] that the scattered field caused by a point source source must satisfy the above Sommerfeld radiation condition. However, the total field (i.e., the Green's function to the rough surface scattering problem) satisfies an analogous condition but with the weighted index $\rho < 1$ in place of $\rho < 0$. Motivated by this fact, we define a modified open waveguide radiation condition by changing the generalized angular spectrum radiation condition of the radiating part of Def. 2.5.

Definition 2.10. *A solution $u \in H^1_{loc}(\Sigma_R)$ of (1) satisfies the modified open waveguide radiation condition with respect to an inner product $(\cdot, \cdot)_{X_j}$ in X_j if u has a decomposition into $u = u_{rad} + u_{prop}$ in Σ_R where u_{prop} satisfied the same condition specified as in Def. 2.5 (a) and u_{rad} fulfills the Sommerfeld radiation condition of Def. 2.8 but with the index $\rho < 1$.*

Below we prove the equivalence of the two open waveguide radiation conditions.

Theorem 2.11. *The open waveguide radiation condition of Def. 2.5 and the modified one given by Def. 2.10 are equivalent.*

Proof. Write $u = u_{rad} + u_{prop}$ where $u_{rad} \in H^1_*(\tilde{D})$ denotes the radiating part and u_{prop} the propagating part. First we suppose that u_{rad} fulfills the generalized angular spectrum radiation condition (7). By arguing analogously to [19, Theorem 6.2] for compact source terms, one can show the asymptotics $u_{rad}(x) = O(|x_1|^{-3/2})$ as $|x_1| \to \infty$ in W_h. This gives $u_{rad} \in H^1_\rho(W_h \cap \Sigma_R)$ for all $h > h_0$, $\rho < 1$ and proves the modified open waveguide radiation condition of Definition 2.10; see [19, Section 6] for details.

Now it remains to justify the generalized angular spectrum radiation condition of u_{rad}, under the assumption that u_{rad} satisfies the Sommerfeld radiation condition of Def. 2.8 but with the index $\rho < 1$. Since $u_{rad}|_{\Gamma_{h_0}} \in H^{1/2}_\rho(\mathbb{R})$ for all $1/2 < \rho < 1$, we recall from [11, Lemma A.2, Appendix] (see also [19]) that the function

$$v(x) = 2\int_{\Gamma_{h_0}} \frac{\partial G(x,y)}{\partial y_2} u_{rad}(y)\, ds(y), \quad x_2 > h_0,$$

satisfies the homogeneous Helmholtz equation together the Sommerfeld radiation conditions 13 and the boundary value $v = u_{rad}$ on $x_2 = h_0$. Hence, the function $w := u_{rad} - v$ satisfies 13 in $x_2 > h_0$ and the boundary value problem

$$\Delta w + k^2 w = \varphi \quad \text{in} \quad x_2 > h_0, \qquad w = 0 \quad \text{on} \quad x_2 = h_0,$$

where φ is given by the right hand side of (9a).This implies that w can be represented as

$$w(x) = \int_{-\sigma_0}^{\sigma_0}\int_{h_0}^{\infty} \left[G(x,y) - G(x,y^*)\right] \varphi(y)\, dy_2 dy_1, \quad x_2 > h_0,$$

with $y^* := (y_1 - 2h_0 - y_2)^\top$. Now, following the proof of [19, Lemma 7.1] one can show that w satisfies the stronger form (10) of the radiation condition (7).This proves the generalized angular spectrum radiation condition of u_{rad}. □

We would like to extend the Sommerfeld radiation condition up to the boundary Γ. However, since Γ is only Lipschitz, in general the derivatives $\partial v/\partial r$ do not exist up to the boundary. We can, however, define a weaker form which models the integral form of the Sommerfeld radiation condition as follows. The connection between these two radiation conditions will be described in Lemma 2.13.

Definition 2.12. *Let a_j be a sequence in $\mathbb{R}$ such that $a_j \to \infty$ and $\tilde{D}_{a_j}$ are Lipschitz domains. A solution $v \in H^1_{loc}(\Sigma_R)$ satisfies the Sommerfeld radiation condition in integral form if*

$$\left\| \frac{\partial v}{\partial r} - ikv \right\|_{H^{-1/2}(C_{a_j})} \longrightarrow 0, \quad j \to \infty,$$

where $r = |x|$.

Lemma 2.13. *If v satisfies the Sommerfeld radiation condition of Definition 2.8 with the index $\rho \geq 0$, then v also fulfills the integral form of the radiation condition defined by Definition 2.12.*

Proof. Without loss of generality we suppose that $a_{j+1} - a_j \geq 1$ for all $j \in \mathbb{N}$. Let $h_0 > 0$ be the number specified in Definition 2.8. We set $h := h_0 + 1$ and choose $\psi \in C^\infty(\mathbb{R}^2)$ such that $\psi(x) = 0$ for $x \in U_h$ and $\psi(x) = 1$ for $x \notin U_{h-\varepsilon}$. We decompose C_{a_j} into $C_{a_j} = \big(C_{a_j} \cap U_h\big) \cup \big(C_{a_j} \setminus U_h\big)$. Then

$$\begin{aligned}
\left\| \frac{\partial v}{\partial r} - ikv \right\|_{H^{-1/2}(C_{a_j})} &= \left\| \psi\left(\frac{\partial v}{\partial r} - ikv\right) \right\|_{H^{-1/2}(C_{a_j})} + \left\| (1-\psi)\left(\frac{\partial v}{\partial r} - ikv\right) \right\|_{H^{-1/2}(C_{a_j})} \\
&\leq \left\| \psi \frac{\partial v}{\partial r} \right\|_{H^{-1/2}(C_{a_j})} + k\left\| \psi v \right\|_{H^{-1/2}(C_{a_j})} \\
&\quad + \left\| (1-\psi)\left(\frac{\partial v}{\partial r} - ikv\right) \right\|_{H^{-1/2}(C_{a_j})}. \qquad (14)
\end{aligned}$$

The last integral converges to zero because, by the Sommerfeld radiation condition of (13),

$$\left\|(1-\psi)\left(\frac{\partial v}{\partial r}-ikv\right)\right\|_{H^{-1/2}(C_{a_j})} \;\le\; C\left\|\frac{\partial v}{\partial r}-ikv\right\|_{L^2(C_{a_j}\cap U_{h-\epsilon})} \;\longrightarrow\; 0$$

as $j\to\infty$. It remains to discuss the first two integrals on the right hand side of (14). Let $E_j : H_0^{1/2}(C_{a_j})\to H^1(D_{a_j+1}\setminus\overline{D_{a_j}})$ be extension operators which are uniformly bounded with respect to j. In fact, given $\varphi\in H_0^{1/2}(C_{a_j})$ we define $E_j\varphi = w_j$ in $D_{a_{j+1}}\setminus\overline{D_{a_j}}$ where w_j is the unique solution to the boundary value problem

$$\begin{aligned}&\Delta w_j = 0 \quad\text{in}\quad D_{a_{j+1}}\setminus\overline{D_{a_j}},\\ &w_j=\varphi \quad\text{on}\quad C_{a_j}, \qquad w_j = 0 \quad\text{on}\quad \partial(D_{a_{j+1}}\setminus\overline{D_{a_j}})\setminus C_{a_j}.\end{aligned}$$

The norm of such an extension operator depends only on the Lipschitz constants of $D_{a_{j+1}}\setminus\overline{D_{a_j}}$, which are uniformly bounded in j. Then we have

$$\begin{aligned}
&\left\|\psi\,\frac{\partial v}{\partial r}\right\|_{H^{-1/2}(C_{a_j})} + k\|\psi v\|_{H^{-1/2}(C_{a_j})}\\
\le\;& \left\|\frac{\partial(\psi v)}{\partial r}\right\|_{H^{-1/2}(C_{a_j})} + \left\|v\,\frac{\partial\psi}{\partial r}\right\|_{H^{-1/2}(C_{a_j})} + k\|\psi v\|_{H^{-1/2}(C_{a_j})}\\
\le\;& \sup_{\|\varphi\|_{H_0^{1/2}(C_{a_j})}=1}\langle\partial_r(\psi v),\varphi\rangle + c\,\|v\|_{H^{1/2}(C_{a_j}\setminus U_h)}\\
=\;& \sup_{\|\varphi\|_{H_0^{1/2}(C_{a_j})}=1}\int\limits_{D_{a_{j+1}}\setminus\overline{D_{a_j}}}\left[\nabla(\psi v)\cdot\nabla\overline{E_j\varphi} - k^2\psi v\,\overline{E_j\varphi}\right]dx + c\,\|v\|_{H^{1/2}(C_{a_j}\setminus U_h)}\\
\le\;& c\,\|v\|_{H^1(Z_j)}
\end{aligned}$$

where $Z_j = \{x\in D_{a_{j+1}} : |x|>a_j\,,\ x_2<h\}$ and $c>0$ is independent of j. Simple estimates show that Z_j is contained in the set $\{x\in D : a_j-\varepsilon<x_1<a_{j+1},\ x_2<h\}$. From $v\in H^1(W_h)$ we conclude that $\|v\|_{H^1(Z_j)}$ tends to zero. □

2.4. **Uniqueness Of Solutions Of The Perturbed And Unperturbed Problems.** First we show that the propagating part u_{prop} of the open waveguide radiation condition 2.5 has to vanish, if $f=0$.

Theorem 2.14. *Let $u\in H^1_{loc,0}(\tilde D)$ be a solution of $\Delta u + k^2u = 0$ in $\tilde D$ satisfying the open waveguide radiation condition of Definition 2.5. Then u_{prop} vanishes; that is, all the coefficients $a_{\ell,j}$ vanish.*

Proof. Choose $\psi_N\in C^\infty(\mathbb{R})$ and $\varphi_H\in C^\infty(\mathbb{R})$ with $\psi_N(x_1)=1$ for $|x_1|\le N$ and $\psi_N(x_1)=0$ for $|x_1|\ge N+1$ and $\varphi_H(x_2)=0$ for $x_2\ge H+1$ and $\varphi_H(x_2)=1$ for $x_2\le H$. For $N>\sigma_0+1$ and $H>h_0+1$ we define the regions $D_{N,H} := \{x\in\tilde D : |x_1|<N,\ x_2<H\}$ and $W^-_{N,H} := \{x\in\tilde D : -N-1<x_1<-N,\ x_2<H\}$ and $W^+_{N,H} := \{x\in\tilde D : N<x_1<N+1,\ x_2<H\}$ and the horizontal line segments $\Gamma_{N,H} := (-N,N)\times\{H\}$. We apply Green's theorem in $D_{N+1,H+1}$ to $v(x) := \psi_N(x_1)\,u(x)$ and $\overline{v(x)}\,\varphi_H(x_2)$. First we note that

$\Delta v \in L^2(D_{N+1,H+1})$ because $\Delta u = -k^2 u$. Furthermore $v\varphi \in H_0^1(D_{N+1,H+1})$, therefore,

$$
\begin{aligned}
0 &= \int\limits_{D_{N+1,H+1}} \big[\nabla v \cdot \nabla(\overline{v}\,\varphi_H) + (\overline{v}\,\varphi_H)\,\Delta v\big]\,dx \\
&= \int\limits_{D_{N+1,H}} \big[|\nabla v|^2 + \overline{v}\,\Delta v\big]\,dx \;+ \int\limits_{D_{N+1,H+1}\setminus D_{N+1,H}} \big[\nabla v \cdot \nabla(\overline{v}\,\varphi_H) + (\overline{v}\,\varphi_H)\,\Delta v\big]\,dx \\
&= \int\limits_{D_{N+1,H}} \big[|\nabla v|^2 + \overline{v}\,\Delta v\big]\,dx \;- \int\limits_{\Gamma_{N+1,H}} \overline{v}\,\frac{\partial v}{\partial x_2}\,ds
\end{aligned}
$$

where we applied the classical Green's theorem in the rectangle $(-N-1, N+1)\times(H, H+1)$ to the second integral for the smooth function v. Therefore,

$$
\begin{aligned}
\int\limits_{\Gamma_{N+1,H}} \psi_N^2\,\overline{u}\,\frac{\partial u}{\partial x_2}\,ds &= \int\limits_{\Gamma_{N+1,H}} \overline{v}\,\frac{\partial v}{\partial x_2}\,ds \;=\; \int\limits_{D_{N+1,H}} \big[\big|\nabla v\big|^2 + \overline{v}\,\Delta v\big]\,dx \\
= \int\limits_{D_{N,H}} \big[\big|\nabla u\big|^2 + \overline{u}\,\Delta u\big]\,dx &+ \int\limits_{W_{N,H}^+} \big[\big|\nabla v\big|^2 + \overline{v}\,\Delta v\big]\,dx \;+ \int\limits_{W_{N,H}^-} \big[\big|\nabla v\big|^2 + \overline{v}\,\Delta v\big]\,dx\,;
\end{aligned}
$$

that is, with $\Delta u = -k^2 u$,

$$
\text{(15)}\quad \operatorname{Im} \int\limits_{\Gamma_{N+1,H}} \psi_N^2\,\overline{u}\,\frac{\partial u}{\partial x_2}\,ds \;=\; \operatorname{Im} \int\limits_{W_{N,H}^+} \big[\big|\nabla v\big|^2 + \overline{v}\,\Delta v\big]\,dx \;+\; \operatorname{Im} \int\limits_{W_{N,H}^-} \big[\big|\nabla v\big|^2 + \overline{v}\,\Delta v\big]\,dx\,.
$$

The decomposition $u = u_{rad} + u_{prop}$ yields four terms in each of the integrals of (15).
(a) First, we look at the two integrals on the right hand side of (15). We define $v^{(1)} := \psi_N u_{rad}$ and $v^{(2)} = \psi_N u_{prop}$ and estimate the terms

$$
a_{N,H}^{\pm}(j,\ell) \;:=\; \int\limits_{W_{N,H}^{\pm}} \big[\nabla\overline{v^{(j)}} \cdot \nabla v^{(\ell)} + \overline{v^{(j)}}\,\Delta v^{(\ell)}\big]\,dx
$$

for $j,\ell \in \{1,2\}$. Then $|a_{N,H}^{\pm}(1,1)|$, $|a_{N,H}^{\pm}(1,2)|$, and $|a_{N,H}^{\pm}(2,1)|$ are estimated as in the proof of [19, Theorem 2.2]:

$$
|a_{N,H}^{\pm}(1,1)| \;\le\; c\,\gamma_{N,H}\,, \qquad |a_{N,H}^{\pm}(1,2)| \;+\; |a_{N,H}^{\pm}(2,1)| \;\le\; c\,\sqrt{\gamma_{N,H}}
$$

with

$$
\text{(16)}\qquad \gamma_{N,H} \;:=\; \|u_{rad}\|_{H^1(Q_N)}^2 \;+\; H^3 \int\limits_{N<|x_1|<N+1} \rho(x_1)^2\,dx_1
$$

and $Q_N := \{x \in \tilde{D} : N < |x_1| < N+1,\; x_2 < h_0 + 1\}$.
For $a_{N,H}^{\pm}(2,2)$ we need to argue differently as in the proof of [19, Theorem 3.2] to avoid the integral over the vertical boundaries of $W_{N,H}^{\pm}$. We recall that

$$
a_{N,H}^{+}(2,2) \;=\; \int\limits_{W_{N,H}^+} \big[|\nabla(\psi_N\,u_{prop})|^2 + (\psi_N\,\overline{u_{prop}})\,\Delta(\psi_N u_{prop})\big]\,dx
$$

and note that $\Delta(\psi_N\, u_{prop}) = -k^2 \psi_N\, u_{prop} + 2\psi_N' \frac{\partial u_{prop}}{\partial x_1} + \psi_N''\, u_{prop}$. Therefore,

$$\begin{aligned}
\operatorname{Im} a_{N,H}^+(2,2) &= 2 \operatorname{Im} \int\limits_{W_{N,H}^+} \psi_N\, \psi_N'\, \overline{u_{prop}}\, \frac{\partial u_{prop}}{\partial x_1}\, dx \;=\; \operatorname{Im} \int\limits_{W_{N,H}^+} \frac{d}{dx_1} \psi_N^2\, \overline{u_{prop}}\, \frac{\partial u_{prop}}{\partial x_1}\, dx \\
&= \operatorname{Im} \int\limits_{W_{N,H}^+} \big[\nabla u_{prop} \cdot \nabla(\psi_N^2\, \overline{u_{prop}}) - k^2 (\psi_N^2\, \overline{u_{prop}})\, u_{prop} \big]\, dx \\
&= \operatorname{Im} \int\limits_{N+Q_\infty} \big[\nabla u_{prop} \cdot \nabla(\psi_N^2\, \overline{u_{prop}}) - k^2 (\psi_N^2\, \overline{u_{prop}})\, u_{prop} \big]\, dx \;-\; \beta_{N,H}^+
\end{aligned}$$

where again $N + Q_\infty = \{x \in \tilde{D} : N < x_1 < N + 2\pi\}$ and

$$|\beta_{N,H}^+| \;=\; \Bigg| \int\limits_{(N+Q_\infty)\setminus W_{N,H}^+} \big[\nabla u_{prop} \cdot \nabla(\psi_N^2\, \overline{u_{prop}}) - k^2 (\psi_N^2\, \overline{u_{prop}})\, u_{prop} \big]\, dx \Bigg| \;\le\; c\, e^{-2\delta H}\,.$$

Now we set $\varphi(x_1) = 1 - (x_1 - N)/(2\pi)$ and observe that $\psi_N^2 - \varphi$ vanishes for $x_1 = N$ and $x_1 = N + 2\pi$. Green's theorem implies that

$$\int\limits_{N+Q_\infty} \big[\nabla u_{prop} \cdot \nabla\big((\psi_N^2 - \varphi)\, \overline{u_{prop}}\big) - k^2 \big((\psi_N^2 - \varphi)\, \overline{u_{prop}}\big)\, u_{prop} \big]\, dx \;=\; 0$$

and thus

$$\begin{aligned}
\operatorname{Im} a_{N,H}^+(2,2) &= \operatorname{Im} \int\limits_{N+Q_\infty} \big[\nabla u_{prop} \cdot \nabla(\varphi\, \overline{u_{prop}}) - k^2 (\varphi\, \overline{u_{prop}})\, u_{prop} \big]\, dx \;-\; \beta_{N,H}^+ \\
&= -\frac{1}{2\pi} \operatorname{Im} \int\limits_{N+Q_\infty} \overline{u_{prop}}\, \frac{\partial u_{prop}}{\partial x_1}\, dx \;-\; \beta_{N,H}^+ \\
&= -\frac{1}{4\pi} \sum_{j\in J} \sum_{\lambda_{\ell,j}>0} \lambda_{\ell,j}\, |a_{\ell,j}|^2 \;-\; \beta_{N,H}^+
\end{aligned}$$

where we used the results of Lemma 2.7 above. The same estimates hold for $a_{N,H}^-(j,\ell)$; that is, the integrals over $W_{N,H}^-$. Therefore, we have shown that

$$\begin{aligned}
(17)\quad & \operatorname{Im} \int\limits_{W_{N,H}^+} \big[|\nabla v|^2 + \overline{v}\, \Delta v \big]\, dx \;+\; \operatorname{Im} \int\limits_{W_{N,H}^-} \big[|\nabla v|^2 + \overline{v}\, \Delta v \big]\, dx \\
&\le\; -\frac{1}{4\pi} \sum_{j\in J} \sum_{\lambda_{\ell,j}>0} \lambda_{\ell,j}\, |a_{\ell,j}|^2 \;+\; \frac{1}{4\pi} \sum_{j\in J} \sum_{\lambda_{\ell,j}<0} \lambda_{\ell,j}\, |a_{\ell,j}|^2 \;+\; c\, e^{-2\delta H} \;+\; c\, \big[\gamma_{N,H} + \sqrt{\gamma_{N,H}}\big]\,.
\end{aligned}$$

(b) Now we look at the left hand side of (15). The line integrals are outside of the layer W_{h_0}. Their estimates in [19] (proof of Theorem 3.2) are independent of the equation or boundary condition below the line $x_2 = h_0$. In [19] we have shown the existence of sequences (N_m) and (H_m) converging to infinity such that $\gamma_{N_m,H_m} \to 0$ and

$$\limsup_{m\to\infty}\left[\operatorname{Im}\int\limits_{\Gamma_{N_m+1,H_m}} \psi_{N_m}^2\,\overline{u}\,\frac{\partial u}{\partial x_2}\,ds\right] \;\geq\; 0\,. \tag{18}$$

From (17) we conclude that

$$\limsup_{m\to\infty}\left[\operatorname{Im}\int\limits_{W^+_{N_m,H_m}} \left[\left|\nabla v\right|^2+\overline{v}\,\Delta v\right]dx \;+\; \operatorname{Im}\int\limits_{W^-_{N_m,H_m}} \left[\left|\nabla v\right|^2+\overline{v}\,\Delta v\right]dx\right]$$

$$\leq\;\; -\frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}>0}\lambda_{\ell,j}\,|a_{\ell,j}|^2 \;+\; \frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}<0}\lambda_{\ell,j}\,|a_{\ell,j}|^2\,.$$

Combining this estimate with (18) and (15) yields that $a_{\ell,j}=0$ for all ℓ and j. □

Below we sketch another proof based on the modified open waveguide radiation condition.

Theorem 2.15. *Let $u\in H^1_{loc,0}(\tilde{D})$ be a solution of $\Delta u+k^2u=0$ in $\tilde{D}$ satisfying the modified open waveguide radiation condition of Definition 2.10. Then u_{prop} vanishes.*

Proof. Choose $a>0$ and suppose without loss of generality that $\tilde{D}_a$ is a Lipschitz domain. Applying Green's formula for u to $\tilde{D}_a$ gives

$$\int_{\tilde{D}_a}|\nabla u|^2-k^2|u|^2\,dx+\int_{C_a}\partial_\nu u\overline{u}\,ds=0.$$

Here we have used the Dirichlet boundary condition on $\partial\tilde{D}$. Taking the imaginary part and using $u=u_{rad}+u_{prop}$ yields

$$0=\operatorname{Im}\int_{C_a}\partial_\nu u\overline{u}\,ds=\operatorname{Im}\int_{C_a}\left[\partial_\nu u_{rad}\overline{u}_{rad}+\partial_\nu u_{rad}\overline{u}_{prop}+\partial_\nu u_{prop}\overline{u}_{rad}+\partial_\nu u_{prop}\overline{u}_{prop}\right]ds$$

Recalling the Sommerfeld radiation condition of u_{rad}, one can show that the integrals involving the term u_{rad} on the right hand side all vanish as $a\to\infty$; see the proof of [12, Theorem 3.1] for details. Therefore, one arrives at

$$0=\lim_{a\to\infty}\operatorname{Im}\int_{C_a}\partial_\nu u_{prop}\,\overline{u}_{prop}\,ds=\lim_{a\to\infty}\operatorname{Im}\int_{\tilde{D}_a}(\Delta+k^2)u_{prop}\,\overline{u}_{prop}\,dx,$$

because u_{prop} vanishes on $\partial\tilde{D}$. Rcalling the representation of u_{prop} (see (6)) and using the definition of $\psi^\pm$, one deduces that (see e.g., [12, Lemma 2.3])

$$\int_{\tilde{D}_a}(\Delta+k^2)u_{prop}\,\overline{u}_{prop}\,dx=\left(\int\limits_{\gamma_a^+}-\int\limits_{\gamma_a^-}\right)\overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial x_1}\,ds\;+\int\limits_{S_a^+\cup S_a^-}\overline{u_{prop}}\,\frac{\partial u_{prop}}{\partial\nu}\,ds \tag{19}$$

where $\gamma_a^\pm=\tilde{D}_a\cap\{x_1=\pm\sigma_0\}$ and $S_a^\pm=\{x\in\tilde{D}_a:|x|=a,\sigma_0-1<\pm x_1<\sigma_0\}$. Note that here have supposed that $\{x\in\tilde{D}_a:\sigma_0-1<\pm x_1<\sigma_0\}$ are both Lipschitz domains by the choice of $\sigma_0>\max\{R,2\pi\}+1$ and that the integrals over $\gamma_a^\pm$ are understood in the dual form between $H^{-1/2}(\gamma_a^\pm)$ and $H_0^{1/2}(\gamma_a^\pm)$. The second term on the right hand side of (19) tends to zero as $a \to \infty$, due to the exponential decay of u_{prop} as $x_2 \to \infty$. For the imaginary part of the first term, we have the limit (see [19, Lemma 2.6])

$$\begin{aligned} \operatorname{Im} \int\limits_{\tilde{D}\cap\{x_1=\pm\sigma_0\}} \overline{u_{prop}} \frac{\partial u_{prop}}{\partial x_1} ds &= \frac{1}{2\pi} \operatorname{Im} \int\limits_{\tilde{D}\cap\{\pm\sigma_0+Q_\infty\}} \overline{u_{prop}} \frac{\partial u_{prop}}{\partial x_1} dx \\ &= \frac{1}{4\pi} \sum_{j\in J} \sum_{\pm\lambda_{\ell,j}>0} \lambda_{\ell,j} \, |a_{\ell,j}|^2. \end{aligned}$$

In this step we have used the fact $\pm\sigma_0 + Q_\infty$ are Lipschitz domains and Lemma 2.6. Finally, taking the imaginary part in (19) and letting $a \to \infty$, we obtain $a_{l,j} = 0$ for all $j \in J$ and $l = 1, 2, \cdots, m_j$. This proves $u_{prop} \equiv 0$. □

Having proved the unique determination of the propagating part, we can show uniqueness of solutions of the unperturbed and perturbed boundary value problems following almost the same lines in the proof of [19, Theorem 3.3]. We omit the proof of Theorem 2.16 below.

Theorem 2.16. *Let the Assumptions 2.2 and 2.3 hold.*

(i) *Let $u \in H^1_{loc,0}(D)$ be a solution of $\Delta u + k^2 u = 0$ in D satisfying the open waveguide radiation condition of Definition 2.5. Then $u \equiv 0$.*

(ii) *In the perturbed case we have $u \equiv 0$, if there are no bound states to the problem (1), that is, any solution $u \in H^1_0(\tilde{D})$ of $\Delta u + k^2 u = 0$ in $\tilde{D}$ must vanish identically.*

In the remaining part we suppose that there are no bound states for the perturbed scattering problem, so that uniqueness always holds true by Theorem 2.16 (ii). Note that this assumption can be removed, if the domain $\tilde{D}$ fulfills the following condition (see [4]):

$$(x_1, x_2) \in \tilde{D} \quad \Longrightarrow \quad (x_1, x_2 + s) \in \tilde{D} \quad \text{for all } s > 0\,. \tag{20}$$

Obviously, the geometrical condition (20) can be fulfilled if the boundary $\tilde{\Gamma}$ is given by the graph of some continuous function. But then also the existence of guided modes is excluded.

3. Construction of the Dirichlet-to-Neumann (DtN) operator

For simplicity we suppose that there is an open arc of the form $C_R := \{x \in D : |x_1 - \pi|^2 + |x_2|^2 = R^2\}$ for some $R > \pi$ such that the domain D_R is Lipschitz (otherwise we can replace C_R by an open curve with a slightly different shape). This implies that the perturbed defect $\Gamma \setminus \tilde{\Gamma}$ always lies below C_R. We refer to Figure 2 for a typical situation. To reduce the scattering problem to a bounded domain, we need Sobolev spaces defined on an open arc. Define the Sobolev spaces (see [22])

$$\begin{aligned} H_0^{1/2}(C_R) &:= \left\{ f \in H^{1/2}(\partial D_R) : f = 0 \text{ on } \partial D \setminus C_R \right\}, \\ H^{1/2}(C_R) &:= \left\{ f|_{C_R} : f \in H^{1/2}(\partial D_R) \right\}. \end{aligned}$$

An important property of $H_0^{1/2}(C_R)$ is that the zero extension of u to ∂D_R belongs to $H^{1/2}(\partial D_R)$. We remark that in the previous definitions the closed boundary ∂D_R can be replaced by other closed boundaries. If $u \in H^1(D_R)$ with $u = 0$ on $\Gamma \cap \overline{D_R}$, then we have the traces $u|_{C_R} \in H_0^{1/2}(C_R)$. The spaces $H_0^{1/2}(C_R)$ and $H^{-1/2}(C_R)$ are (anti-linear) dual spaces in the sense that

$$\langle \phi, \psi \rangle_{H_0^{1/2}(C_R), H^{-1/2}(C_R)} \;=\; \langle \tilde{\phi}, \psi \rangle_{H^{1/2}(\partial D_R), H^{-1/2}(D_R)}\,,$$

where $\tilde{\phi}$ denotes the zero extension of ϕ to ∂D_R. We further remark that for any $a > R$ there exists a bounded extension operator E from $H_0^{1/2}(C_R)$ into $H_0^1(D_a)$. Indeed, extending $\psi \in H_0^{1/2}(C_R)$ by zero in $\partial D_R \cap \Gamma$ we observe that this extension is in $H^{1/2}(\partial D_R)$. By well known results for Lipschitz domains there exists a bounded extension operator E_1 from $H^{1/2}(\partial D_R)$ into $H^1(D_R)$. In the same way one extends $\psi \in H_0^{1/2}(C_R)$ by zero in $\partial D_a \setminus \overline{D_R}$ and constructs an extension operator E_2 from $H^{1/2}(\partial(D_a \setminus D_R))$ into $H^1(D_a \setminus D_R)$ with zero boundary values for $|x| = a$.

Below we recall the definition of the Floquet-Bloch transform to be used later.

Definition 3.1. *For $g \in C_0^\infty(\mathbb{R})$, the Floquet-Bloch transform F is defined by*

$$(Fg)(x_1, \alpha) \;:=\; \sum_{n \in \mathbb{Z}} g(x_1 + 2\pi n)\, e^{-i2\pi n\alpha}\,, \qquad x_1 \in \mathbb{R},\ \alpha \in [-1/2, 1/2]\,.$$

The Floquet-Bloch transform F extends to an unitary operator from $L^2(\mathbb{R})$ to $L^2((0, 2\pi) \times (-1/2, 1/2))$. If g depends on two variables x_1 and x_2 then the symbol F means the Floquet-Bloch transform with respect to x_1.

In the next subsection we prepare several auxiliary results before constructing the DtN operator.

3.1. **Existence Results For Some Unperturbed Problems.** The first result is well known and a simple application of the Theorem of Riesz. Define the weighted Sobolev spaces $H^s_{(\rho)} = \{u \in H_0^s(D) : w_\rho u \in H^s(D)\}$ where $w_\rho(x) = e^{\rho|x|}$ for $\rho \geq 0$.

Theorem 3.2. *Let $\varphi \in H^{-1/2}(C_R)$ and $\rho \in (0, 1)$. Then there exists a unique solution $v \in H_0^1(D)$ of*

$$\int_D [\nabla v \cdot \nabla \overline{\psi} + v\,\overline{\psi}]\, dx \;=\; \int_{C_R} \varphi\, \overline{\psi}\, ds \quad \text{for all } \psi \in H_0^1(D)\,. \tag{21}$$

Note that we have written the dual form $\langle \varphi, \psi \rangle$ on the right hand side as integral. Here we need that the trace $\psi|_{C_R} \in H_0^{1/2}(C_R)$. Furthermore, $v \in H^1_{(\rho)}(D)$ and $\varphi \mapsto v$ is bounded from $H^{-1/2}(C_R)$ into $H^1_{(\rho)}(D)$ and even compact from $H^{-1/2}(C_R)$ into $L^2_{(\rho')}(D)$ for all $\rho' < \rho$.

Proof: The left hand side is just the inner product in $H^1(D)$, and the right hand side is estimated by

$$\Big|\int_{C_R} \varphi\, \overline{\psi}\, ds\Big| \;\leq\; \|\varphi\|_{H^{-1/2}(C_R)}\, \|\psi\|_{H_0^{1/2}(C_R)} \;\leq\; c\|\varphi\|_{H^{-1/2}(C_R)}\, \|\psi\|_{H^1(D)}.$$

Therefore, Riesz's theorem implies uniqueness and existence of a solution in $H_0^1(D)$. Set $\tilde{v} = w_\rho v$ and $\tilde{\psi} = \frac{1}{w_\rho}\psi$. Then $\nabla\psi = \nabla w_\rho \tilde{\psi} + w_\rho \nabla\tilde{\psi}$ and $\nabla v = -\frac{\tilde{v}}{w_\rho^2}\nabla w_\rho + \frac{1}{w_\rho}\nabla\tilde{v}$.

Substituting this into the variational equation yields

$$\int\limits_D \left[\nabla\tilde v\cdot\nabla\overline{\tilde\psi} + \overline{\tilde\psi}\,\frac{\nabla w_\rho}{w_\rho}\cdot\nabla\tilde v - \tilde v\,\frac{\nabla w_\rho}{w_\rho}\cdot\nabla\overline{\tilde\psi} - \frac{|\nabla w_\rho|^2}{w_\rho^2}\,\tilde v\,\overline{\tilde\psi} + \tilde v\,\overline{\tilde\psi}\right] dx \;=\; \int\limits_{C_R} \varphi\,\overline{\tilde\psi}\,w_\rho\,ds\,.$$

We observe that the left hand side defines a sesqui-linear form on $H^1(D)$ which is coercive for $\rho < 1$ because $\frac{|\nabla w_\rho|}{w_\rho} = \rho$. The right hand side defines again a bounded linear form on $H^1(D)$. Therefore, Lax-Milgram yields existence and uniqueness. This proves that $v \in H^1_{(\rho)}(D)$ and that $\varphi \mapsto v$ is bounded from $H^{-1/2}(C_R)$ into $H^1_{(\rho)}(D)$.

Finally we show that $H^1_{(\rho)}(D)$ is compactly imbedded in $L^2_{(\rho')}(D)$ for all $\rho' < \rho$. Let (v_j) be a sequence in $H^1_{(\rho)}(D)$ which converges weakly to zero. Set again $\tilde v_j = w_\rho v_j$. Then $(\tilde v_j)$ converges weakly to zero in $H^1(D)$ and is thus bounded. Therefore, there exists $c > 0$ with $\|\tilde v_j\|_{L^2(D)} \le c$ for all j. We estimate for any $a > 0$

$$\begin{aligned}\int\limits_{D\setminus D_a} w_{\rho'}^2(x)\,|v_j(x)|^2\,dx \;&=\; \int\limits_{D\setminus D_a} w_\rho^2(x)\,|v_j(x)|^2\,e^{-2(\rho-\rho')|x|}\,dx \\ &\le\; e^{-2(\rho-\rho')a}\int\limits_D w_\rho^2(x)\,|v_j(x)|^2\,dx \;\le\; c^2\,e^{-2(\rho-\rho')a}\,.\end{aligned}$$

Given $\varepsilon > 0$ we choose $a > 0$ with $c^2\,e^{-2(\rho-\rho')a} < \frac{\varepsilon^2}{2}$ and keep r fixed. Since $(\tilde v_j)$ tends to zero weakly in $H^1(D)$ it tends to zero weakly in $H^1(D_a)$. Therefore, $\|\tilde v_j\|_{L^2(D_a)}$ tends to zero and thus also $\int_{D_a} w_{\rho'}^2|v_j|^2dx$ because on D_a the norms $\|w_\eta v\|_{L^2(D_a)}$ are all equivalent. Thus, for sufficiently large j the term $\int_{D_a} w_{\rho'}^2|v_j|^2dx$ is less than $\frac{\varepsilon^2}{2}$. □

The proofs of most existence results for the Helmholtz equation in periodic structures are based on the following result for quasi-periodic problems. For a proof we refer to [19, Theorems 4.2, 4.3, and Remark 4.4] adopted to the present situation.

Theorem 3.3. *Let Assumptions 2.2 and 2.3 hold and let $g_\alpha \in L^2(Q_\infty)$ for $\alpha \in [-1/2, 1/2]$ depend continuously differentiable on α in $[-1/2, 1/2]$. Let there exist $\hat c > 0$ and $\delta > 0$ with $|g_\alpha(x)| + |\partial g_\alpha(x)/\partial\alpha| \le \hat c e^{-\delta x_2}$ for almost all $x \in Q_\infty$ with $x_2 > h_0$ and all $\alpha \in [-1/2, 1/2]$. Furthermore, let $G \in H^{-1}(Q_{h_0}) = H^1_0(Q_{h_0})^*$ and assume that for any propagative wave number $\hat\alpha_j \in [-1/2, 1/2]$ the orthogonality condition*

$$(22)\qquad \langle G, \overline{\hat\phi}\rangle \;+\; \int\limits_{Q_\infty} g_{\hat\alpha_j}(x)\,\overline{\hat\phi(x)}\,dx \;=\; 0$$

hold for all modes $\hat\phi \in X_j$ corresponding to the propagative wave number $\hat\alpha_j$. Here, $\langle\cdot,\cdot\rangle$ denotes the dual (bi-linear) form.

Then for every $\alpha \in [-1/2, 1/2]$ there exists an $\alpha-$quasi-periodic solution $v_\alpha \in H^1_{\alpha,loc,0}(D)$ of the equation

$$(23)\qquad \Delta v_\alpha + k^2 v_\alpha \;=\; -g_\alpha \;-\; G \quad \text{in } Q_\infty$$

satisfying the generalized Rayleigh radiation condition

$$(24)\qquad \sum_{n\in\mathbb{Z}}\left|\frac{\partial v_{\alpha,n}(x_2)}{\partial x_2} - i\sqrt{k^2-(\alpha+n)^2}\,v_{\alpha,n}(x_2)\right|^2 \;\to\; 0\,, \qquad x_2 \to +\infty\,.$$

Here, $v_{\alpha,n}(x_2) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} v_\alpha(x_1,x_2)e^{-i(n+\alpha)x_1}dx_1$ *are the Fourier coefficients of* $v_\alpha(\cdot,x_2)$*, and (23) is understood in the variational sense*

$$\int_{Q_\infty}\left[\nabla v_\alpha\cdot\nabla\overline{\psi} - k^2 v_\alpha\,\overline{\psi}\right]dx \;=\; \langle G,\overline{\psi}\rangle \;+\; \int_{Q_\infty} g_\alpha\,\overline{\psi}\,dx$$

for all $\psi\in H^1_{\alpha,loc,0}(D)$ *which vanish for* $x_2>h$ *for some* $h>h_0$.
Furthermore, v_α *can be chosen to depend continuously on* α*, and for every* $h>h_0$ *there exists* $c_h>0$ *with*

$$\|v_\alpha\|_{H^1(Q_h)} \;\le\; c_h\Big[\sup_{\beta\in[-1/2,1/2]}\|g_\beta\|_{L^{(1,2)}(Q_\infty)} + \sup_{\beta\in[-1/2,1/2]}\|\partial g_\beta/\partial\beta\|_{L^{(1,2)}(Q_\infty)} \;+\; \|G\|_{H^{-1}(Q_{h_0})}\Big]$$

for all $\alpha\in[-1/2,1/2]$ *where we used the notation* $\|\phi\|_{L^{(1,2)}(Q_\infty)} := \|\phi\|_{L^1(Q_\infty)} + \|\phi\|_{L^2(Q_\infty)}$.

We will apply this result to the following two problems.
Given $\varphi\in H^{-1/2}(C_R)$, consider the problem of determining $u\in H^1_{loc}(D)$ such that

$$\Delta u + k^2 u = 0 \text{ in } D\setminus C_R\,,\quad u=0 \text{ on } \Gamma\,,\quad \frac{\partial u_-}{\partial\nu} - \frac{\partial u_+}{\partial\nu} = \varphi \text{ on } C_R\,, \tag{25}$$

and that u satisfies the open waveguide radiation condition of Definition 2.5. Here the normal direction ν is supposed to direct into the exterior Σ_R. Well-posedness of the variational formulation corresponding to the problem (25) is stated as follows.

Theorem 3.4. *Let* $\varphi\in H^{-1/2}(C_R)$*. Then there exists a unique solution* $u\in H^1_{loc,0}(D)$ *of*

$$\int_D [\nabla u\cdot\nabla\overline{\psi} - k^2 u\,\overline{\psi}]\,dx \;=\; \int_{C_R}\varphi\,\overline{\psi}\,ds \quad \text{for all } \psi\in H^1_{0,c}(D) \tag{26}$$

satisfying the open waveguide radiation condition. Here,

$$H^1_{0,c}(D) \;:=\; \left\{\psi\in H^1_0(D) : \text{there exists } a>0 \text{ with } \psi(x)=0 \text{ for } |x|>a\right\}.$$

Furthermore, the mapping $\varphi\mapsto u|_{C_R}$ *is bounded from* $H^{-1/2}(C_R)$ *into* $H^{1/2}_0(C_R)$.

Proof: Uniqueness follows directly from Theorem 2.14, part (i). To prove existence, we suppose without loss of generality that C_R is chosen to lie in Q_{h_0} and define the coefficients $a_{\ell,j}$ explicitly as

$$a_{\ell,j} \;:=\; \frac{2\pi i}{|\lambda_{\ell,j}|}\int_{C_R}\varphi(x)\,\overline{\hat{\phi}_{\ell,j}(x)}\,ds(x)\,,\;\; \ell=1,\dots,m_j\,,\;\; j\in J\,. \tag{27}$$

Then the propagating part u_{prop} is defined, and the radiating part u_{rad} has to satisfy

$$\begin{aligned}
&\int_D [\nabla u_{rad}\cdot\nabla\overline{\psi} - k^2 u_{rad}\,\overline{\psi}]\,dx \\
&= \; -\int_D [\nabla u_{prop}\cdot\nabla\overline{\psi} - k^2 u_{prop}\,\overline{\psi}]\,dx \;+\; \int_{C_R}\varphi\,\overline{\psi}\,ds \\
&= \; \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\int_D \varphi_{\ell,j}\,\overline{\psi}\,dx \;+\; \int_{C_R}\varphi\,\overline{\psi}\,ds \quad \text{for all } \psi\in H^1_{0,c}(D)
\end{aligned} \tag{28}$$

and the generalized angular spectrum radiation condition (7). Here, $\varphi_{\ell,j}$ are given by (9b). Defining the distribution $f_\varphi \in H^{-1}(D)$ as

$$\langle f_\varphi, \psi\rangle \;:=\; \int\limits_{C_R} \varphi\,\overline{\psi}\,ds \quad \text{for all} \quad \psi\in H^1_0(D)\,,$$

where the right hand side is understood as the duality between $H^{-1/2}(C_R)$ and $H^{1/2}_0(C_R)$, we observe that the variational equation represents the differential equation

$$\Delta u_{rad} + k^2 u_{rad} \;=\; -\sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\,\varphi_{\ell,j} \;-\; f_\varphi\,.$$

One now applies Theorem 3.3 to $g_\alpha = \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\,F\varphi_{\ell,j}$ and $G = f_\varphi$. The orthogonality condition (22) is satisfied by the choice of the coefficients (27) (see [19, Lemma 5.1]). Therefore, for all $\alpha\in[-1/2,1/2]$ there exists a solution $\hat v(\cdot,\alpha)\in H^1_{\alpha,loc,0}(D)$ of the $\alpha-$quasi-periodic problems

$$\int\limits_{Q_\infty}\left[\nabla\hat v(\cdot,\alpha)\cdot\nabla\overline{\hat\psi} - k^2\hat v(\cdot,\alpha)\,\overline{\hat\psi}\right]dx \tag{29}$$

$$=\;\sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\int\limits_{Q_\infty}(F\varphi_{\ell,j})(\cdot,\alpha)\,\overline{\hat\psi}\,dx \;+\; \int\limits_{C_R}\varphi\,\overline{\hat\psi}\,ds$$

for all $\hat\psi\in H^1_\alpha(D)$ which vanish for $x_2>h$ for some $h>h_0$ satisfying the generalized Rayleigh radiation condition (24). Furthermore, $\hat v(\cdot,\alpha)$ depends continuously on α and for every $h>h_0$ there exists $c_h>0$ with

$$\|\hat v(\cdot,\alpha)\|_{H^1(Q_h)} \;\le\; c_h\left[\sum_{j\in J}\sum_{\ell=1}^{m_j}|a_{\ell,j}| \;+\; \sup_{\|\psi\|_{H^1(Q_\infty)}=1}\left|\int\limits_{C_R}\varphi\,\overline{\psi}\,ds\right|\right] \;\le\; c_h\,\|\varphi\|_{H^{-1/2}(C_R)}\,. \tag{30}$$

By the properties of the Floquet-Bloch transform the inverse transform

$$u_{rad}(x) \;:=\; (F^{-1}\hat v)(x) \;=\; \int\limits_{-1/2}^{1/2}\hat v(x,\alpha)\,d\alpha$$

is in $H^1_*(D)$. Furthermore, taking $\psi\in C_0^\infty(D)$ we substitute $\hat\psi := (F\psi)(\cdot,\alpha)$ into the variational equation (29) and integrate with respect to α; that is,

$$\int\limits_{-1/2}^{1/2}\int\limits_{Q_\infty}\left[\nabla\hat v(x,\alpha)\cdot\nabla\overline{(F\psi)(x,\alpha)} - k^2\hat v(x,\alpha)\,\overline{(F\psi)(x,\alpha)}\right]dx\,d\alpha$$

$$=\;\sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\int\limits_{-1/2}^{1/2}\int\limits_{Q_\infty}(F\varphi_{\ell,j})(x,\alpha)\,\overline{(F\psi)(x,\alpha)}\,dx\,d\alpha + \int\limits_{-1/2}^{1/2}\int\limits_{C_R}\varphi(x)\,\overline{(F\psi)(x,\alpha)}\,ds(x)\,d\alpha$$

Noting that $\int_{-1/2}^{1/2}(F\psi)(\cdot,\alpha)\,ds = \psi$ and using the unitarity of the Floquet-Bloch transform we observe that this is exactly the equation (28). Boundedness of $\varphi\to u|_{C_R}$ is now easily seen from (30) and the unitarity of F and the fact that u_{prop} depends explicitly on φ through $a_{\ell,j}$. $\square$

A second application is the following result where the source fails to be compactly supported.

Theorem 3.5. *Let $f \in L^2_{(\rho)}(D)$ for some $\rho \in (0,1)$. Then there exists a unique solution $w \in H^1_{loc,0}(D)$ of $\Delta w + k^2 w = -f$ in D satisfying the open waveguide radiation condition. Furthermore, for every $a > R$ the mappings $f \mapsto w|_{D_a}$ and $f \mapsto w|_{C_R}$ are bounded from $L^2_{(\rho)}(D)$ into $H^1(D_a)$ and $H^{1/2}_0(C_R)$, respectively.*

Proof: Since f decays exponentially, its Floquet-Bloch transform Ff is well defined and continuously differentiable with respect to α. Instead of (29) we now solve

$$\int_{Q_\infty} \left[\nabla \hat{w}(\cdot,\alpha)\cdot\nabla\overline{\hat{\psi}} - k^2\hat{w}(\cdot,\alpha\,\overline{\hat{\psi}}\right] dx$$

$$= \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\int_{Q_\infty}(F\varphi_{\ell,j})(\cdot,\alpha)\,\overline{\hat{\psi}}\,dx \;+\; \int_{Q_\infty}(Ff)(\cdot,\alpha)\,\overline{\hat{\psi}}\,dx$$

for all $\hat{\psi} \in H^1_\alpha(D)$ which vanish for $x_2 > h$ for some $h > h_0$ and the generalized Rayleigh radiation condition (24).
The coefficients $a_{\ell,j}$ have to be chosen as

$$a_{\ell,j} \;=\; \frac{2\pi i}{|\lambda_{\ell,j}|}\int_{Q_\infty}(Ff)(x,\hat{\alpha}_j)\,\overline{\hat{\phi}_{\ell,j}(x)}\,dx\,,\quad \ell = 1,\cdots,m_j\,,\; j\in J\,,$$

so that the right hand side is always orthogonal to the nullspace of the homogeneous equation. Using the estimate

$$\begin{aligned}
\|(Ff)(\cdot,\beta)\|^2_{L^2(Q_\infty)} &= \int_{Q_\infty}\bigl|(Ff)(x,\beta\bigr|^2 dx \;\le\; \int_{Q_\infty}\Bigl[\sum_{\ell\in\mathbb{Z}}|f(x_1+2\pi\ell,x_2)|\Bigr]^2 dx \\
&= \int_{Q_\infty}\Bigl[\sum_{\ell\in\mathbb{Z}}(1+\ell^2)^{-1/2}\bigl|(1+\ell^2)^{1/2}f(x_1+2\pi\ell,x_2)\bigr|\Bigr]^2 dx \\
&\le \int_{Q_\infty}\sum_{\ell\in\mathbb{Z}}\frac{1}{1+\ell^2}\sum_{\ell\in\mathbb{Z}}(1+\ell^2)|f(x_1+2\pi\ell,x_2)|^2\,dx \\
&\le c\int_D (1+x_1^2)\,|f(x)|^2\,dx \;\le\; c\,\|f\|^2_{L^2_{(\rho)}(D)}
\end{aligned}$$

and analogously for $\|(Ff)(\cdot,\beta)\|^2_{L^1(Q_\infty)}$ and the derivatives with respect to β we can repeat the proof of Theorem 3.4. $\square$

3.2. **The DtN Operator.** Now we turn to the construction of the Dirichlet-to-Neumann operator on the artificial boundary C_R. In the remaining part of this paper we make the following assumption.

Assumption 3.6. *Assume that k^2 is not the Dirichlet eigenvalue of $-\Delta$ in the Lipschitz domain D_R and there are no bound states of the Helmholtz equation over the domain Σ_R; that is, if $u \in H_0^1(\Sigma_R)$ solves $\Delta u + k^2 u = 0$ in Σ_R, then u must vanish identically.*

As usual, the DtN operator Λ should be defined as follows.

Definition 3.7. *The Dirichlet-to-Neumann operator $\Lambda : H_0^{1/2}(C_R) \to H^{-1/2}(C_R)$ is defined by $\Lambda\, g = \partial_\nu u|_{C_R}$ where $u \in H^1_{loc}(\Sigma_R)$ is the unique solution to*

$$\Delta u + k^2 u = 0 \text{ in } \Sigma_R \quad u = g \text{ on } C_R\,, \quad u = 0 \text{ on } \Gamma \cap \partial\Sigma_R\,, \tag{31}$$

which fulfills the open waveguide radiation condition of Definition 2.5. Here the unit normal vector ν at C_R is supposed to direct into Σ_R.

The above definition assumes already the solvability of a boundary value problem in the perturbed region Σ_R – which to show is the purpose of the forthcoming Section 4. However, the perturbed region Σ_R is a subset of D (in contrast to the more general perturbation $\tilde{D}$) which allows the application of an integral equation approach with the Dirichlet-Green's function of D. Before we explain the construction we note that an explicit representation in form of a series can be obtained if Γ is a straight line parallel to the x_1-axis. In this exceptional case the propagating part (guided waves) vanishes identically and the radiating part fulfills the classical Sommerfeld radiation condition. Consequently, the function $g \in H_0^{1/2}(C_R)$ can be expanded into $g(\theta) = \sum_{n\in\mathbb{N}_0} g_n \sin n\theta$ with $\theta \in (0,\pi)$ and the DtN operator takes the explicit form

$$(\Lambda g)(\theta) \;=\; \sum_{n\in\mathbb{N}_0} g_n \frac{k\, H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} \sin n\theta \quad \text{for } \theta \in (0,\pi)\,.$$

Here, $H_0^{(1)}$ denotes the Hankel function of the first kind of order zero. In the general case that Γ is a periodic curve, we will express the field in Σ_R as a single layer potential with density φ and the Green's function as kernel. As usual, φ is determined from g by solving an integral equation for the single layer boundary operator. We divide our arguments into two steps.

(A) Construction of the single layer boundary operator. As a motivation we recall that for smooth data the single layer boundary operator with the Green's function as kernel is given by $S\varphi = u|_{C_R}$ where $u \in H^1_{loc}(D)$ satisfies the transmission problem (25) and the open waveguide radiation condition. In this way we avoid the explicit use of the Green's function. For given $\varphi \in H^{-1/2}(C_R)$, the variational form of (25) is given by (26) and has been studied in Theorem 3.4.
We take the solution of this transmission problem as the definition of the single layer operator, namely $S\varphi := u|_{C_R}$ where $u \in H^1_{loc,0}(D)$ is the unique solution of (26) satisfying the open waveguide radiation condition. Then S is bounded from $H^{-1/2}(C_R)$ into $H_0^{1/2}(C_R)$ by Theorem 3.4. To show the injectivity of S, we suppose that $S\varphi = 0$. Then $u = 0$ in Σ_R and $u = 0$ in D_R by the Assumption 3.6 and the uniqueness result of Theorem 2.16. From the variational equation (26) we conclude that $\int_{C_R} \varphi\, \overline{\psi}\, ds = 0$ for all ψ which implies that φ vanishes. This proves the injectivity of S. Next we show that S is boundedly invertible. Let S_i be the operator corresponding to wave number $k = i$. Then, setting $S_i\varphi = v|_{C_R}$, we get $\langle \varphi, S_i\varphi\rangle = \int_{C_R} \varphi\, \overline{v}\, ds$ by the definition of the dual form $\langle\cdot,\cdot\rangle$ (see Theorem 3.2)

and $v \in H_0^1(D)$ solves (21). Setting $\psi = v\,\phi_a$ in (21) where $\phi_a \in C_0^\infty(D)$ satisfies $\phi_a = 1$ for $|x| \le a$ and letting a tend to infinity shows that

$$\langle \varphi, S_i \varphi \rangle \;=\; \int\limits_{C_R} \varphi\, \overline{v}\, ds \;=\; \int\limits_{D} [|\nabla v|^2 + |v|^2]\, dx \;=\; \|v\|^2_{H^1(D)}\,.$$

Next we note that $\|\varphi\|_{H^{-1/2}(C_R)} = \sup\bigl\{|\langle\varphi,\psi\rangle| : \|\psi\|_{H_0^{1/2}(C_R)} \le 1\bigr\}$. For $\psi \in H_0^{1/2}(C_R)$ with $\|\psi\|_{H_0^{1/2}(C_R)} \le 1$ we set $\tilde{\psi} = E\psi$ with the extension operator E from $H_0^{1/2}(C_R)$ into $H_0^1(D_a)$ for some $a > R$ and estimate

$$|\langle\varphi,\psi\rangle| \;=\; \Bigg|\int\limits_{C_R} \varphi\,\overline{\psi}\, ds\Bigg| = \Bigg|\int\limits_{D} [\nabla v\cdot\nabla\overline{\tilde{\psi}} + v\,\overline{\tilde{\psi}}]\, dx\Bigg| \le\; c\,\|\tilde{\psi}\|_{H^1(D)}\,\|v\|_{H^1(D)} \;\le\; c\,\|E\|\,\|v\|_{H^1(D)}$$

for $\|\psi\|_{H_0^{1/2}(C_R)} \le 1$ and thus $\|\varphi\|_{H^{-1/2}(C_R)} \le c\,\|E\|\,\|v\|_{H^1(D)}$. Combining this with the previous estimate yields coercivity of S_i; that is,

$$\langle \varphi, S_i\varphi\rangle \;\ge\; \frac{1}{c^2\|E\|^2}\,\|\varphi\|^2_{H^{-1/2}(C_R)}\,.$$

Now we show that $S - S_i$ is compact. We observe that $(S - S_i)\varphi = w|_{C_R}$ where $w = u - v \in H^1_{loc}(D)$ satisfies

$$\Delta w + k^2 w \;=\; -(k^2+1)\,v \text{ in } D\,, \quad w = 0 \text{ on } \Gamma\,,$$

and the open waveguide radiation conditions. Here, v corresponds to the solution of (21) with $k = i$ as before. By Theorem 3.2 we know that $\varphi \mapsto v$ is compact from $H^{-1/2}(C_R)$ into $L^2_{(\rho')}(D)$ for all $\rho' < \rho$. Furthermore, by Theorem 3.5 (for ρ' replacing ρ) the mapping $(1+k^2)v \mapsto w|_{C_R}$ is bounded from $L^2_{(\rho')}(D)$ into $H_0^{1/2}(C_R)$. Combining this yields compactness of $\varphi \mapsto w|_{C_R}$; that is, compactness of $S - S_i$ from $H^{-1/2}(C_R)$ into $H_0^{1/2}(C_R)$.

Therefore, the operator equation $S\varphi = g$ can be written as $S_i\varphi + (S - S_i)\varphi = g$. This shows that S is a Fredholm operator with index zero. By the Fredholm alternative, the injectivity implies the invertibility of S.

(B) Construction of the Dirichlet-to-Neumann operator. Given $g \in H_0^{1/2}(C_R)$ we define $\varphi := S^{-1}g \in H^{-1/2}(C_R)$. Then, by definition, $g = S\varphi = u|_{C_R}$ where u satisfies (26); in particular $\Delta u + k^2 u = 0$ in Σ_R and $u = 0$ on $\Gamma \cap \partial\Sigma_R$, complemented by the open waveguide radiation condition. Consequently, the Neumann boundary data can be defined by Green's first formula; that is, the DtN operator Λ from $H_0^{1/2}(C_R)$ into $H^{-1/2}(C_R) = \bigl(H_0^{1/2}(C_R)\bigr)^*$ can be defined as follows.

Definition 3.8. *Let $a > R$ be fixed. Then $\Lambda : H_0^{1/2}(C_R) \to H^{-1/2}(C_R) = \bigl(H_0^{1/2}(C_R)\bigr)^*$ is defined as*

$$\text{(32)} \qquad \langle \Lambda g, \psi\rangle \;=\; -\int\limits_{D_a\setminus D_R} \bigl[\nabla u\cdot\nabla\overline{(E\psi)} - k^2 u\,\overline{(E\psi)}\bigr]\, dx\,, \quad \psi \in H_0^{1/2}(C_R)\,,$$

where $E : H_0^{1/2}(C_R) \to H_0^1(D_a)$ is again a fixed extension operator and $u \in H^1_{loc}(D)$ is the single layer potential with density $\varphi := S^{-1}g \in H^{-1/2}(C_R)$; that is, the unique solution of (26) studied in Theorem 3.4.

We note that the definition is independent of $a > R$ or the choice of the extension operator E. This follows from Green's identity $\int_{D_a\setminus D_R}\big[\nabla u \cdot \nabla(\overline{\psi_1} - \overline{\psi_2}) - k^2 u\,(\overline{\psi_1} - \overline{\psi_2})\big]\,dx = 0$ for all $\psi_j \in H_0^1(D_a)$ with $\psi_1 = \psi_2$ on C_R.

We finish this section by proving some mapping properties of Λ.

Lemma 3.9. *The DtN operator $\Lambda : H_0^{1/2}(C_R) \to H^{-1/2}(C_R)$ is bounded. Moreover, the operator $-\Lambda$ can be decomposed into the sum of a coercive operator and a compact operator.*

Proof. By the trace lemma and (32), the boundedness of Λ follows from the estimate

$$\|\Lambda g\|_{H^{-1/2}(C_R)} \;=\; \sup_{\|\psi\|_{H_0^{1/2}(C_R)}=1} |\langle \Lambda g, \psi\rangle| \;\leq\; c\,\|u\|_{H^1(D_a\setminus D_R)} \;\leq\; c\,\|g\|_{H^{1/2}(C_R)}\,,$$

where we have used the boundedness of the extension operator E and the continuous dependence of u from g. Define $\Lambda_i : H_0^{1/2}(C_R) \to H^{-1/2}(C_R)$ as the DtN operator for the wave number $k = i$; that is,

$$\langle \Lambda_i g, \psi\rangle \;=\; -\int\limits_{D_a\setminus D_R} [\nabla v \cdot \nabla\overline{\tilde\psi} + v\,\overline{\tilde\psi}]\,dx\,, \quad \psi \in H_0^{1/2}(C_R)\,,$$

where $v \in H_0^1(D)$ solves (21) for $\varphi := S_i^{-1}g \in H^{-1/2}(C_R)$, and $\tilde\psi \in H_0^1(D_a)$ is an extension of ψ. The operator $-\Lambda_i$ is coercive over $H_0^{1/2}(C_R)$. Indeed, choose $\phi_a \in C^\infty(\mathbb{R}^2)$ with $\phi_a = 1$ for $|x| < R$ and $\phi_a = 0$ for $|x| > a-1$ and set $\tilde\psi = v\,\phi_a$ for $a > R+1$. Then $\tilde\psi \in H_0^1(D_a)$ and thus

$$-\langle \Lambda_i g, g\rangle \;=\; \|v\|^2_{H^1(D_{a-1}\setminus D_R)} \;+\; \int\limits_{D_a\setminus D_{a-1}} [\nabla v \cdot \nabla\overline{(v\phi_a)} + \phi_a|v|^2]\,dx\,.$$

Now we let a tend to infinity and use that $v \in H^1(D)$. Therefore,

$$-\langle \Lambda_i g, g\rangle \;=\; \|v\|^2_{H^1(D\setminus D_R)} \;\geq\; c\,\|v\|^2_{H^{1/2}(C_R)} \;=\; c\,\|g\|^2_{H^{1/2}(C_R)}$$

where we used the boundedness of the trace operator in the inequality. Furthermore, the operator $\Lambda - \Lambda_i$ is compact. Indeed, this follows from

$$\langle (\Lambda - \Lambda_i)\,g, \psi\rangle \;=\; -\int\limits_{D_a\setminus D_R} \big[\nabla(u-v)\cdot\nabla\overline{(E\psi)} - (k^2u + v)\,\overline{E\psi}\big]\,dx\,, \quad \psi \in H_0^{1/2}(C_R)\,,$$

and the compactness of the mapping $g \mapsto (u-v)|_{D_a\setminus D_R}$ from $H_0^{1/2}(C_R)$ into $H^1(D_a \setminus D_R)$ (by the same arguments as in the proof of the compactness of $S - S_i$) and the boundedness of $g \mapsto k^2u + v$ from $H_0^{1/2}(C_R)$ into $H^1(D_a \setminus D_R)$ and the compact embedding of $H^1(D_a \setminus D_R)$ into $L^2(D_a \setminus D_R)$. $\square$

4. Existence of Solutions of the Perturbed Problem

In this section we investigate well-posedness of time-harmonic scattering of an incoming wave u^{in} from a locally perturbed periodic curve $\tilde{\Gamma} = \partial\tilde{D}$ of Dirichlet kind; see Figure 2. We consider three kinds of incoming waves:

(i) Point source wave: $u^{in}(x) := \Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|)$ with the source position $y \in \tilde{D}$. Without loss of generality we suppose that $y \in \tilde{D}_R$.

(ii) Plane wave: $u^{in}(x) = e^{ikx\cdot\hat{\theta}}$ where $\hat{\theta} = (\sin\theta, -\cos\theta)$ is the incident direction with some incident angle $\theta \in (-\pi/2, \pi/2)$. In this case the incoming wave is incident onto $\tilde{\Gamma}$ from above, and the parameter $\alpha := k\sin\theta$ is supposed to be not a propagative wavenumber (see Definition 2.1 (ii)).

(iii) $u^{in}(x) = \hat{\phi}_{\ell,j}(x)$ is a right (resp. left) going surface wave at the propagative wavenumber $\hat{\alpha}_j$ for some $j \in J$ which corresponds to the spectral problem (5) with the eigenvalue $\lambda_{\ell,j} > 0$ (resp. $\lambda_{\ell,j} < 0$).

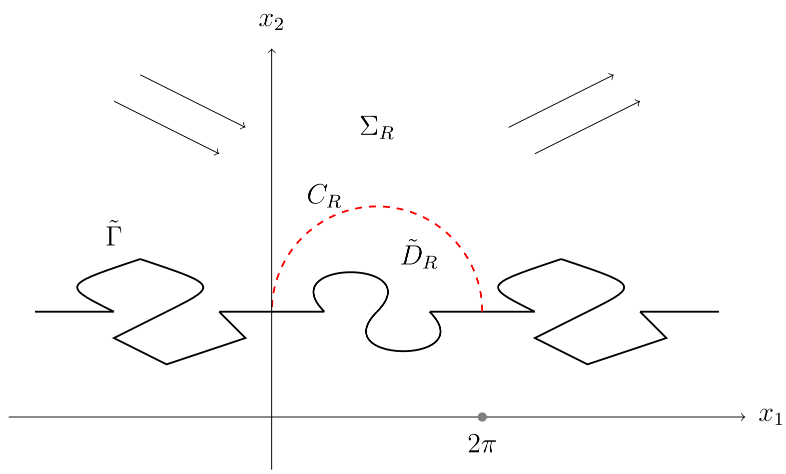


Figure 2. Illustration of wave scattering from perfectly reflecting periodic curves with a local perturbation.

We denote by u^{sc}_{unpert} the unperturbed scattered field, defined in D, which is caused by the unperturbed curve Γ. In Σ_R the total field u can be decomposed into $u = u^{in} + u^{sc}_{unpert} + u^{sc}_{pert}$, and u^{sc}_{pert} can be considered as the scattered part induced by the defect. The field u^{sc}_{pert} is supposed to fulfill the open waveguide radiation condition of Definition 2.5 for all of the cases (i), (ii), (iii).

Define the spaces

$$Y_R \ := \ \left\{ v \in H^1(\tilde{D}_R) : v = 0 \text{ on } \tilde{\Gamma} \cap \partial\tilde{D}_R \right\},$$

where $y \in \tilde{D}$. Well-posedness of our scattering problems will be stated separately for different incoming waves.

Theorem 4.1 (Well-posedness for point source waves). *Let $u^{in} := \Phi(\cdot, y)$ be an incoming point source wave with $y \in \tilde{D}_R$. Then the locally perturbed scattering problem admits a unique solution u such that $u - u^{in} \in H^1_{loc}(\tilde{D})$ and u satisfies the open waveguide radiation conditions of Definitions 2.5 and 2.10.*

In this theorem, the total field u is required to satisfy the open waveguide radiation condition of Definition 2.5, because u is nothing else but the Green's function of the perturbed problem. We remark that in Σ_R the scattered field $u - u^{in}$ does not fulfill this radiation condition, since $u^{in} = \Phi(\cdot, y)$ does not belong to $H^1(W_h \cap \Sigma_R)$ for any $h > h_0$. If both Γ and $\tilde{\Gamma}$ can be represented as graphs of Lipschitz functions, the propagating part u_{prop} vanishes identically (see [4, 5]) and thus $u = u_{rad}$. In such a case, it was verified in [11, Theorem 2.2] that $u = u(\cdot, y) \in H^1_\rho(W_h \cap \Sigma_R)$ with $R > |y_1 - \pi|$ for all $\rho < 1$ and that $u(\cdot, y) - \Phi(\cdot, y) \in H^1_\rho(W_h)$ for all $\rho < 0$. In addition, both u and $u - \Phi$ satisfy the Sommerfeld radiation condition of Definition 2.8. The above results of Theorem 4.1 have generalized those of [11] to non-graph curves where guided (propagating) waves may occur. On the other hand, the technical assumption made in [11, Section 2.3] that Γ should contain at least one line segment in each period was removed in this paper by constructing a new form of the DtN operator; see subsection 3.2.

Proof of Theorem 4.1. Since the incident field is singular at y, we transform our scattering problem to an equivalent source problem of the form (1). Introduce a smooth cut-off function $\chi : \mathbb{R}^2 \to \mathbb{R}$ such that $\chi(x) = 1$ for $|x - y| < \epsilon/2$ and $\chi(x) = 0$ for $|x - y| \geq \epsilon$. Here $\epsilon > 0$ is chosen to be less than the distance between y and $\partial\tilde{D}_R$. We make the ansatz on the total field u as

$$u(x) \;=\; \chi(x)\, \Phi(x, y) \;+\; v(x, y)\,, \quad x \in \tilde{D}, \quad x \neq y\,.$$

Then the scattering problem is equivalent of finding $v(\cdot, y) \in H^1_{0,loc}(\tilde{D})$ such that

$$\begin{cases} \Delta_x v(\cdot, y) + k^2 v(\cdot, y) \;=\; -g_y \text{ in } \tilde{D}, \quad v(\cdot, y) = 0 \text{ on } \tilde{\Gamma}, \\ v(\cdot, y) \text{ satisfies the open waveguide radiation condition of Definition 2.5} \end{cases}$$

with

$$g_y \;:=\; \Delta\chi\, \Phi(\cdot, y) + 2\nabla\chi \cdot \nabla_x \Phi(\cdot, y) \in L^2(\tilde{D})\,.$$

Note that the source term g_y is compactly supported in $\tilde{D}_R$. By the DtN operator Λ, this problem can be reduced to an equivalent boundary value problem over the truncated domain $\tilde{D}_R$. Consequently, we get the following variational formulation. Determine $v \in Y_R$ such that

$$\int_{\tilde{D}_R} \nabla v \cdot \overline{\nabla\psi} - k^2 v\, \overline{\psi}\, dx \;-\; \int_{C_R} \Lambda v\, \overline{\psi} ds \;=\; \int_{\tilde{D}_R} g_y\, \overline{\psi}\, ds \quad \text{for all} \quad \psi \in Y_R\,. \tag{33}$$

Here the integral over C_R is understood as the duality between $H^{-1/2}(C_R)$ and $H^{1/2}_0(C_R)$. In view of Lemma 3.9, the sesqui-linear form defined by the left hand side of (33) is strongly elliptic, leading to a Fredholm operator with index zero over Y_R. By Theorem 2.16 we have uniqueness and thus also existence of $v(\cdot, y) \in Y_R$ by the Fredholm alternative. This solution can be extended to the exterior Σ_R by solving the problem of Theorem 3.2 with $\varphi := S^{-1}(v|_{C_R}) \in H^{-1/2}(C_R)$.

Finally we note that $u - u^{in} = v + (\chi - 1)\Phi(\cdot, y) \in H^1_{loc}(\tilde{D})$ because χ vanishes in a neighborhood of y, and $u = v$ in Σ_R because χ vanishes in Σ_R. This ends the proof. $\square$

Remark 4.2. *If we decompose the field u into $u = u^{in} + u^{sc}_{unpert} + u^{sc}_{pert}$ in Σ_R then we observe that also u^{sc}_{pert} satisfies the open radiation condition because u and $u^{in} + u^{sc}_{unpert}$ do, the latter because it is the total field corresponding to the unperturbed problem.*

We proceed with the scattering problem for plane waves.

Theorem 4.3 (Well-posedness for plane waves)**.** *Let $\alpha := k\sin\theta$ be not a propagative wavenumber (see Definition 2.1 (ii)). Then the perturbed scattering problem for a plane wave incidence $u^{in}(x) = e^{ikx\cdot\hat{\theta}}$ admits a unique solution $u = u^{in} + u^{sc} \in H^1_{loc,0}(\tilde{D})$ such that the scattered part u^{sc} has a decomposition in the form $u^{sc} = u^{sc}_{unpert} + u^{sc}_{pert}$ in the region Σ_R where $u^{sc}_{unpert} \in H^1_{\alpha,loc}(D)$ is the scattered field corresponding to the unperturbed problem that satisfies the upward Rayleigh expansion* (2b) *with the quasi-periodic parameter $\alpha = k\sin\theta$. The part $u^{sc}_{pert} \in H^1_{loc}(\Sigma_R)$ fulfills the open waveguide radiation conditions defined by Def. 2.5 and Def. 2.10.*

Proof. In the unperturbed case, uniqueness and existence of the field $u^{sc}_{unpert} \in H^1_{\alpha,loc}(D)$ can be justified using standard variational arguments in the truncated periodic cell Q_h (for some $h > h_0$) by enforcing the $\alpha-$quasi-periodic DtN mapping on the artificial boundary Γ_h. Uniqueness follows from the assumption that $\alpha = k\sin\theta$ is not a propagative wavenumber, and existence is a consequence of the Fredholm alternative.

Set $\tilde{u}^{in} = u^{in} + u^{sc}_{unpert} \in H^1_{\alpha,0}(D)$. This field is well defined in Σ_R. We make the ansatz for the perturbed problem in the form $u = \tilde{u}^{in} + u^{sc}_{pert}$ in Σ_R and $u = u^{in} + u^{sc}$ in $\tilde{D}_R$. Since $u^{sc}_{pert} \in H^1_{loc}(\Sigma_R)$ is required to fulfill the open waveguide radiation condition and $u^{sc}_{pert} = 0$ on $\Gamma \setminus \tilde{D}_R$, it has to satisfy

$$\partial_\nu u^{sc}_{pert}\big|_+ \;=\; \Lambda(u^{sc}_{pert}|_+)\,, \quad \text{thus} \quad \partial_\nu u\big|_- \;=\; \partial_\nu \tilde{u}^{in}\big|_+ \;+\; \Lambda(u|_- - \tilde{u}^{in}|_+) \quad \text{on } C_R$$

where $|_+$ and $|_-$ denote the traces from Σ_R and $\tilde{D}_R$, respectively. Therefore, we have to determine the total field $u \in Y_R$ such that

$$\int\limits_{\tilde{D}_R} \nabla u \cdot \overline{\nabla\psi} - k^2 u\overline{\psi}\,dx \;-\; \int\limits_{C_R} \Lambda u\,\overline{\psi} ds \;=\; \int\limits_{C_R} [\partial_\nu \tilde{u}^{in} - \Lambda\tilde{u}^{in}]\,\overline{\psi}\,ds \tag{34}$$

for all $\psi \in Y_R$. Application of Lemma 3.9, Theorem 2.16 and the Fredholm alternative yields the uniqueness and existence of $u \in Y_R$. This also gives the scattered field $u^{sc} = u - u^{in} \in H^1(\tilde{D}_R)$ and the trace of the perturbed scattered field $g := u^{sc}_{pert}|_{C_R} = (u^{sc} - u^{sc}_{unpert})|_-$ on C_R. Finally, u^{sc}_{pert} can be extended to Σ_R by solving the problem of Theorem 3.2 with $\varphi = S^{-1}(g)$. □

Remark 4.4. *Suppose in Theorem 4.3 that $k\sin\theta = \hat{\alpha}_j$ is a propagative wavenumber for some fixed $j \in J$. Then it is well known that there exists still a $\hat{\alpha}_j-$quasi-periodic solution $u_{unpert,0} = u^{in} + u^{sc}_{unpert}$ of the unperturbed problem. However, the solution is not unique, and the general solution is given by*

$$u_{unpert} \;=\; u_{unpert,0} \;+\; \sum_{\ell=1}^{m_j} c_\ell\,\hat{\phi}_{\ell,j} \quad \textit{in } D \tag{35}$$

where $\hat{\phi}_{\ell,j} \in X_j$ (see (3) and (5)) and $c_\ell \in \mathbb{C}$ are arbitrary. In our paper [12] *we derive a new radiation condition based on the limiting absorption principe to prove uniqueness of the unperturbed scattering problem, even if $k \sin\theta$ is a propagative wavenumber.*

Now we consider the case that $u^{in} = \hat{\phi}_{\ell,j}$ for some $\ell \in \{1, \ldots, m_j\}$ and $j \in J$ is an incoming surface wave corresponding to the propagative wavenumber $\hat{\alpha}_j$; that is,

$$\begin{cases} \Delta u^{in} + k^2 u^{in} = 0 \quad \text{in} \quad D, \quad u^{in} = 0 \quad \text{on} \quad \Gamma, \\ u^{in} \text{ is } \hat{\alpha}_j\text{-quasi-periodic in } x_1 \text{ and exponentially decays in the } x_2\text{-direction.} \end{cases}$$

Since u^{in} vanishes already on Γ and satisfies the radiation condition we conclude that the variational formulation for $u \in Y_R$ takes the same form as in (34) with $\tilde{u}^{in} = u^{in}$. Analogously to the proof of Theorem 4.3, we obtain

Theorem 4.5 (Well-posedness for incoming surface waves)**.** *Given an incoming surface wave $u^{in} = \hat{\phi}_{\ell,j}$ for some $\ell \in \{1, \ldots, m_j\}$ and $j \in J$, the perturbed scattering problem admits a unique solution $u = u^{in} + u^{sc} \in H^1_{loc,0}(\tilde{D})$ such that $u^{sc} \in H^1_{loc}(\tilde{D})$ fulfills the open waveguide radiation conditions of Def. 2.5 and Def. 2.10.*

By Theorem 4.5, each surface wave $\hat{\phi}_{\ell,j}$ produces a non-trivial scattered field to the locally defected problem. Combining Theorems 4.3, 4.5 and Remark 4.4, we can get a general solution for plane wave incidence when $k \sin\theta$ is a progagative wavenumber.

Corollary 4.6. *Let u^{in} be a plane wave and suppose that $k \sin\theta = \hat{\alpha}_j$ is a propagative wavenumber for some fixed $j \in J$. The general solution of the perturbed scattering problem for plane wave incidence takes the form*

$$u \; = \; u_{unpert,0} \; + \; u^{sc}_{pert} \; + \; \sum_{\ell=1}^{m_j} c_\ell \, \hat{\phi}_{\ell,j} \; + \; \sum_{\ell=1}^{m_j} c_\ell \, u^{sc}_\ell \quad \text{in } \Sigma_R \,. \tag{36}$$

Here, u^{sc}_{pert} is the open waveguide radiation solution determined in Theorem 4.3 excited by the incoming reference wave $\tilde{u}^{in} = u_{unpert,0} = u^{in} + u^{sc}_{unpert}$, and u^{sc}_ℓ is the scattered field specified in Theorem 4.5 with $u^{in} := \hat{\phi}_{\ell,j}$.

5. Scattering by Neumann curves and by periodically arrayed obstacles

With slight changes our solvability results presented in Section 4 carry over to periodic and locally perturbed periodic curves of Neumann kind. Below we only remark the necessary modifications.

In the Neumann case, $\alpha \in [-1/2, 1/2]$ is called a *propagative wave number* if there exists a non-trivial $\phi \in H^1_{\alpha,loc}(D)$ such that

$$\Delta\phi + k^2\phi \; = \; 0 \text{ in } D \,, \quad \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \Gamma \,,$$

and ϕ satisfies the Rayleigh expansion (2b). Here ν denotes the normal direction at Γ pointing into D. Under the Assumption 2.2, one can still prove that there exist at most a finite number of propagative wavenumvers in the interval $[-1/2, 1/2]$. The finite dimensional eigenspace X_j can be defined similarly to (3) but with the Neumann boundary condition on Γ. The definition of the space $H^1_*(\Sigma_R)$ should be replaced by

$$H^1_*(\Sigma_R) \; := \; \left\{ u \in H^1_{loc}(\Sigma_R) : \partial_\nu u = 0 \text{ on } \Gamma \cap \partial\Sigma_R, \; u \in H^1(W_h \cap \Sigma_R) \text{ for all } h > h_0 \right\}.$$

In this case, a bound state of the perturbed scattering problem is defined as a solution $u \in H^1(\tilde{D})$ to the Helmholtz equation $(\Delta + k^2)u = 0$ in $\tilde{D}$ satisfying the Neumann boundary condition $\partial_\nu u = 0$ on $\tilde{\Gamma}$. Assuming that there are no bound states in $\tilde{D}$, one can prove uniqueness to the perturbed scattering problem analogously to Theorem 2.16. To construct the DtN operator, we consider the problem of determining $u \in H^1_{loc}(D)$ such that, for $\phi \in H_0^{-1/2}(C_R)$,

$$\Delta u + k^2 u = 0 \text{ in } D \setminus C_R\,, \quad \partial_\nu u = 0 \text{ on } \Gamma\,, \quad \frac{\partial u_-}{\partial \nu} - \frac{\partial u_+}{\partial \nu} = \varphi \text{ on } C_R\,, \tag{37}$$

and that u satisfies the open waveguide radiation condition in Σ_R. The variational form of this transmission problem is to determine $u \in H^1_{loc}(D)$ such that

$$\int_D [\nabla u \cdot \nabla \overline{\psi} - k^2 u \overline{\psi}]\, dx \;=\; \int_{C_R} \varphi\, \overline{\psi}\, ds \quad \text{for all } \psi \in H^1_c(D)\,, \tag{38}$$

together with the open waveguide radiation condition. Here,

$$H^1_c(D) \;:=\; \{\psi \in H^1(D) : \text{there exists } a > 0 \text{ with } \phi(x) = 0 \text{ for all } |x| > a\}.$$

Note that the right hand side of (38) is understood as the duality between $H_0^{-1/2}(C_R)$ and $H^{1/2}(C_R)$. The mapping $S\varphi = u|_{C_R}$ defines the single layer operator under the Neumann boundary condition. Choose the open arc C_R such that the mixed boundary value problem

$$(\Delta + k^2)u = 0 \text{ in } D_R\,, \quad u = 0 \text{ on } C_R\,, \quad \partial_\nu u = 0 \text{ on } \partial D_R \setminus C_R\,,$$

admits the trivial solution only. We make the assumption that every solution $u \in H^1(\Sigma_R)$ to the exterior boundary value problem

$$(\Delta + k^2)u = 0 \text{ in } \Sigma_R\,, \quad u = 0 \text{ on } C_R\,, \quad \partial_\nu u = 0 \text{ on } \Gamma \cap \partial\Sigma_R\,,$$

must vanish identically, that is, there are no bound states to this special perturbation problem. The previous two conditions ensure that the single layer operator $S : H_0^{-1/2}(C_R) \to H^{1/2}(C_R)$ is injective and boundedly invertible. The DtN operator Λ from $H^{1/2}(C_R)$ into $H_0^{-1/2}(C_R)$ takes the explicit form

$$\langle \Lambda g, \psi \rangle \;=\; -\int_{D_a \setminus D_R} [\nabla u \cdot \nabla \overline{\tilde{\psi}} - k^2 u\, \overline{\tilde{\psi}}]\, dx\,, \quad \psi \in H^{1/2}(C_R)\,,$$

where $\tilde{\psi} = E\psi$ is a bounded extension operator from $H^{1/2}(C_R)$ to $H^1_0(D_a \setminus D_R)$ for some $a > R$. Here u is the single layer potential with density $\varphi := S^{-1} g \in H_0^{-1/2}(C_R)$; that is, the open waveguide radiation solution to the boundary value problem (37). Mapping properties of Λ can be proved in the same way as Lemma 3.9. Finally, well-posedness results for scattering of point source waves, plane waves and surface waves from locally perturbed Neumann curves can be verified in the same manner as in the proofs of Theorems 4.1, 4.3 and 4.5,

Let us now consider the TE and TM polarizations of time-harmonic electromagnetic scattering by periodically arrayed obstacles. Define the boundary conditions $\mathcal{B}u := u$ in the TE case and $\mathcal{B}u := \partial_\nu u$ in the TM case. Let $\Omega \subset \mathbb{R} \times (-H, H)$ be a domain

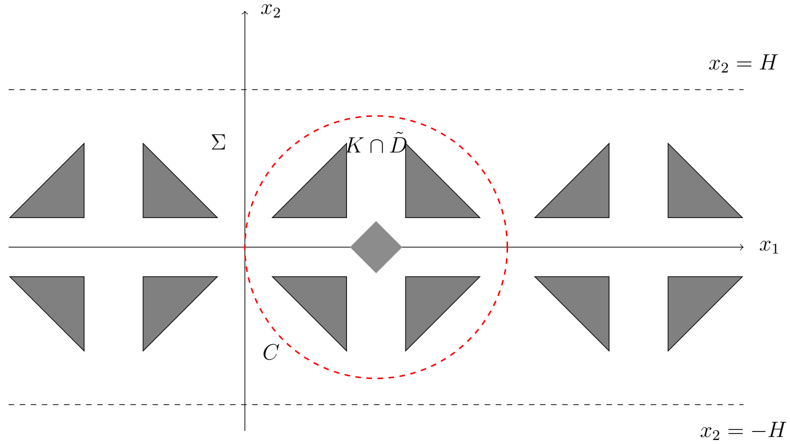


FIGURE 3. Illustration of the artificial boundary $C := \partial K \subset D$ (in this case a circle) on which the DtN operator Λ (see Definition 3.7) is defined for scattering by periodically arrayed obstacles with a local defect.

which is $2\pi-$periodic with respect to x_1 such that the exterior $D := \mathbb{R}^2 \setminus \overline{\Omega}$ is connected. Then $\alpha \in [-1/2, 1/2]$ is called a *propagative wave number* if there exists a non-trivial $\phi \in H^1_{\alpha,loc}(D)$ such that

$$\Delta\phi + k^2\phi \;=\; 0 \text{ in } D\,, \quad \mathcal{B}\phi = 0 \text{ on } \partial D\,,$$

and ϕ satisfies the Rayleigh expansions

$$\phi(x) \;=\; \sum_{\ell\in\mathbb{Z}} \phi_\ell^\pm \, e^{i(\ell+\alpha)x_1}\, e^{\pm i\sqrt{k^2-(\ell+\alpha)^2}(x_2\mp H)} \quad \text{for } x_2 \gtrless \pm H$$

for some $\phi_\ell^\pm \in \mathbb{C}$. Then the spaces X_j of modes and their basis $\{\hat{\phi}_{\ell,j} : \ell = 1, \dots, m_j\}$ are defined as in (3)–(5). Furthermore, let Ω be locally defected such that the periodic domain D is replaced by a perturbed connected domain $\tilde{D}$. We assume that there exists a bounded Lipschitz domain K which contains the defect $(D\setminus\tilde{D})\cup(\tilde{D}\setminus D)$ and such that $C := \partial K$ is contained in D. Defining $\Sigma := D \setminus \overline{K}$ and the Sobolev space

$$H^1_*(\Sigma) \;:=\; \big\{u \in H^1_{loc}(\Sigma) : \mathcal{B}u = 0 \text{ on } \partial D \cap \partial\Sigma\,,\; u \in H^1(W_h \cap \Sigma) \text{ for all } h > H\big\}\,,$$

where now $W_h := \mathbb{R} \times (-h, h)$. Then the radiation conditions of Definitions 2.5, 2.8, and 2.12 carry over. C, K, and Σ correspond to C_R, D_R, and Σ_R, respectively. A situation where C can be chosen as a circle C_R is sketched in Figure 3. The Dirichlet-to-Neumann operator Λ is again defined as $\Lambda g = \partial_\nu u|_C$ where $u \in H^1_{loc}(\Sigma)$ is the unique solution of

$$\Delta u + k^2 u = 0 \text{ in } \Sigma\,, \quad u = g \text{ on } C\,, \quad \mathcal{B}u = 0 \text{ on } \partial D \cap \partial\Sigma\,, \tag{40}$$

together with the open waveguide radiation condition. We remark that the domain and range space of Λ relies on the boundary condition under consideration. With proper assumptions on the domain K, one can construct an invertible single layer operator $S\varphi = u|_C$, where $u \in H^1_{loc}(D)$ is the radiating solution of the transmission problem

$$\Delta u + k^2 u = 0 \text{ in } D \setminus C\,, \quad \mathcal{B}u = 0 \text{ on } \partial D\,, \quad \frac{\partial u_-}{\partial\nu} - \frac{\partial u_+}{\partial\nu} = \varphi \text{ on } C\,.$$

Then one can define the DtN operator via Green's formula, analogously to the scattering by Dirichlet and Neumann curves. The well-posedness results of Section 4 can be justified in the same manner.

Remark 5.1. *Exact boundary conditions (DtN maps) were also constructed for wave propagating in a closed periodic waveguide* [13] *and in a photonic crystal* [7] *containing a local perturbation. In comparision with* [7]*, the DtN map defined by* (40) *applies to artificial boundary curves of arbitrary shape (although circular curves are used in this paper) and the medium is periodic in one direction only. The exact boundary condition of* [7] *is defined along square-shaped artificial boundaires, and the medium is periodic in two directions. In this paper the DtN map relies heavily on the open waveguide radiation condition of Def. 2.5.*

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