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The factorization method for inverse obstacle scattering with conductive boundary condition

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Abstract
The inverse acoustic scattering by a penetrable obstacle with a general conductive boundary condition is considered. Having established the well posedness of the direct problem by a variational method, we study the factorization method for recovering the location and the shape of the obstacle. One by-product of the method is an explicit proof of uniqueness of the inverse scattering problem under certain assumptions. Some numerical experiments are also presented to demonstrate the feasibility and effectiveness of the factorization method.

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper, we are concerned with an inverse scattering problem which is derived as the TM-mode from the time-harmonic Maxwell system, where the scattering medium is covered by a thin layer with very high conductivity. This model leads to the so-called conductive transmission conditions, which can be regarded as a generalization of the well-known transmission boundary condition. The case of an impenetrable obstacle covered by a thin layer leads to an impedance boundary condition. Recently, in [3], there has been published a result on inverse scattering for a generalized impedance boundary condition, where the surface impedance involves a second-order surface operator, which provides a better model for complicated surface materials. The conductive boundary condition has been known for a long time in the study of electromagnetic induction in the earth ([30, 31] provide the physical explanation of the condition). We refer to [1] for the derivation of the conductive boundary condition for the full Maxwell system, where an inhomogeneity is covered by an infinitely thin (the electric field would not penetrate into an ideal conductor of positive thickness) highly conducting layer. The attention to such inverse problems is stipulated among others by the

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recent interest in using inflatable objects, covered, for example, by a thin aluminum sheet as a decoy. For the inhomogeneity represented by a constant refractive index, the direct and inverse scattering problems have been studied in [2, 14, 15]. In this work, we assume the refractive index to be an $L^\infty(D)$ function, where D represents the inhomogeneity.

For the TM-mode and an obstacle being an infinite cylinder, we arrive at the scalar Helmholtz equation with corresponding boundary conditions in $\mathbb{R}^2$. For convenience, we will treat the problem in $\mathbb{R}^3$. However, all of the results hold also for the two-dimensional case with possibly different constants, because we do not use any particular feature of $\mathbb{R}^3$.

Let $u^i = \mathrm{e}^{\mathrm{i}kx\cdot d}$ be the incident plane wave which is described by the positive wave number $k = \omega/c$ with frequency ω, sound speed c and incident direction $d \in S^2$. Here and in the following, $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ denotes the unit sphere in $\mathbb{R}^3$. Then, the scattering problem for an inhomogeneous medium with a thin layer of high conductivity is to find the total field u such that

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^3\backslash\overline{D}, \tag{1.1}$$

$$\Delta u + k^2(1+q)u = 0 \qquad \text{in } D, \tag{1.2}$$

and the conductive boundary condition

$$u_+ - u_- = 0, \quad \frac{\partial u_+}{\partial \nu} - \lambda \frac{\partial u_-}{\partial \nu} + \mu u_+ = 0 \qquad \text{on } \partial D, \tag{1.3}$$

where ν is the unit outward normal to the boundary ∂D. Here, $u_\pm$ and $\partial u_\pm/\partial \nu$ denote the limit of u and $\partial u/\partial \nu$ from the exterior (+) and interior (−), respectively. The assumptions on D, q (or n), λ and μ will be given in the following sections. Furthermore, the scattered field $u^s = u - u^i$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0, \qquad r = |x|, \tag{1.4}$$

which holds uniformly with respect to $\hat{x} = x/|x| \in S^2$.

Roughly speaking, the Sommerfeld radiation condition (1.4) imposes that u^s behave as a spherical wave propagating away from the obstacle D. More precisely, it is known (see [11]) that u^s has the asymptotic behavior of an outgoing spherical wave

$$u(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|}\left\{u^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\} \text{ as } |x| \to \infty \tag{1.5}$$

uniformly in all directions $\hat{x} := x/|x|$, where the function $u^\infty(\hat{x})$ defined on the unit sphere S^2 is known as the far-field pattern with $\hat{x}$ denoting the observation direction.

Let $u^\infty(\hat{x}, d)$ be the far-field pattern corresponding to the observation direction $\hat{x}$ and the incident direction d. The *inverse problem* we consider is to determine D from knowledge of the far-field patterns $u^\infty(\hat{x}, d)$ for all $\hat{x}, d \in S^2$.

In this paper, we will study the applicability of the factorization method (FM) for the problem above. The FM has been introduced by Kirsch in [16] for scattering by impenetrable sound-soft or sound-hard obstacles. The essential idea of the FM is to decide for a given sampling point $z \in \mathbb{R}^3$ whether or not the equation

$$\widetilde{F}g = \mathrm{e}^{-\mathrm{i}kz\cdot\hat{x}}, \qquad \hat{x} \in S^2,$$

is solvable in $L^2(S^2)$, which in turn is equivalent to whether or not the given point z belongs to D. Here, $\widetilde{F}$ is a self-adjoint operator which can be computed from the far-field operator $F : L^2(S^2) \to L^2(S^2)$ given by

$$(Fg)(\hat{x}) = \int_{S^2} u^\infty(\hat{x}, d)g(d)\,\mathrm{d}s(d) \qquad \text{for } \hat{x} \in S^2. \tag{1.6}$$

The FM has been already employed for some special cases of (1.1)–(1.4). The reader can consult the papers by Kirsch [17, 18] for $\lambda = 1$ and $\mu = 0$, by Kirsch and Liu [24] for $\lambda \neq 1$ and $\mu = 0$ and by Kirsch and Kleefeld [23] for $q = 0$, $\lambda = 1$ and $\mu = ic$ where $c \geqslant 0$. In this paper, we will study the FM for the general case under assumption 2.1.

For the case of constant $q \in \mathbb{C}$, it is known that the inverse problem admits at most one solution [12, 13], i.e., the location and shape of the obstacle D can be uniquely recovered from knowledge of the far-field pattern $u^\infty(\hat{x}, d)$ for all $\hat{x}, d \in S^2$. In fact, this also follows as a corollary of theorems 3.5 and 3.14 below, which thus provides a novel proof of this uniqueness result.

This paper is organized as follows. In the next section, we will study the well posedness (existence, uniqueness and stability) of the direct problem by a variational method. Section 3 is devoted to a study of the FM for recovering the shape and location of the penetrable obstacle D. Some numerical simulations in two dimensions will be presented in section 4 to justify the validity of our method.

2. The direct scattering problem

This section is devoted to the solution of the direct acoustic scattering problem (1.1)–(1.4). Let D denote a bounded Lipschitz domain and $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$. Define the Sobolev spaces

$$H^1(D) := \{u : u \in L^2(D), |\nabla u| \in L^2(D)\},$$
$$H^1_{loc}(\mathbb{R}^3 \backslash \overline{D}) := \{u : u \in H^1(B_R \backslash \overline{D}) \text{ for every } R > 0, \text{ such that } \overline{D} \subset B_R\}.$$

Furthermore, $H^{1/2}(\partial D)$ is the trace space of $H^1(D)$ and $H^{-1/2}(\partial D)$ is the dual space of $H^{\frac{1}{2}}(\partial D)$.

Let $n \in L^\infty(\mathbb{R}^3)$, $n := 1 + q$, denote the index of refraction. Here and throughout this paper, we make the following general assumptions on D, q (or n), λ and μ.

Assumption 2.1.

(a) Let $D \subset \mathbb{R}^3$ be bounded domains, such that the exterior $D^e := \mathbb{R}^3 \backslash \overline{D}$ of $\overline{D}$ is connected. Assume that the boundary ∂D is smooth enough, such that both the embedding of $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$ and the embedding of $H^1(D)$ into $L^2(D)$ are compact.

(b) The contrast function $q \in L^\infty(D)$ satisfies $\Re(1+q) > 0$ and $\Im(q) \geqslant 0$ almost everywhere (a.e.) in D.

(c) λ is a non-zero complex constant, such that

(1) there exists $\hat{c} > 0$, such that $\Re(\lambda) \geqslant \hat{c}$,
(2) $\Im(\lambda) \leqslant 0$,
(3) $\Im(\lambda n) \geqslant 0$ a.e. in D.

(d) $\mu \in L^\infty(\partial D)$ with $\Re(\mu) \leqslant 0$ and $\Im(\mu) \geqslant 0$ a.e. on ∂D.
We extend q by zero (or n by $n = 1$) outside of D.

Since the incident field u^i satisfies the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in all of $\mathbb{R}^3$, the scattered field u^s solves the following problem with the boundary data $f_2 = (\lambda - 1)\frac{\partial u^i}{\partial \nu} - \mu u^i$ on ∂D and the source term $f = k^2 q u^i$ in D:

$$\Delta u^s + k^2 u^s = 0 \qquad \text{in } \mathbb{R}^3 \backslash \overline{D}, \tag{2.1}$$

$$\Delta u^s + k^2 n u^s = -f \qquad \text{in } D, \tag{2.2}$$

$$u_+^s - u_-^s = 0, \qquad \frac{\partial u_+^s}{\partial \nu} - \lambda \frac{\partial u_-^s}{\partial \nu} + \mu u_+^s = f_2 \qquad \text{on } \partial D, \tag{2.3}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0 \qquad \text{uniformly in all directions x/r.} \tag{2.4}$$

For $f \in L^2(D)$ and $f_2 \in H^{-1/2}(\partial D)$, the solution $u^s \in H^1_{loc}(\mathbb{R}^3)$ of (2.1)–(2.3) has to be understood in the weak sense; that is,

$$\iint_{\mathbb{R}^3\backslash\overline{D}} [\nabla u^s \cdot \nabla\overline{\varphi} - k^2 u^s \overline{\varphi}]\,\mathrm{d}x + \lambda \iint_D [\nabla u^s \cdot \nabla\overline{\varphi} - k^2 n u^s \overline{\varphi}]\,\mathrm{d}x - \int_{\partial D} \mu u^s \overline{\varphi}\,\mathrm{d}s$$
$$= \lambda \iint_D f\overline{\varphi}\,\mathrm{d}x - \int_{\partial D} f_2 \overline{\varphi}\,\mathrm{d}s \tag{2.5}$$

for any test function $\varphi \in H^1(\mathbb{R}^3)$ with compact support. A well-known regularity result for elliptic differential equations [28] yields that u^s is even analytic in D^e. In particular, radiation condition (2.4) makes sense.

Theorem 2.2. *For any $f \in L^2(D)$ and $f_2 \in H^{-1/2}(\partial D)$, there exists at most one solution $v \in H^1_{loc}(\mathbb{R}^3)$ of (2.1)–(2.4), or, equivalently (2.5), (2.4) has a unique solution in $H^1_{loc}(\mathbb{R}^3)$.*

Proof. Let v be the difference of two solutions. Then v solves (2.5), (2.4) with $f_2 = 0$ on ∂D and $f = 0$ in D. To prove the uniqueness, we show that v vanishes in all of $\mathbb{R}^3$.

Choose a ball B_R centered at the origin big enough such that $\overline{D} \subset B_R$. Let $\phi \in C^\infty(\mathbb{R}^3)$, such that $\phi(x) = 1$ for $|x| < R$ and $\phi(x) = 0$ for $|x| \geqslant R+1$. Then, setting $\varphi = \phi v$, we obtain for (2.5)

$$\iint_{R\leqslant x\leqslant R+1} [\nabla v \cdot \nabla\overline{\varphi} - k^2 v\overline{\varphi}]\,\mathrm{d}x + \iint_{B_R\backslash\overline{D}} [|\nabla v|^2 - k^2|v|^2]\,\mathrm{d}x$$
$$+ \lambda \iint_D [|\nabla v|^2 - k^2 n|v|^2]\,\mathrm{d}x - \int_{\partial D} \mu|v|^2 \mathrm{d}s = 0.$$

From interior regularity results (2.4), v is analytic outside of B_R. Applying Green's first theorem to the first integral yields

$$-\int_{|x|=R} \frac{\partial v}{\partial \nu}\overline{v}\,\mathrm{d}s + \iint_{B_R\backslash\overline{D}} [|\nabla v|^2 - k^2|v|^2]\,\mathrm{d}x + \lambda \iint_D [|\nabla v|^2 - k^2 n|v|^2]\,\mathrm{d}x - \int_{\partial D} \mu|v|^2\,\mathrm{d}s = 0.$$

From this and the assumptions on μ, λ and n given in assumption 2.1, it follows that

$$\Im \int_{\partial B_R} v \frac{\partial \overline{v}}{\partial \nu}\,\mathrm{d}s \geqslant 0.$$

Theorem 2.12 in [11] now shows $v = 0$ in $\mathbb{R}^3\backslash B_R$. By analytic continuation $v = 0$ in D^e and thus the trace of v also vanishes on ∂D. Thus, $v \in H^1(\mathbb{R}^3)$ is a weak solution of $\Delta v + k^2 n v = 0$ in $\mathbb{R}^3$ and is identically zero outside some ball. The unique continuation principle (see e.g., theorem 6.4 in [21]) implies that v vanishes in all of $\mathbb{R}^3$. This completes the proof. □

Next, using variational techniques, we prove existence for (2.1)–(2.4) and show that the solution depends continuously on the data f and f_2. We write the problem (2.1)–(2.4) as an equivalent problem in a bounded domain. For this, we introduce the Dirichlet-to-Neumann operator (DtN)

$$\Lambda : v \mapsto \frac{\partial \tilde{v}}{\partial \nu} \text{ on } \partial B_R,$$

mapping v to $\partial\tilde{v}/\partial\nu$, where $\tilde{v}$ solves the exterior Dirichlet problem for the Helmholtz equation $\Delta\tilde{v} + k^2\tilde{v} = 0$ in $\mathbb{R}^3\backslash B_R$ with the Dirichlet boundary data $\tilde{v}|_{\partial B_R} = v$. Again here, B_R is a ball of radius R, such that $\overline{D} \subset B_R$. We will need the following important property of the DtN operator (see e.g. [11, pp. 116–7] for details).

Lemma 2.3. *The DtN operator Λ is a bounded linear operator from $H^{1/2}(\partial B_R)$ to $H^{-1/2}(\partial B_R)$. Furthermore, there exists a bounded operator $\Lambda_0 : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ satisfying that*

$$-\int_{\partial B_R} \Lambda_0 w\overline{w}\,\mathrm{d}s \geqslant c\|w\|^2_{H^{1/2}(\partial B_R)}$$

for some constant $c > 0$, such that $\Lambda - \Lambda_0 : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is compact.

We now reformulate problem (2.1)–(2.3) as follows: given $f \in L^2(D)$ and $f_2 \in H^{-1/2}(\partial D)$, find $v \in H^1(B_R)$ satisfying (2.1)–(2.3) and the boundary condition

$$\frac{\partial v}{\partial \nu} = \Lambda v \qquad \text{on } \partial B_R. \tag{2.6}$$

Again, problem (2.1)–(2.3), (2.6) has to be understood in the weak sense; that is, $v \in H^1(B_R)$ solves

$$\iint_{B_R\setminus\overline{D}}[\nabla v\cdot\nabla\overline{\varphi} - k^2 v\overline{\varphi}]\,\mathrm{d}x - \int_{\partial B_R}\Lambda v\overline{\varphi}\,\mathrm{d}s + \lambda\iint_D[\nabla v\cdot\nabla\overline{\varphi} - k^2 nv\overline{\varphi}]\,\mathrm{d}x$$
$$-\int_{\partial D}\mu v\overline{\varphi}\,\mathrm{d}s = \lambda\iint_D f\overline{\varphi}\,\mathrm{d}x - \int_{\partial D} f_2\overline{\varphi}\,\mathrm{d}s. \tag{2.7}$$

In exactly the same way as in the proof of lemma 5.22 in [4], one can show that a solution v to problem (2.1)–(2.3) and (2.6) can be extended to a solution to the scattering problem (2.1)–(2.4) and conversely for a solution v to the scattering problem (2.1)–(2.4), v, restricted to B_R, solves problem (2.1)–(2.3) and (2.6). Therefore, by theorem 2.2, the problem (2.1)–(2.3) and (2.6) has at most one solution. We now have the following result on the well posedness of the problem (2.1)–(2.3) and (2.6).

Theorem 2.4. *Let $f \in L^2(D)$ and $f_2 \in H^{-1/2}(\partial D)$. Then problem (2.7) has a unique solution $v \in H^1(B_R)$. Furthermore,*

$$\|v\|_{H^1(B_R)} \leqslant C(\|f_2\|_{H^{-1/2}(\partial D)} + \|f\|_{L^2(D)}) \tag{2.8}$$

with a positive constant C independent of f and f_2.

Proof. We write (2.7) as

$$a(v,\varphi) = b(\varphi) \quad \forall\varphi \in H^1(B_R), \tag{2.9}$$

with

$$a(v,\varphi) = \iint_{B_R\setminus\overline{D}}[\nabla v\cdot\nabla\overline{\varphi} - k^2 v\overline{\varphi}]\,\mathrm{d}x - \int_{\partial B_R}\Lambda v\overline{\varphi}\,\mathrm{d}s$$
$$+\lambda\iint_D[\nabla v\cdot\nabla\overline{\varphi} - k^2 nv\overline{\varphi}]\,\mathrm{d}x - \int_{\partial D}\mu v\overline{\varphi}\,\mathrm{d}s,$$

and

$$b(\varphi) = \lambda\iint_D f\overline{\varphi}\,\mathrm{d}x - \int_{\partial D} f_2\overline{\varphi}\,\mathrm{d}s.$$

We write $a = a_1 + a_2$, where

$$a_1(v,\varphi) = \iint_{B_R\setminus\overline{D}}[\nabla v\cdot\nabla\overline{\varphi} + v\overline{\varphi}]\,\mathrm{d}x - \int_{\partial B_R}\Lambda_0 v\overline{\varphi}\,\mathrm{d}s + \lambda\iint_D[\nabla v\cdot\nabla\overline{\varphi} + v\overline{\varphi}]\,\mathrm{d}x$$
$$-\int_{\partial D}\mu v\overline{\varphi}\,\mathrm{d}s$$

and

$$a_2(v,\varphi) = -\iint_{B_R\setminus\overline{D}}[1+k^2]v\overline{\varphi}\,\mathrm{d}x - \int_{\partial B_R}(\Lambda-\Lambda_0)u\overline{\varphi}\,\mathrm{d}s - \lambda\iint_{B_R\setminus\overline{D}}[1+k^2 n]v\overline{\varphi}\,\mathrm{d}x,$$

where Λ_0 is the operator defined in lemma 2.3. By the boundedness of Λ_0 and the trace theorem, a_1 is bounded. By the Riesz representation theorem, there exists a bounded linear operator $A_1 : H^1(B_R) \to H^1(B_R)$, such that

$$a_1(v, \varphi) = (A_1 v, \varphi) \quad \text{for all } \varphi \in H^1(B_R).$$

On the other hand, by assumption 2.1 and lemma 2.3, for all $v \in H^1(B_R)$,

$$\begin{aligned}\Re[a_1(v, v)] &= \|v\|^2_{H^1(B_R\setminus\overline{D})} - \int_{\partial B_R} \Lambda_0 v\overline{\varphi}\,\mathrm{d}s + \Re(\lambda)\|v\|^2_{H^1(D)} - \int_{\partial D} \Re(\mu) v\overline{\varphi}\,\mathrm{d}s\\ &\geqslant \|v\|^2_{H^1(B_R\setminus\overline{D})} + c\|v\|^2_{H^{\frac{1}{2}}(\partial B_R)} + \hat{c}\|v\|^2_{H^1(D)}\\ &\geqslant \min\{1, \hat{c}\}\|v\|^2_{H^1(B_R)},\end{aligned}$$

that is, a_1 is strictly coercive. The Lax–Milgram theorem (see theorem 13.26 in [26]) implies that the operator $A_1 : H^1(B_R) \to H^1(B_R)$ has a bounded inverse. By the Riesz representation theorem again, there exists a bounded linear operator $A_2 : H^1(B_R) \to H^1(B_R)$, such that

$$a_2(v, \varphi) = (A_2 v, \varphi) \quad \text{for all } \varphi \in H^1(B_R).$$

By the compactness of $\Lambda - \Lambda_0$ and Rellich's embedding theorem (that is, that the embedding of $H^1(B_R)$ into $L^2(B_R)$) is compact, it follows that A_2 is compact. By the Riesz representation theorem again, there is a function $\widetilde{v} \in H^1(B_R)$, such that

$$b(\varphi) = (\widetilde{v}, \varphi) \quad \text{for all } \varphi \in H^1(B_R).$$

Thus, the variational formulation (2.9) is equivalent to the problem

$$\text{find } v \in H^1(B_R) \text{ such that } A_1 v + A_2 v = \widetilde{v}, \tag{2.10}$$

where A_1 is bounded and strictly coercive and A_2 is compact. The Riesz–Fredholm theory and the uniqueness result (theorem 2.2) imply that problem (2.10) or, equivalently, problem (2.9) has a unique solution. Estimate (2.8) follows from the fact that $\|\widetilde{v}\|_{H^1(B_R)} = \|b\|$ is bounded by $\|f\|_{L^2(D)} + \|f_2\|_{H^{-1/2}(\partial D)}$. □

3. Factorization of the far-field operator

In this section, we turn to the inverse problem of recovering the shape of D from knowledge of the far-field patterns $u^\infty(\hat{x}, d)$ for all $\hat{x}, d \in S^2$.

To prove the applicability of the FM, we proceed in following steps:

- derive a factorization of the far-field operator of the form $F = GTG^*$,
- characterize D by test functions,
- link the test functions and the data operator F by the FM.

The main difficulty is to show the middle operator T satisfies the assumptions required by theorem 2.15 ([22]). For rigorous justification of the FM, we had to distinguish between two cases, $\lambda = 1$ and $\lambda \neq 1$ and derived different factorizations of F for each of them.

We first prove the FM for the case $\lambda = 1$. As in the case of scattering by an inhomogeneous medium [22], we assume that the contrast q is locally bounded below.

Assumption 3.1. *$|q|$ is locally bounded below; that is, for every compact subset $M \subset D$, there exists $c > 0$ (depending on M), such that*

$$|q(x)| \geqslant c \text{ for almost all } x \in M.$$

Furthermore, let $\Gamma \subset D$ be relatively open, such that $\mu \neq 0$ on Γ and $\mu = 0$ on $\partial D\backslash\Gamma$. We rewrite problem (2.1)–(2.4) as follows. Let $f \in L^2(D)$ and $f_2 \in H^{-1/2}(\partial D)$ be given. Find $u \in H^1_{loc}(\mathbb{R}^3)$, such that

$$\Delta u + k^2(1+q)u = -k^2\frac{q}{\sqrt{|q|}}f \quad \text{in } \mathbb{R}^3\backslash\partial D, \tag{3.1}$$

$$\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} + \mu u = -f_2 \text{ on } \Gamma, \qquad \frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} = 0 \text{ on } \partial D\backslash\Gamma, \tag{3.2}$$

$$u_+ = u_- \qquad \text{on } \partial D, \tag{3.3}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0 \quad \text{uniformly in all directions x/r.} \tag{3.4}$$

Define the data-to-pattern operator $G : L^2(D) \times H^{-1/2}(\Gamma) \to L^2(S^2)$ by

$$G : \begin{pmatrix} f \\ f_2 \end{pmatrix} \mapsto v^\infty,$$

where v^∞ is the far-field pattern of the solution to (3.1)–(3.4), and the operator $H : L^2(S^2) \to L^2(D) \times H^{-1/2}(\Gamma)$, $Hg = \begin{pmatrix} H_1 g \\ H_2 g \end{pmatrix}$, where $H_1 : L^2(S^2) \to L^2(D)$ and $H_2 : L^2(S^2) \to H^{-1/2}(\Gamma)$ are given by

$$(H_1\psi)(x) = \sqrt{|q(x)|}\int_{S^2}\psi(\theta)\,\mathrm{e}^{\mathrm{i}kx\cdot\theta}\,\mathrm{d}s(\theta), x \in D, \tag{3.5}$$

and

$$(H_2\varphi)(x) = \mu(x)\int_{S^2}\varphi(\theta)\,\mathrm{e}^{\mathrm{i}kx\cdot\theta}\,\mathrm{d}s(\theta), x \in \Gamma. \tag{3.6}$$

Then, by the superposition principle, $F = GH$.

The adjoint of H, $H^* : L^2(D) \times H^{1/2}(\Gamma) \to L^2(S^2)$ is given by

$$H^*\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}(\hat{x}) = \iint_D \varphi_1(y)\sqrt{|q(y)|}\,\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}y + \int_\Gamma \varphi_2(y)\overline{\mu(y)}\,\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\,\mathrm{d}s(y), \hat{x} \in S^2.$$

From the asymptotic behavior of the fundamental solution, it follows that $H^*(\varphi_1, \varphi_2)^\top$ is the far field w^∞ of the function w, which is the sum of the volume and the single-layer potentials

$$w(x) = \iint_D \varphi_1(y)\sqrt{|q(y)|}\Phi(x,y)\,\mathrm{d}y + \int_\Gamma \varphi_2(y)\overline{\mu(y)}\Phi(x,y)\,\mathrm{d}y, x \in \mathbb{R}^3\backslash\partial D.$$

By properties of the volume [20] and the single-layer potentials [28], $w \in H^1_{loc}(\mathbb{R}^3)$ is radiating and satisfies

$$\Delta w + k^2 w = -\varphi_1\sqrt{|q|} \text{ in } \mathbb{R}^3\backslash\partial D, \tag{3.7}$$

$$w_+ = w_- \qquad \text{on } \partial D, \tag{3.8}$$

$$\frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} = -\varphi_2\overline{\mu} \text{ on } \Gamma, \quad \frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} = 0 \text{ on } \partial D\backslash\Gamma \tag{3.9}$$

in the weak sense.

We also can write (3.7)–(3.9) as

$$\Delta w + k^2(1+q)w = -k^2\frac{q}{\sqrt{|q|}}\left(\frac{\overline{q}}{k^2|q|}\varphi_1 - \sqrt{|q|}w\right) \text{ in } \mathbb{R}^3\backslash\partial D, \tag{3.10}$$

$$w_+ = w_- \qquad \text{on } \partial D, \tag{3.11}$$

$$\frac{\partial w_+}{\partial\nu}-\frac{\partial w_-}{\partial\nu}+\mu w=-(\varphi_2\overline{\mu}-\mu w)\ \text{ on }\Gamma,\ \frac{\partial w_+}{\partial\nu}-\frac{\partial w_-}{\partial\nu}=0\text{ on }\partial D\backslash\Gamma. \tag{3.12}$$

Since, (3.7)–(3.9) is uniquely solvable, $H^*(\varphi_1,\varphi_2)^\top=w^\infty=G(\frac{\bar{q}\varphi_1}{k^2|q|}-\sqrt{|q|}w,\varphi_2\overline{\mu}-\mu w)^\top$ for all $(\varphi_1,\varphi_2)^\top\in L^2(D)\times H^{1/2}(\Gamma)$. We define an operator $T: L^2(D)\times H^{1/2}(\Gamma)\to L^2(D)\times H^{-1/2}(\Gamma)$ by

$$T\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\begin{pmatrix}\frac{\bar{q}\varphi_1}{k^2|q|}-\sqrt{|q|}w\\ \varphi_2\overline{\mu}-\mu w\end{pmatrix}.$$

Then $H^*=GT$ or $H=T^*G^*$. Thus, the far-field operator F can be represented as $F=GT^*G^*$.

The next theorem provides a link between D and the range of the data-to-pattern operator G.

Theorem 3.2. *For any $z\in\mathbb{R}^3$, define ϕ_z by*

$$\phi_z(\hat{x}):=\mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z},\ \hat{x}\in S^2. \tag{3.13}$$

Then,

$$z\in D\Longleftrightarrow\phi_z\in\mathscr{R}(G). \tag{3.14}$$

Proof. First we assume that $z\in D$. Let $B[z,e]\subset D$ be some closed ball centered at z with radius ε. Choose a cut-off function $\psi\in C^\infty(\mathbb{R})$ with $\psi(t)=1$ for $|t|>\varepsilon$ and $\psi(t)=0$ for $|t|\leqslant\varepsilon/2$. Define $v\in C^\infty(\mathbb{R}^3)$ by

$$v(x):=\psi(|x-z|)\frac{\mathrm{e}^{\mathrm{i}k|x-z|}}{4\pi|x-z|},\ x\in\mathbb{R}^3.$$

The far-field pattern of v is given by $v^\infty=\phi_z$. Also, v solves (3.1)–(3.4) with $f=-\Delta v+k^2(1+q)v$ and $f_2=-\mu v$. Therefore, $G(f,f_2)^\top=\phi_z$.

Let now $z\notin D$ and assume on the contrary that there exists $(f,f_2)^T\in H^{1/2}(\partial D)\times H^{-1/2}(\partial D)$, such that $G(f,f_2)^T=\phi_z$. Let u be the solution of (3.1)–(3.4) determined by f and f_1, and $u^\infty=G(f_1,f_2)^T$ be its far-field pattern. Since, ϕ_z is the far-field pattern of $\Phi(\cdot,z)$, by Rellich's lemma and analytic continuation $u(x)=\Phi(x,z)$ for all $x\in D^e\backslash\{z\}$. If $z\in D^e$, this contradicts the fact that u is analytic in D^e while $\Phi(\cdot,z)$ is singular at $x=z$. If $z\in\partial D$, we have that $u_+\in H^{1/2}(\partial D)$ by trace theorem. However, $\Phi(x,z)$ cannot be in $H^{1/2}(\partial D)$. We observe that $\nabla\Phi(x,z)=\mathcal{O}(1/|x-z|^2)$ as $x\to z$, and thus $\Phi(\cdot,z)$ is neither in $H^1(D)$ nor in $H^1_{loc}(D^e)$. The proof is finished. □

Let $\langle\cdot,\cdot\rangle$ denote the dual form

$$\left\langle\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle=(\psi_1,\varphi_1)_{L^2(D)}+\int_\Gamma\psi_2\overline{\varphi_2}\,\mathrm{d}s$$

for all $\binom{\psi_1}{\psi_2}\in L^2(D)\times H^{-1/2}(\Gamma)$ and $\binom{\varphi_1}{\varphi_2}\in L^2(D)\times H^{1/2}(\Gamma)$; the integral $\int_\Gamma\psi_2\overline{\varphi_2}\,\mathrm{d}s$ has to be understood as the dual form in the dual system $\langle H^{-1/2}(\Gamma),H^{1/2}(\Gamma)\rangle$.

In following, we collect properties of G and T in order to apply the functional analytic result first stated by Kirsch [16] and further refined by Lechleiter [27]. This result will enable us to express the range of the unknown operator G by the range of the far-field operator F. We make the following additional assumption on the contrast q.

Assumption 3.3. *There exist $t\in[0,\pi]$ and $c_0>0$, such that*

$$\Re[\mathrm{e}^{\mathrm{i}t}\overline{q(x)}]\geqslant c_0|q(x)|\text{a.e. in }D.$$

For a real-valued q, this condition is satisfied (with $c_0 = 1$ and $t = 0$ if q is positive or $t = \pi$ if q is negative). If q is complex valued and $\Re q \geqslant 0$, we can set $t = \pi/4$. For the case $\Re q \leqslant 0$, lemma 4.11 in [22] provides a sufficient condition for existence of $t \in [0, \pi]$.

Theorem 3.4.

(a) *G is compact with dense range.*

(b) *Let t and c_0 be such that the assumption 3.3 is satisfied. Then, the middle operator T has a decomposition of the form $T = C + K$, where K is compact and the real part $\Re[e^{it}C] : L^2(D) \times H^{1/2}(\Gamma) \to L^2(D) \times H^{-1/2}(\Gamma)$ is coercive, in particular,*

$$\Re\left[e^{it}\left\langle C\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle\right] \geqslant \frac{1}{2}\min\left\{\frac{c_0}{k^2}, 1\right\}\left\|\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\|^2_{L^2(D)\times H^{1/2}(\Gamma)} \tag{3.15}$$

$$\text{for all}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \in L^2(D) \times H^{1/2}(\Gamma). \tag{3.16}$$

(c) *For $\Im T$ holds*

$$\left\langle \Im T\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle \leqslant 0 \text{ for all } \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \in L^2(D) \times H^{1/2}(\Gamma).$$

(d) *T is one-to-one.*

Proof.

(a) The proof for the compactness of G is analogous to the arguments provided in the proof of lemma 1.13 in [22] and is omitted for brevity.

We consider the L^2-adjoint G^* of G and show that it is injective, which implies the denseness of the range of G. Recall that u is a solution of (3.1)–(3.4) with $f \in L^2(D)$ and $f_2 \in H^{-1/2}(\Gamma)$. Let v be a solution of the boundary value problem defined in (3.20)–(3.23) with $\lambda = 1$ and boundary data $(\overline{v_g}, \partial\overline{v_g}/\partial\nu + \mu\overline{v_g})$, where v_g is the trace of the Herglotz wave function; that is, a function given by

$$v_g(y) = \int_{S^2} e^{ik\hat{x}\cdot y} g(\hat{x})\, ds(\hat{x}), \qquad y \in \partial D.$$

Here and in the following, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Then, we obtain that the adjoint operator $G^* : L^2(S^2) \to L^2(D) \times H^{1/2}(\partial D)$ is given by

$$G^* g = \begin{pmatrix} v \\ v_- \end{pmatrix}. \tag{3.17}$$

Let, $f \in L^2(D)$, $f_2 \in H^{-1/2}(\Gamma)$ and $g \in L^2(S^2)$. Then,

$$\begin{aligned}(G(f_1, f_2)^T, g)_{L^2(S^2)} &= \int_{S^2} u^\infty(\hat{x})\overline{g(\hat{x})}\, ds(\hat{x})\\
&= \int_{S^2}\left(\int_{\partial D}\left[u_+(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial\nu(y)} - e^{-ik\hat{x}\cdot y}\frac{\partial u_+(y)}{\partial\nu}\right] ds(y)\right)\overline{g(\hat{x})}\, ds(d)\\
&= \int_{\partial D}\left[u_+(y)\frac{\partial\overline{v_g(y)}}{\partial\nu} - \overline{v_g(y)}\frac{\partial u_+(y)}{\partial\nu}\right] ds(y)\\
&= \int_{\partial D}\left[u_+(y)\left(\frac{\partial v_-(y)}{\partial\nu} - \frac{\partial v_+(y)}{\partial\nu} - \mu(y)v_-(y)\right)\right.\\
&\quad\left. -(v_-(y) - v_+(y))\frac{\partial u_+(y)}{\partial\nu}\right] ds(y)\end{aligned}$$

$$= \int_{\partial D}\left[u_+(y)\left(\frac{\partial v_-(y)}{\partial \nu} - \mu(y)v_-(y)\right) - v_-(y)\frac{\partial u_+(y)}{\partial \nu}\right] \mathrm{d}s(y)$$
$$= \int_{\partial D}\left[u_+(y)\left(\frac{\partial v_-(y)}{\partial \nu} - \mu(y)v_-(y)\right)\right.$$
$$\left. -v_-(y)\left(\frac{\partial u_-(y)}{\partial \nu} - \mu(y)u_+(y)\right)\right] \mathrm{d}s(y) + \int_\Gamma v_-(y)f_2(y)\,\mathrm{d}s(y)$$
$$= \int_{\partial D}\left[u_+(y)\frac{\partial v_-(y)}{\partial \nu} - v_-(y)\frac{\partial u_-(y)}{\partial \nu}\right] \mathrm{d}s(y) + \int_\Gamma v_-(y)f_2(y)\,\mathrm{d}s(y)$$
$$= \iint_D u(-k^2(1+q(x))v(x)) + v(x)(k^2(1+q(x))u(x))$$
$$+ v(x)f(x)\,\mathrm{d}x + \int_\Gamma v_-(y)f_2(y)\,\mathrm{d}s(y)$$
$$= \iint_D v(x)f(x)\,\mathrm{d}x + \int_\Gamma v_-(y)f_2(y)\,\mathrm{d}s(y),$$

where we have used the conductive boundary conditions (3.22) and (3.2) in the fourth and the sixth equality for v and u, respectively. The fifth equality holds because both u and v are radiating solutions in D^e. In the eighth equality, we have applied Green's theorem, (3.1) and (3.21). Thus, $G^*g = (\overline{v}, \overline{v_-})^\top$ for all $g \in L^2(S^2)$.

We proceed by showing that the adjoint operator G^* is injective. Let, $g \in L^2(S^2)$ be such that $G^*g = 0$, i.e., $(v|_D, v_-)^\top = (0,0)^\top$. Thus, $v_- = \partial v_-/\partial\nu = 0$ on ∂D and by boundary conditions (3.22), $v_+ = -\overline{v_g}$ and $\partial v_+/\partial \nu = -\partial\overline{v_g}/\partial\nu$ on ∂D. We define a function w by

$$w = \begin{cases} -\overline{v_g} & \text{in } \overline{D}, \\ v & \text{in } D^e. \end{cases} \tag{3.18}$$

Then, w is an entire solution of the Helmholtz equation satisfying the radiation condition. From this we conclude that $w = 0$ in $\mathbb{R}^3$ and thus $v_g = 0$ in D. By theorem 3.15 in [11], $g = 0$, which implies that G^* is injective.

(b) We decompose T into the sum $T = C + K$ with $C\binom{\varphi_1}{\varphi_2} = \binom{\frac{\overline{q}}{k^2|q|}\varphi_1}{\varphi_2}$ and $K\binom{\varphi_1}{\varphi_2} = -\binom{\sqrt{|q|}w}{(1-\overline{\mu})\varphi_2+\mu w}$. It has been shown in the previous section that the mapping $\varphi_1 \mapsto w|_D$ from $L^2(D)$ into $H^1(D)$ is bounded. The operator K is compact because of the compact embeddings of $H^1(D)$ into $L^2(D)$ and $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$. For $\Re[\mathrm{e}^{\mathrm{i}t}C]$, we have

$$\Re\left[\mathrm{e}^{\mathrm{i}t}\left\langle C\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle\right] = \frac{1}{k^2}\iint_D \Re\left[\mathrm{e}^{\mathrm{i}t}\frac{\overline{q}}{|q|}\right]|\varphi_1|^2\,\mathrm{d}x + \int_\Gamma |\varphi_2|^2 ds$$
$$\geqslant \frac{c_0}{k^2}\|\varphi_1\|^2_{L^2(D)} + \|\varphi_2\|^2_{H^{1/2}(\Gamma)}$$
$$\geqslant \frac{1}{2}\min\left\{\frac{c_0}{k^2}, 1\right\}\left\|\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\|^2_{L^2(D)\times H^{1/2}(\Gamma)}.$$

(c) Let $\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \in L^2(D)\times H^{1/2}(\Gamma)$. Then,

$$\left\langle T\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle = \iint_D\left(\frac{1}{k^2}\frac{\overline{q}}{|q|}\varphi_1 - \sqrt{|q|}w\right)\overline{\varphi_1}\,\mathrm{d}x + \int_\Gamma(\overline{\mu}\varphi_2 - \mu w)\overline{\varphi_2}\,\mathrm{d}s$$
$$= \iint_D \frac{1}{k^2}\frac{\overline{q}}{|q|}|\varphi_1|^2\,\mathrm{d}x + \int_\Gamma \overline{\mu}|\varphi_2|^2\,\mathrm{d}s - \iint_D\sqrt{|q|}w\overline{\varphi_1}\,\mathrm{d}x - \int_\Gamma \mu w\overline{\varphi_2}\,\mathrm{d}s.$$

Application of Green's theorem in D and in $\{x \notin D, |x| < R\}$ yields

$$\begin{aligned}\left\langle T\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle &= \iint_D \frac{1}{k^2}\frac{\overline{q}}{|q|}|\varphi_1|^2\,\mathrm{d}x + \int_\Gamma \overline{\mu}|\varphi_2|^2\,\mathrm{d}s + \iint_D [-|\nabla w|^2 + k^2|w|^2]\,\mathrm{d}x\\ &\quad + \int_\Gamma \overline{\frac{\partial w_-}{\partial \nu}} w\,\mathrm{d}s - \int_\Gamma \mu w\overline{\varphi_2}\,\mathrm{d}s(y)\\ &= \iint_D \frac{1}{k^2}\frac{\overline{q}}{|q|}|\varphi_1|^2\,\mathrm{d}x + \int_\Gamma \overline{\mu}|\varphi_2|^2\,\mathrm{d}s\\ &\quad + \iint_D [-|\nabla w|^2 + k^2|w|^2]\,\mathrm{d}x + \int_\Gamma \overline{\frac{\partial w_+}{\partial \nu}} w\,\mathrm{d}s\\ &= \iint_D \frac{1}{k^2}\frac{\overline{q}}{|q|}|\varphi_1|^2\,\mathrm{d}x + \int_\Gamma \overline{\mu}|\varphi_2|^2\,\mathrm{d}s\\ &\quad + \iint_{B_R} [-|\nabla w|^2 + k^2|w|^2]\,\mathrm{d}x + \int_{|x|=R} \overline{\frac{\partial w}{\partial \nu}} w\,\mathrm{d}s.\end{aligned}$$

Finally, by the radiation condition, the last term converges to $-\mathrm{i}k/(4\pi)^2\int_{S^2}|w^\infty|^2\,\mathrm{d}s$. Therefore, since by assumption 2.1 $\Im q \geqslant 0$ and $\Im\mu \geqslant 0$, we have

$$\Im\left\langle T\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right\rangle = \iint_D \frac{1}{k^2}\Im\frac{\overline{q}}{|q|}|\varphi_1|^2\,\mathrm{d}x + \int_\Gamma \Im\overline{\mu}|\varphi_2|^2\,\mathrm{d}s - \frac{k^2}{(4\pi)^2}\int_{S^2}|w^\infty|^2 \leqslant 0.$$

(d) Let $\binom{\varphi_1}{\varphi_2} \in L^2(D)\times H^{1/2}(\Gamma)$, such that $T\binom{\varphi_1}{\varphi_2} = \binom{0}{0}$. Then, (3.10)–(3.12) becomes

$$\begin{aligned}&\Delta w + k^2(1+q)w = 0 \text{ in } \mathbb{R}^3\backslash\partial D,\\ &w_+ = w_- \quad \text{on } \partial D,\\ &\frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} + \mu w = 0 \text{ on } \partial D.\end{aligned}$$

From the uniqueness of the solution to (2.1)–(2.4), it follows that w vanishes in all of $\mathbb{R}^3$. Thus, $\frac{k^2\overline{q}}{|q|}\varphi_1 = 0$ in D and $\overline{\mu}\varphi_2 = 0$ on Γ. Since $|q|$ is locally bounded below and $\mu \neq 0$ on Γ, $(\varphi_1, \varphi_2)^\top = (0, 0)$. □

Now we can state the first main result of this section.

Theorem 3.5. *Let the assumptions 2.1, 3.1 and 3.3 hold. For $z \in \mathbb{R}^3$, define $\phi_z \in L^2(S^2)$ by (3.13). Then,*

$$z \in D \Longleftrightarrow \phi_z \in \mathscr{R}\big(F_\sharp^{1/2}\big),$$

and consequently

$$z \in D \Longleftrightarrow \sum_{j=1}^\infty \frac{|\langle\phi_z, \psi_j\rangle_{L^2(S^2)}|^2}{|\lambda_j|} < \infty, \tag{3.19}$$

where $F_\sharp = |\Re[\mathrm{e}^{\mathrm{i}t}F]| + |\Im F|$ and (λ_j, ψ_j) is its eigensystem. In other words, the sign of the function

$$W(z) = \left[\sum_{j=1}^\infty \frac{|\langle\phi_z, \psi_j\rangle_{L^2(S^2)}|^2}{|\lambda_j|}\right]^{-1}$$

is the characteristic function of D.

We proceed with the case when λ is complex valued and $\lambda \neq 1$. Since u^i solves the Helmholtz equation in all of $\mathbb{R}^3$, a function $u := (u^i|_D + u^s|_D, u^s|_{D^e})$ solves the following boundary value problem with $f_1 = u^i$ and $f_2 = \frac{\partial u^i}{\partial \nu} + \mu u^i$. Given $f_1 \in H^{1/2}(\partial D)$ and $f_2 \in H^{-1/2}(\partial D)$ find $u|_D \in H^1(D), u|_{D^e} \in H^1_{loc}(D^e)$, such that

$$\Delta u + k^2 u = 0 \qquad \text{in } D^e, \tag{3.20}$$

$$\Delta u + k^2 n u = 0 \qquad \text{in } D, \tag{3.21}$$

$$u_+ - u_- = -f_1, \quad \frac{\partial u_+}{\partial \nu} - \lambda \frac{\partial u_-}{\partial \nu} + \mu u_+ = -f_2 \qquad \text{on } \partial D, \tag{3.22}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0 \qquad r = |x|. \tag{3.23}$$

The well posedness of (3.20)–(3.23) can be established following the same steps as in the proof of theorem 2.2 and theorem 2.4.

Now we derive a factorization of the far-field operator F. Define the data-to-pattern operator $G : H^{1/2}(\partial D) \times H^{-1/2}(\partial D) \to L^2(S^2)$ by

$$G\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = v^\infty,$$

where v^∞ is the far-field pattern of the solution to (3.20)–(3.23). Besides, we also define an auxiliary operator $H : L^2(S^2) \to H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ by

$$(Hg)(x) = \begin{pmatrix} \int_{S^2} \mathrm{e}^{\mathrm{i}kx\cdot d} g(d)\,\mathrm{d}s(d) \\ \int_{S^2} \left(\frac{\partial \mathrm{e}^{\mathrm{i}kx\cdot d}}{\partial \nu(x)} + \mu \mathrm{e}^{\mathrm{i}kx\cdot d}\right) g(d)\,\mathrm{d}s(d) \end{pmatrix}, \quad x \in \partial D. \tag{3.24}$$

From the superposition principle, we observe that

$$F = GH. \tag{3.25}$$

For the subsequent analysis, we need the following two assumptions regarding the wave number k.

Assumption 3.6.

- *k^2 is not a Dirichlet eigenvalue of the following boundary value problem:*

$$\Delta u + k^2 n u = 0 \quad \text{in } D, \tag{3.26}$$

$$u = f \quad \text{on } \partial D, \tag{3.27}$$

i.e., the only solution $u \in H^1(D)$ of (3.26),(3.27) for $f = 0$ is the trivial one.

- *k^2 is not a Neumann eigenvalue of the following boundary value problem:*

$$\Delta u + k^2 n u = 0 \quad \text{in } D, \tag{3.28}$$

$$\lambda \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial D, \tag{3.29}$$

i.e., the only solution $u \in H^1(D)$ of (3.28),(3.29) for $g = 0$ is the trivial one.

We make yet another assumption on the wave number k in order to ensure the injectivity of the far-field operator.

Theorem 3.7. *Assume that k^2 is not an eigenvalue of the following interior transmission problem in D, i.e., the only solution $(u, v) \in H^1(D) \times H^1(D)$ of*

$$\Delta u + k^2 n u = 0, \quad \Delta v + k^2 v = 0 \qquad \text{in } D, \tag{3.30}$$

$$u = v, \quad \lambda \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} + \mu v \qquad \text{on } \partial D \tag{3.31}$$

is the trivial one $u = v = 0$. Then, F is injective and its range $\mathscr{R}(F)$ is dense in $L^2(S^2)$.

Proof. The proof follows from the same arguments as in theorem 4.4 of [22] and is omitted here for the sake of brevity. □

Remark 3.8. With the help of Green's theorem and the unique continuation principle, one can show that there are no transmission eigenvalues if $\Im(\lambda) < 0$ or $\Im(\lambda n) > 0$ on some subset $D_0 \subset D$. We refer to Päivärinta and Sylvester [29], Kirsch [20], Cakoni and her collaborators [5–9] for further existence results.

With the two assumptions, we can define the interior DtN operator $T : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ by

$$Tf = \lambda \frac{\partial u}{\partial \nu} \quad \text{on } \partial D, \tag{3.32}$$

where u solves the interior Dirichlet boundary value problem (3.26)–(3.27) with boundary data $f \in H^{1/2}(\partial D)$ and the interior Neumann-to-Dirichlet operator $T^{-1} : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ by

$$T^{-1}g = u \quad \text{on } \partial D, \tag{3.33}$$

where u solves the interior Neumann boundary value problem (3.28)–(3.29) with boundary data $g \in H^{-1/2}(\partial D)$. We note that T and T^{-1} are bounded linear operators.

Properties of G are collected in the following lemma.

Theorem 3.9. *Assume that k^2 is not a Dirchlet eigenvalue of the boundary value problem (3.26)–(3.27).*

(a) The data-to-pattern operator $G : H^{1/2}(\partial D) \times H^{-1/2}(\partial D) \to L^2(S^2)$ is compact with dense range in $L^2(S^2)$.

(b) The kernel space of G is given by $\mathscr{N}(G) = \{(f, Tf)^T : f \in H^{1/2}(\partial D)\}$.

(c) For any $z \in \mathbb{R}^3$, define ϕ_z by

$$\phi_z(\hat{x}) := \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot z}, \hat{x} \in S^2. \tag{3.34}$$

Then,

$$z \in D \Longleftrightarrow \phi_z \in \mathscr{R}(G). \tag{3.35}$$

Proof.

(a) This follows by the similar arguments as in the case of $\lambda = 1$ (see theorem 3.4(a)).

(b) We show first that $(f, Tf) \subseteq N(G)$, i.e., that the far field of $u|_D \in H^1(D)$, $u|_{D^e} \in H^1_{loc}(D^e)$, satisfying (3.20)–(3.23) with $f_1 = f$ and $f_2 = Tf$ is identically zero.

Define $\tilde{u}|_D \in H^1(D)$, $\tilde{u}|_{D^e} \in H^1_{loc}(D^e)$ by

$$\tilde{u} = \begin{cases} v & \text{in } D, \\ 0 & \text{in } D^e, \end{cases} \tag{3.36}$$

with $v \in H^1(D)$ being the solution of

$$\Delta v + k^2 n v = 0 \quad \text{in } D,$$
$$v = f \quad \text{on } \partial D.$$

Then, $\lambda \partial \tilde{u}_- / \partial \nu = Tf$ and, since (3.20)–(3.23) is uniquely solvable, $\tilde{u} = u$ is the solution of (3.20)–(3.23) with $f_1 = f$ and $f_2 = Tf$. Thus, $G(f, Tf) = \tilde{u}^\infty = 0$.

It remains to prove that the set $\{(f, Tf)^T : f \in H^{1/2}(\partial D)\}$ forms the entire kernel of G. Let $(f, g)^T \in H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ be from $\mathscr{N}(G)$. By linearity, we have that $G(0, g - Tf)^T = 0$. This means that the far-field patterns of the solution u of the boundary value problem (3.20)–(3.23) with boundary data $(f_1, f_2) = (0, g - Tf)$ vanish, and therefore $u = 0$ in D^e by an application of Rellich's lemma and unique continuation. From this we conclude that $u_- = 0$ and $\lambda \partial u_-/\partial \nu = g - Tf$ on ∂D. Assumption 3.6 implies that $g - Tf = 0$.

(c) Let first $z \in D$ and define

$$v(x) := \Phi(x, z) = \frac{e^{ik|x-z|}}{4\pi |x - z|}, \quad x \in D^e.$$

The far-field pattern of v is given by $v^\infty = \phi_z$. Define $u|_D \in H^1(D)$, $u|_{D^e}$ in $H^1_{loc}(D^e)$ by

$$u = \begin{cases} 0 & \text{in } D, \\ v & \text{in } D^e. \end{cases} \tag{3.37}$$

Then, u solves (3.20)–(3.23) with $f_1 = -v_+$ and $f_2 = -\partial v_+/\partial \nu - \mu v_+$. Thus, since u and v coincide in the exterior of D, we have that $u^\infty = v^\infty = \phi_z$. Therefore, for chosen f_1 and f_2 holds $G(f_1, f_2) = \phi_z$.

The rest of the proof follows by the same arguments used in theorem 3.2. □

Define, for any $\varphi \in H^{-1/2}(\partial D)$, the single-layer potential $S_L\varphi$ by

$$(S_L\varphi)(x) := \int_{\partial D} \varphi(y)\Phi(x, y)\, ds(y), \quad x \in \mathbb{R}^3 \backslash \partial D, \tag{3.38}$$

and for any $\psi \in H^{1/2}(\partial D)$, the double-layer potential $K_L\varphi$ by

$$(K_L\psi)(x) := \int_{\partial D} \psi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, ds(y), \quad x \in \mathbb{R}^3 \backslash \partial D, \tag{3.39}$$

where Φ denotes the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$ and is given by

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi |x - y|}, \quad x, y \in \mathbb{R}^3, \ x \neq y.$$

We return now to the factorization of the far-field operator F. The adjoint operator $H^* : H^{-1/2}(\partial D) \times H^{1/2}(\partial D) \to L^2(S^2)$ of H (see 3.24) is given by

$$\left(H^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right)(\hat{x}) = \int_{\partial D} \left(e^{-ik\hat{x}\cdot y}\varphi(y) + \left(\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} + \overline{\mu(y)}\, e^{-ik\hat{x}\cdot y} \right) \psi(y) \right) dy, \quad \hat{x} \in S^2. \tag{3.40}$$

By the asymptotic behavior of the fundamental solution, $H^*(\varphi, \psi)^T$ is just the far-field pattern of

$$u(x) = (S_L\varphi)(x) + [(K_L + S_L\overline{\mu})\psi](x), \quad x \in D^e.$$

The theory of potentials yields (see e.g. [28]) that $u \in H^1_{loc}(D^e)$ is a radiating solution of the Helmholtz equation (3.20) with traces

$$u_+ = S\varphi + K\psi + S\overline{\mu}\psi + \frac{1}{2}\psi, \quad \frac{\partial u_+}{\partial \nu} = K'\varphi + N\psi + K'\overline{\mu}\psi - \frac{1}{2}\varphi - \frac{\overline{\mu}}{2}\psi \quad \text{on } \partial D. \tag{3.41}$$

where for any $\varphi \in H^{-1/2}(\partial D), \psi \in H^{1/2}(\partial D)$, the boundary integral operators $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$, $K : H^{1/2}(\partial D) \to H^{1/2}(\partial D)$, $K' : H^{-1/2}(\partial D) \to H^{-1/2}(\partial D)$ and $N : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ on ∂D are given by

$$(S\varphi)(x) = \int_{\partial D} \varphi(y)\Phi(x, y)\, ds(y), \quad x \in \partial D,$$

$$(K\psi)(x) = \int_{\partial D} \psi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, ds(y), \quad x \in \partial D,$$

$$(K'\varphi)(x) = \int_{\partial D} \psi(y) \frac{\partial \Phi(x,y)}{\partial \nu(x)} \, \mathrm{d}s(y), \quad x \in \partial D,$$

$$(N\psi)(x) = \frac{\partial}{\partial \nu(x)} \int_{\partial D} \psi(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \, \mathrm{d}s(y), \quad x \in \partial D.$$

We extend u by a solution of the equation (3.21) in D such that $u|_D \in H^1(D)$ and $u|_{D^e} \in H^1_{loc}(D^e)$ satisfies (3.20)–(3.23) with boundary data $(f_1, f_2) = (u_- - u_+, \lambda \partial u_-/\partial\nu - \partial u_+/\partial\nu - \mu u_+)$ on ∂D. From the definition of the data-to-pattern operator G, we obtain $G(u_- - u_+, \lambda\partial u_-/\partial\nu - \partial u_+/\partial\nu - \mu u_+)^T = H^*(\varphi, \psi)^T$. Since $(u_-, \lambda\partial u_-/\partial\nu)^T \in \mathscr{N}(G)$, we obtain

$$H^* = G\begin{pmatrix} -S & -K - S\overline{\mu} - \frac{1}{2}I \\ -K' + \frac{1}{2}I - \mu S & -N - K'\overline{\mu} + \frac{\overline{\mu}}{2}I - \mu K - \mu S\overline{\mu} - \frac{\mu}{2}I \end{pmatrix}. \tag{3.42}$$

Combining (3.42) with the earlier established decomposition (3.25), we arrive at the following factorization of the far-field operator:

$$F = G\begin{pmatrix} -S & -K - S\overline{\mu} - \frac{1}{2}I \\ -K' + \frac{1}{2}I - \mu S & -N - K'\overline{\mu} + \frac{\overline{\mu}}{2}I - \mu K - \mu S\overline{\mu} - \frac{\mu}{2}I \end{pmatrix}^* G^*. \tag{3.43}$$

For the FM, we need to show that the real part of the middle operator can be represented as a sum of a compact and a coercive operator. Similar to the case of the mixed boundary value problem ([22]), due to the fact that the real part of S is the sum of a positively coercive and a compact part, while the real part of N is the sum of a negatively coercive and a compact part, the coercivity condition for the middle operator cannot be established. We follow the idea in Charalambopoulus *et al*'s paper [10] and decompose the operator G, such that only its injective part is retained:

$$\begin{aligned} G &= G\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ &= \frac{1}{2}G\begin{pmatrix} I & T^{-1} \\ T & I \end{pmatrix} + \frac{1}{2}G\begin{pmatrix} I & -T^{-1} \\ -T & I \end{pmatrix} \\ &= \frac{1}{2}G\begin{pmatrix} I \\ T \end{pmatrix}\begin{pmatrix} I & T^{-1} \end{pmatrix} + \frac{1}{2}G\begin{pmatrix} T^{-1} \\ -I \end{pmatrix}\begin{pmatrix} T & -I \end{pmatrix} \\ &= G_1\begin{pmatrix} T & -I \end{pmatrix}. \end{aligned} \tag{3.44}$$

Here, we have used the fact that the range of the operator $(I, T)^T : H^{1/2}(\partial D) \to H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ belongs to the kernel $\mathscr{N}(G)$ by theorem 3.9. The operator $G_1 : H^{-1/2}(\partial D) \to L^2(S^2)$ is given by

$$G_1 = \frac{1}{2}G\begin{pmatrix} T^{-1} \\ -I \end{pmatrix}. \tag{3.45}$$

From the derivation of (3.44) and the definition of G_1, we see that G_1 inherits the range of G. Substituting decomposition (3.44) of the operator G into factorization (3.43) of the operator F, we obtain a new factorization for F as follows:

$$\begin{aligned} F &= G_1\begin{pmatrix} T & -I \end{pmatrix}\begin{pmatrix} -S & -K - S\overline{\mu} - \frac{1}{2}I \\ -K' + \frac{1}{2}I - \mu S & -N - K'\overline{\mu} + \frac{\overline{\mu}}{2}I - \mu K - \mu S\overline{\mu} - \frac{\mu}{2}I \end{pmatrix}^* \begin{pmatrix} T^* \\ -I \end{pmatrix} G_1^* \\ &= G_1 M^* G_1^*, \end{aligned} \tag{3.46}$$

where the operator M^* is the adjoint of the operator $M : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ defined by

$$\begin{aligned} M = &-TST^* + TK + TS\overline{\mu} + \frac{T}{2} + K'T^* - N - K'\overline{\mu} - \frac{T^*}{2} + \frac{\overline{\mu}}{2}I \\ &+ \mu ST^* - \mu K - \mu S\overline{\mu} - \frac{\mu}{2}I. \end{aligned} \tag{3.47}$$

We summarize these results in the following theorem.

Theorem 3.10. *Assume that k^2 is neither a Dirchlet eigenvalue of the boundary value problem (3.26)–(3.27) nor a Neumann eigenvalue of the boundary value problem (3.28)–(3.29). Then, the far-field operator $F : L^2(S^2) \to L^2(S^2)$ from (1.6) has a factorization of the form*

$$F = G_1 M^* G_1^*,$$

where the operator $G_1 : H^{-1/2}(\partial D) \to L^2(S^2)$ is given by (3.45) and the operator $M : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is defined by (3.47). G_1 is compact, injective with dense range in $L^2(S^2)$. For any $z \in \mathbb{R}^3$, let ϕ_z be given by (3.13). Then,

$$z \in D \Longleftrightarrow \phi_z \in \mathscr{R}(G_1). \tag{3.48}$$

We write $M = M_0 + M_1$ with

$$M_0 = -TST^* + TK + \frac{T}{2} + K'T^* - N - \frac{T^*}{2}, \tag{3.49}$$

$$M_1 = TS\overline{\mu} - K'\overline{\mu} + \frac{\overline{\mu}}{2}I + \mu ST^* - \mu K - \mu S\overline{\mu} - \frac{\mu}{2}I. \tag{3.50}$$

$M_1 : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is compact because the operators S, K and K' are compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$ and the embedding from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$ is compact. For the convenience of the subsequent analysis, we will indicate the dependence of M_0 on the wave number k by writing $M_0(k)$, similarly for the operators S, K, K', N and T. The differences $S(k) - S(i)$, $K(k) - K(i)$, $K'(k) - K'(i)$, $N(k) - N(i)$ are compact since the kernels of these operators are sufficiently smooth. We note that $T(k)f - T(i)f = \lambda \partial\widetilde{u}/\partial\nu$, where $\widetilde{u} = u_k - u_i$ solves

$$\Delta\widetilde{u} + k^2 n\widetilde{u} = -(1 + k^2)nu_i \quad \text{in } D, \qquad \widetilde{u} = 0 \quad \text{on } \partial D$$

in the weak sense, with u_k and u_i being solutions of the boundary value problem (3.26)–(3.27), corresponding to wave numbers k and i, respectively. The boundedness of $f \mapsto u_i$ and $u_i \mapsto \widetilde{u}$ from $H^{1/2}(\partial D)$ into $H^1(D)$ and from $L^2(D)$ into $H^1(D)$, respectively, and the compactness of the embedding $H^1(D)$ into $L^2(D)$ implies that $T(k) - T(i)$ is compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. These compactness results imply that the difference of $M_0(k) - M_0(i)$ is compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. So far we have shown $M_0(k) = M_0(i) + (M_0(k) - M_0(i))$, where $M_0(k) - M_0(i)$ is compact.

For the FM, we need a further property of the operator $M_0(i)$ and introduce three more DtN operators $\Lambda^+(i)$, $\Lambda^-(i)$ and T_{n_0}. Precisely, the exterior DtN operator $\Lambda^+(i) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is defined by $\Lambda^+(i)f = \partial u_+/\partial\nu$ on ∂D, where u solves the exterior Dirichlet boundary value problem (1.1) and (1.4) with $k = i$ and boundary data $f \in H^{1/2}(\partial D)$, while the interior DtN operator $\Lambda^-(i) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is defined by $\Lambda^-(i)f = \partial v_-/\partial\nu$ on ∂D, where v solves the equation $\Delta v - v = 0$ in D with boundary data $f \in H^{1/2}(\partial D)$. $T_{n_0} : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is defined by $T_{n_0}f = \lambda\partial w_-/\partial\nu$ on ∂D, where w solves the following boundary value problem

$$\Delta w - n_0 w = 0 \quad \text{in } D, \qquad w = f \quad \text{on } \partial D \tag{3.51}$$

with some chosen constant $n_0 > 0$. We note that $\Lambda^\pm(i) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ are self-adjoint operators.

In the subsequent analysis, we shall use c to denote a generic positive constant whose values may change in different inequalities but remain always bounded away from infinity.

Lemma 3.11. *The operator $M_0(i)$ has a new representation*

$$M_0(i) = -BS(i)B^* + B + R, \tag{3.52}$$

where $B = T_{n_0} - \Lambda^+(i)$ *and* $R = -(T(i) - T_{n_0})S(i)(T(i) - \Lambda^+(i))^* - (T_{n_0} - \Lambda^+(i))S(i)(T(i) - T_{n_0})^* + (T(i) - T_{n_0})$ *have the property that* $\Re B$ *is coercive and* R *is compact from* $H^{1/2}(\partial D)$ *into* $H^{-1/2}(\partial D)$.

Proof. Using the operator identities (see lemma 2.6 in [24])

$$S(i)[\Lambda^-(i) - \Lambda^+(i)] = I = [\Lambda^-(i) - \Lambda^+(i)]S(i), \tag{3.53}$$

$$\Lambda^+(i)S(i) = -\tfrac{1}{2}I + K'(i), \tag{3.54}$$

$$S(i)\Lambda^-(i) = \tfrac{1}{2}I + K(i), \tag{3.55}$$

$$\Lambda^-(i)S(i)\Lambda^+(i) = \Lambda^+(i)S(i)\Lambda^-(i) = N(i), \tag{3.56}$$

one can derive new representation (3.52).

To prove the coercivity property of the operator $B : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$, let v be the radiating solution of $\Delta v - v = 0$ in D^e with boundary data $v_+ = f \in H^{1/2}(\partial D)$ on ∂D and w the solution of (3.51). Then, using Green's theorem in D and in $D_R := \{x \in D^e : |x| < R\}$, we obtain

$$\begin{aligned}
\langle Bf, f\rangle &= \langle T_{n_0}f, f\rangle - \langle \Lambda^+(i)f, f\rangle \\
&= \left\langle \lambda \frac{\partial w_-}{\partial \nu}, w_- \right\rangle - \left\langle \frac{\partial v_+}{\partial \nu}, v_+ \right\rangle \\
&= \lambda \iint_D (n_0|w|^2 + |\nabla w|^2)\,\mathrm{d}x + \iint_{D_R} (|v|^2 + |\nabla v|^2)\,\mathrm{d}x - \int_{|x|=R} \frac{\partial v}{\partial \nu}\overline{v}\,\mathrm{d}s \\
&= \lambda \iint_D (n_0|w|^2 + |\nabla w|^2)\,\mathrm{d}x + \iint_{D_R} (|v|^2 + |\nabla v|^2)\,\mathrm{d}x + \int_{|x|=R} |v|^2\,\mathrm{d}s + \mathcal{O}\left(\frac{1}{R}\right),
\end{aligned}$$

and thus as $R \to \infty$ (note that v decays exponentially)

$$\langle Bf, f\rangle = \lambda \iint_D (n_0|w|^2 + |\nabla w|^2)\,\mathrm{d}x + \iint_{D^e} (|v|^2 + |\nabla v|^2)\,\mathrm{d}x. \tag{3.57}$$

Since, $n_0 > 0$ and $\Re(\lambda) \geqslant \hat{c} \geqslant 0$, the trace theorem now yields the existence of $c > 0$, such that

$$\Re\langle Bf, f\rangle \geqslant \|v\|^2_{H^1(D^e)} \geqslant c\|v\|^2_{H^{1/2}(\partial D)} = c\|f\|^2_{H^{1/2}(\partial D)}.$$

Analogously to the proof of the compactness of the operator $T(k) - T(i)$, we obtain the compactness result for $T(i) - T_{n_0}$. Thus, the operator R is also compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. □

Lemma 3.12. *Denote by* $\widetilde{M}_0(i) := -BS(i)B^* + B$. *Then,* $\widetilde{M}_0(i)$ *has the following properties.*

(a) If $\Re(\lambda) - 1 < 0$, *then* $\Re\widetilde{M}_0(i)$ *is positively coercive, i.e., there exists* $c > 0$ *with*

$$\Re\langle \widetilde{M}_0(i)f, f\rangle \geqslant c\|f\|^2_{H^{1/2}(\partial D)} \quad \text{for all } f \in H^{1/2}(\partial D).$$

(b) If $\Re(\lambda) - 1 > 0$, *then* $\Re\widetilde{M}_0(i)$ *is negatively coercive, i.e, there exists* $c > 0$ *with*

$$-\Re\langle \widetilde{M}_0(i)f, f\rangle \geqslant c\|f\|^2_{H^{1/2}(\partial D)} \quad \text{for all } f \in H^{1/2}(\partial D).$$

(c) If $\Re(\lambda) - 1 = 0$ *and* $\Im(\lambda) \neq 0$, *then* $\Re\widetilde{M}_0(i)$ *is negatively coercive.*

Proof. Define, for any $f \in H^{1/2}(\partial D)$, the single-layer potentials u and v by $u(x) = S_L(i)B^*f$ in $\mathbb{R}^3\backslash\partial D$ and $v(x) = S_L(i)S^{-1}(i)f$ in D^e, respectively. Then, $u|_D \in H^1(D)$ and $u|_{D^e} \in H^1(D^e)$ is a solution of the equation $\Delta u - u = 0$ in $\mathbb{R}^3\backslash\partial D$ and, in terms of traces, $u_\pm$ and $\partial u_\pm/\partial\nu$ are given by

$$u_\pm = S(i)B^*f \quad \text{and} \quad B^*f = \frac{\partial u_-}{\partial\nu} - \frac{\partial u_+}{\partial\nu} \quad \text{on } \partial D, \tag{3.58}$$

and $v \in H^1(D^e)$ is a solution of $\Delta v - v = 0$ in D^e with

$$v = f \quad \text{on } \partial D. \tag{3.59}$$

Using Green's theorem in D, we see that $T^*_{n_0} : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is defined by $T^*_{n_0} f = \overline{\lambda}\partial w/\partial\nu$, where w satisfies

$$\Delta w - n_0 w = 0 \quad \text{in } D,$$

$$w = f \quad \text{on } \partial D. \tag{3.60}$$

From the definition of the operator B, we conclude that

$$B^*f = \overline{\lambda}\frac{\partial w}{\partial\nu} - \frac{\partial v}{\partial\nu} \quad \text{on } \partial D, \tag{3.61}$$

where we have used the fact that the operator $\Lambda^+(i)$ is self-adjoint.

Using the traces introduced in (3.58)–(3.61), we can obtain the following three representations:

$$\begin{aligned}\langle\widetilde{M}_0(i)f, f\rangle &= -\left\langle u, \overline{\lambda}\frac{\partial w}{\partial\nu} - \frac{\partial v}{\partial\nu}\right\rangle + \left\langle f, \frac{\partial u_-}{\partial\nu} - \frac{\partial u_+}{\partial\nu}\right\rangle \\ &= -\left\langle u, \overline{\lambda}\frac{\partial w}{\partial\nu}\right\rangle + \left\langle u, \frac{\partial v}{\partial\nu}\right\rangle + \left\langle w, \frac{\partial u_-}{\partial\nu}\right\rangle - \left\langle v, \frac{\partial u_+}{\partial\nu}\right\rangle \\ &= -\left\langle u, \overline{\lambda}\frac{\partial w}{\partial\nu}\right\rangle + \left\langle u, \frac{\partial v}{\partial\nu}\right\rangle + \left\langle w, \frac{\partial u_-}{\partial\nu}\right\rangle - \left\langle v, \frac{\partial u_+}{\partial\nu}\right\rangle \\ &= -\left\langle u, \overline{\lambda}\frac{\partial w}{\partial\nu}\right\rangle + \left\langle w, \frac{\partial u_-}{\partial\nu}\right\rangle,\end{aligned} \tag{3.62}$$

$$\begin{aligned}\langle\widetilde{M}_0(i)f, f\rangle &= -\left\langle u, \frac{\partial u_-}{\partial\nu} - \frac{\partial u_+}{\partial\nu}\right\rangle + \left\langle f, \frac{\partial u_-}{\partial\nu} - \frac{\partial u_+}{\partial\nu}\right\rangle \\ &= -\left\langle u, \frac{\partial u_-}{\partial\nu}\right\rangle + \left\langle u, \frac{\partial u_+}{\partial\nu}\right\rangle + \left\langle w, \frac{\partial u_-}{\partial\nu}\right\rangle - \left\langle v, \frac{\partial u_+}{\partial\nu}\right\rangle,\end{aligned} \tag{3.63}$$

$$\begin{aligned}\langle\widetilde{M}_0(i)f, f\rangle &= -\left\langle u, \overline{\lambda}\frac{\partial w}{\partial\nu} - \frac{\partial v}{\partial\nu}\right\rangle + \left\langle f, \overline{\lambda}\frac{\partial w}{\partial\nu} - \frac{\partial v}{\partial\nu}\right\rangle \\ &= -\left\langle u, \overline{\lambda}\frac{\partial w}{\partial\nu}\right\rangle + \left\langle u, \frac{\partial v}{\partial\nu}\right\rangle + \left\langle w, \overline{\lambda}\frac{\partial w}{\partial\nu}\right\rangle - \left\langle v, \frac{\partial v}{\partial\nu}\right\rangle.\end{aligned} \tag{3.64}$$

Note that the last equality in (3.62) holds since both u and v decay exponentially in D^e.

Subtracting (3.63) from (3.64), we have that

$$\begin{aligned}0 = &-\left\langle v - u, \frac{\partial(v - u_+)}{\partial\nu}\right\rangle + \left\langle u, \frac{\partial u_-}{\partial\nu}\right\rangle - \left\langle w, \frac{\partial u_-}{\partial\nu}\right\rangle \\ &-\left\langle u, \overline{\lambda}\frac{\partial w}{\partial\nu}\right\rangle + \left\langle w, \overline{\lambda}\frac{\partial w}{\partial\nu}\right\rangle.\end{aligned} \tag{3.65}$$

(a) We first consider the case when $\Re(\lambda) - 1 < 0$. Furthermore, we choose $n_0 < 1$. Adding (3.62) to (3.65), using Green's theorem and the fact that both u and v decay exponentially as $|x| \to \infty$, we see that

$$\begin{aligned}\Re\langle \widetilde{M}_0(i)f, f\rangle &= \iint_{D^e} (|u-v|^2 + |\nabla(u-v)|^2)\,\mathrm{d}x + \Re(\lambda)\\ &\quad\times \iint_D [n_0|u-w|^2 + |\nabla(u-w)|^2]\,\mathrm{d}x\\ &\quad+ \iint_D [(1-\Re(\lambda)n_0)|u|^2 + (1-\Re(\lambda))|\nabla u|^2]\,\mathrm{d}x\\ &\geqslant \min\{1-\Re(\lambda)n_0, 1-\Re(\lambda)\}\|u\|^2_{H^1(D)}\\ &\geqslant c\|S(i)B^* f\|^2_{H^{1/2}(\partial D)}\\ &\geqslant c\|B^* f\|^2_{H^{-1/2}(\partial D)}\\ &\geqslant c\frac{|\langle B^* f, f\rangle|^2}{\|f\|^2_{H^{1/2}(\partial D)}}\\ &\geqslant c\frac{|\Re\langle Bf, f\rangle|^2}{\|f\|^2_{H^{1/2}(\partial D)}}\\ &\geqslant c\|f\|^2_{H^{1/2}(\partial D)},\end{aligned}$$

where we have used the trace theorem in the second inequality, the fact that the operator $S(i)$ is continuously invertible in the third inequality, the Cauchy–Schwarz inequality in the fourth inequality and the coercivity property of the operator $\Re(B)$ (see lemma 3.11) in the last inequality.

(b) Considering now the case of $\Re(\lambda) - 1 > 0$. We choose $n_0 > 1$. Subtracting (3.62) from (3.65), again using Green's theorem and the fact that u and v decay exponentially as $|x| \to \infty$, one obtains

$$\begin{aligned}-\Re\langle \widetilde{M}_0(i)f, f\rangle &= \iint_{D^e} (|u-v|^2 + |\nabla(u-v)|^2)\,\mathrm{d}x + \iint_D (|u-w|^2 + |\nabla(u-w)|^2)\,\mathrm{d}x\\ &\quad+ \iint_D [(\Re(\lambda)n_0 - 1)|w|^2 + (\Re(\lambda)-1)|\nabla w|^2]\,\mathrm{d}x\\ &\geqslant \min\{\Re(\lambda)n_0 - 1, \Re(\lambda)-1\}\|w\|^2_{H^1(D)}\\ &\geqslant c\|f\|^2_{H^{1/2}(\partial D)}, \end{aligned} \tag{3.66}$$

where the last inequality is justified by the trace theorem.

(c) Let $\Re(\lambda) - 1 = 0$ but $\Im(\lambda) \neq 0$. From (3.66), we observe that

$$-\Re\langle \widetilde{M}_0(i)f, f\rangle \geqslant (n_0 - 1)\|w\|^2_{L^2(D)},$$

which implies that $-\Re[\widetilde{M}(i)]$ is non-negative and injective for some chosen $n_0 > 1$. Define $\widetilde{B} := T_{n_0} - \Lambda^-(i)$. One readily sees that $\widetilde{B}$ is compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. With the help of operator identity (3.53), from the definition of $B = T_{n_0} - \Lambda^+(i)$, we derive that $\widetilde{B} = B - S(i)^{-1}$ and from this

$$\widetilde{M}_0(i) = -\widetilde{B}S(i)\widetilde{B}^* - \widetilde{B}^*.$$

Then,

$$-\Re[\widetilde{M}_0(i)] = \Re(\widetilde{B})S(i)\Re(\widetilde{B}) + \Re(\widetilde{B}) + \Im(\widetilde{B})S(i)\Im(\widetilde{B}).$$

From (3.57), we derive that

$$-\Im\langle Bf, f\rangle = -\Im(\lambda)\iint_D (n_0|w|^2 + |\nabla w|^2)\,\mathrm{d}x$$
$$\geqslant -\Im(\lambda)\min\{n_0, 1\}\|w\|^2_{H^1(D)}$$
$$\geqslant c\|f\|^2_{H^{1/2}(\partial D)},$$

where we again have used the trace theorem. By the Lax–Milgram theorem, we conclude that $\Im(\widetilde{B}) = \Im(B)$ is bijective. This, together with the fact that $S(i)$ is bijective (e.g., see lemma 1.14 in [22]), imply that $\Im(\widetilde{B})S(i)\Im(\widetilde{B})$ is bijective. Thus, the operator $-\Re[\widetilde{M_0}(i)]$ is an isomorphism from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. Noting that $-\Re[\widetilde{M_0}(i)]$ is a bounded non-negative operator, we have (see inequality (4.5) in [19])

$$-\Re\langle\widetilde{M_0}(i)f, f\rangle \geqslant \frac{\|-\Re[\widetilde{M_0}(i)]f\|^2_{H^{-1/2}(\partial D)}}{\|\Re[\widetilde{M_0}(i)]\|} \geqslant c\|f\|^2_{H^{1/2}(\partial D)}. \tag{3.67}$$

□

Lemma 3.13. *Assume that k^2 is neither a Dirichlet eigenvalue of boundary value problem (3.26)–(3.27) nor an eigenvalue of the interior transmission problem (3.30)–(3.31). Then $\Im\langle Mf, f\rangle < 0$ for all $f \in H^{1/2}(\partial D)$ with $f \neq 0$.*

Proof. Define, for any $f \in H^{1/2}(\partial D)$, the combination v of the single-layer and double-layer potentials by

$$v = (S_L T^* - K_L - S_L\overline{\mu})f \quad \text{in } \mathbb{R}^3\backslash\partial D.$$

Then, $v \in H^1(D)$ and $v|_{D^e} \in H^1_{loc}(D^e)$ is a radiating solution of the Helmholtz equation

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3\backslash\partial D,$$

and

$$v_\pm = \left(ST^* - K - S\overline{\mu} \mp \frac{1}{2}I\right)f \quad\text{and}\quad \frac{\partial v_\pm}{\partial\nu} = \left(K'T^* - N - K'\overline{\mu} \mp \frac{1}{2}T^* \pm \frac{\overline{\mu}}{2}I\right)f \quad \text{on } \partial D \tag{3.68}$$

in terms of traces. Consequently, we have that

$$v_- - v_+ = f \quad\text{and}\quad \frac{\partial v_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu} = (T^* - \overline{\mu})f \quad \text{on } \partial D. \tag{3.69}$$

From the definition (3.47) of the operator M, we may rewrite

$$M = -(T-\mu)\left[ST^* - K - S\overline{\mu} - \frac{1}{2}I\right] + \left[K'T^* - N - K'\overline{\mu} - \frac{1}{2}T^* + \frac{\overline{\mu}}{2}I\right].$$

Using the relations given in (3.68) and (3.69), we have

$$\begin{aligned}
\langle Mf, f\rangle &= -\left\langle (T-\mu)v_+ - \frac{\partial v_+}{\partial\nu}, f\right\rangle \\
&= -\left\langle v_+, \frac{\partial v_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu}\right\rangle + \left\langle \frac{\partial v_+}{\partial\nu}, v_- - v_+\right\rangle \\
&= -\left\langle v_+, \frac{\partial v_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu}\right\rangle - \left\langle \frac{\partial v_+}{\partial\nu}, v_+\right\rangle + \left\langle \frac{\partial v_+}{\partial\nu}, v_-\right\rangle \\
&= -\left\langle v_+, \frac{\partial v_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu}\right\rangle - \left\langle \frac{\partial v_+}{\partial\nu}, v_+\right\rangle + \left\langle \frac{\partial v_-}{\partial\nu} - T^*f + \overline{\mu}f, v_-\right\rangle \\
&= -\left\langle v_+, \frac{\partial v_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu}\right\rangle - \left\langle \frac{\partial v_+}{\partial\nu}, v_+\right\rangle + \left\langle \frac{\partial v_-}{\partial\nu}, v_-\right\rangle - \langle T^*f - \overline{\mu}f, f + v_+\rangle \\
&= -2\Re\left[\left\langle v_+, \frac{\partial v_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu}\right\rangle\right] - \left\langle \frac{\partial v_+}{\partial\nu}, v_+\right\rangle + \left\langle \frac{\partial v_-}{\partial\nu}, v_-\right\rangle \\
&\quad - \langle f, Tf\rangle + \overline{\mu}\langle f, f\rangle.
\end{aligned} \tag{3.70}$$

We now look at the imaginary part for each term on the right-hand side (rhs) of (3.70). The first term on the rhs of (3.70) is real valued. Using Green's theorem in $D_R := \{x \in D^e : |x| < R\}$, we obtain

$$-\left\langle \frac{\partial v_+}{\partial \nu}, v_+ \right\rangle = \iint_{D_R} [v\Delta\overline{v} + |\nabla v|^2]\,\mathrm{d}x - \int_{|x|=R} \frac{\partial v}{\partial \nu}\overline{v}\,\mathrm{d}s$$
$$= \iint_{D_R} [-k^2|v|^2 + |\nabla v|^2]\,\mathrm{d}x - \mathrm{i}k\int_{|x|=R} |v|^2\,\mathrm{d}s + \mathcal{O}\left(\frac{1}{R}\right)$$

as R tends to infinity. Taking the imaginary part yields

$$-\Im\left\langle \frac{\partial v_+}{\partial \nu}, v_+ \right\rangle = -k\lim_{R\to\infty}\int_{|x|=R}|v|^2\,\mathrm{d}s = -\frac{k}{(4\pi)^2}\int_{S^2}|v^\infty|^2\,\mathrm{d}s \leqslant 0.$$

Using the Green's theorem in D again, we find

$$\left\langle v_-, \frac{\partial v_-}{\partial \nu} \right\rangle = \iint_D [v\Delta\overline{v} + |\nabla v|^2]\,\mathrm{d}x = \iint_D [-k^2|v|^2 + |\nabla v|^2]\,\mathrm{d}x,$$

and therefore the third term on the rhs of (3.70) is real valued. Recall that the DtN operator $T : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is defined by $Tf = \lambda\partial u/\partial\nu$, where u solves the interior Dirichlet boundary value problem (3.26)–(3.27) with boundary data $f \in H^{1/2}(\partial D)$. Using Green's theorem in D, we have

$$\langle f, Tf\rangle = \overline{\lambda}\int_{\partial D} u\overline{\frac{\partial u}{\partial \nu}}\,\mathrm{d}s = \overline{\lambda}\iint_D [-k^2\overline{n}|u|^2 + |\nabla u|^2]\,\mathrm{d}x.$$

Taking the imaginary part yields

$$-\Im\langle f, Tf\rangle = \iint_D [-k^2\Im(\lambda n)|u|^2 + \Im(\lambda)|\nabla u|^2]\,\mathrm{d}x,$$

and, since by assumption, $\Im(\lambda) \leqslant 0$ and $\Im(\lambda n) \geqslant 0$, the imaginary part of the fourth term on the rhs of (3.70) is non-positive. The assumption $\Im(\mu) \geqslant 0$ implies that the imaginary part of the fifth term on the rhs of (3.70) is non-positive.

Together with the above analysis, we have in fact proved that $\Im\langle Mf, f\rangle \leqslant 0$ for all $f \in H^{1/2}(\partial D)$. Let us now assume that $\Im\langle Mf, f\rangle = 0$ for some $f \in H^{1/2}(\partial D)$. Then $v^\infty = 0$. Theorem 2.13 in [11] now shows that $v = 0$ in D^e. Then, from the boundary condition (3.69), we see that

$$v_- = f \quad \text{and} \quad \frac{\partial v_-}{\partial \nu} = T^*f - \overline{\mu}f \quad \text{on } \partial D.$$

By Green's theorem, we have that the adjoint operator $T^* : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is given by

$$T^*f = \overline{\lambda}\frac{\partial u}{\partial \nu} \quad \text{on } \partial D,$$

where u solves the boundary value problem

$$\Delta u + k^2\overline{n}u = 0 \quad \text{in } D, \qquad u = f \quad \text{on } \partial D.$$

Thus $(\overline{u}, \overline{v})$ solves the interior conductive problem (3.30)–(3.31) and consequently $u = v = 0$ by taking into account the assumption on k^2. From this we conclude that $f = 0$. □

Combining theorem 3.10, lemma 3.12 and lemma 3.13, with the help of the well-known range identity theorem (see e.g., theorem 2.15 in [22]), we obtain the following main result of this paper.

Theorem 3.14. *In addition to assumption 3.6, we further assume that k^2 is not an eigenvalue of the interior transmission problem (3.30)–(3.31). For $z \in \mathbb{R}^3$, define $\phi_z \in L^2(S^2)$ by (3.13). Then*

$$z \in D \Longleftrightarrow \phi_z \in \mathscr{R}\big(F_\sharp^{1/2}\big),$$

and consequently

$$z \in D \Longleftrightarrow \sum_{j=1}^{\infty} \frac{|\langle \phi_z, \psi_j \rangle_{L^2(S^2)}|^2}{|\lambda_j|} < \infty. \tag{3.71}$$

where (λ_j, ψ_j) is an eigensystem of the operator $F_\sharp : L^2(S^2) \to L^2(S^2)$ given by

$$F_\sharp = |\Re F| + |\Im F|. \tag{3.72}$$

So again, the sign of the function

$$W(z) = \left[\sum_{j=1}^{\infty} \frac{|\langle \phi_z, \psi_j \rangle_{L^2(S^2)}|^2}{|\lambda_j|} \right]^{-1}$$

is just the characteristic function of D.

4. Numerical results

In this section, we study the applicability of our method through some numerical simulations in $\mathbb{R}^2$. In each example, the forward data was generated for a kite-shaped object by coupling of the finite element and boundary integral equation method as suggested in [25]. For the numerical treatment of the integral equations, we applied the Nystrom method with 128 quadrature points, for the finite element method, we used the MATLAB PDE toolbox.

In all of the examples, the computed data set is represented by a $\mathbb{C}^{64\times 64}$ matrix F, where each entry is the far-field pattern $u^\infty(\theta_j, \theta_l)$, $j, l \in \{1, \dots 64\}$, with $\theta_j = 2\pi j/64$ and $\theta_l = 2\pi l/64$ denoting the corresponding incident direction of the plane wave and the observation point, respectively. Furthermore, we compute the matrix

$$F_\sharp = |\Re[e^{it} F]| + |\Im F|,$$

which represents a discretized version of the operator $F_\sharp$ defined in (3.19) (t is chosen to match the assumption 3.3) or in (3.72) (here, $t = 0$). The real and imaginary part of a matrix $A \in \mathbb{C}^{N\times N}$ is given by

$$\Re(A) = \frac{A + A^*}{2} \quad \text{and} \quad \Im(A) = \frac{A - A^*}{2i},$$

respectively. We define the absolute value of a matrix $A \in \mathbb{C}^{N\times N}$ with a singular value decomposition $A = U\Lambda V^*$ as

$$|A| = U|\Lambda|V^*,$$

with $|\Lambda| = \text{diag}|\lambda_j|$, $j = 1, \dots N$. For our reconstructions, we used a grid $\mathcal{G}$ of 200×200 equally spaced sampling points on the rectangle $[-4, 4] \times [-4, 4]$. Let $\{(\sigma_n, \psi_n) : n = 1, \dots, 64\}$ represent the eigensystem of the matrix $F_\sharp$. Then, the analogous W of the indicator function in (3.73) is given by

$$W(z) := \left[\sum_{j=1}^{64} \frac{|\phi_z^* \psi_n|^2}{|\sigma_n|} \right]^{-1}, z \in \mathcal{G},$$

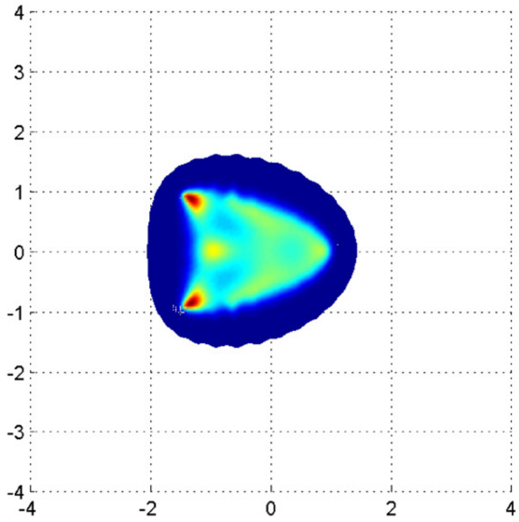

Figure 1. Reconstruction of the inhomogeneity with $n(x, y) = 1.1 + i(x^4 + y^4)$, $k = 4$, $\mu(x, y) = i - 2x^2$ and $\lambda = 1$.

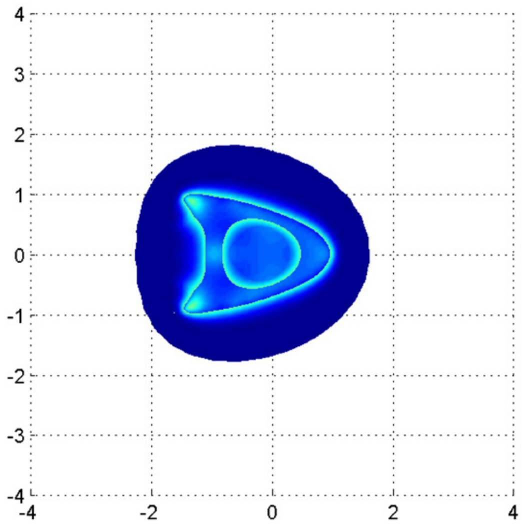

Figure 2. Reconstruction of the inhomogeneity with $n(x, y) = 1 + x^2 + i(x^2 + y^2 + 2)$, $k = 3$, $\mu(x, y) = -2 + i(x^2 + y^2)$ and $\lambda = 1 - i$.

where $\phi_z = (e^{-ik\theta_1 \cdot z}, e^{-ik\theta_2 \cdot z}, \ldots, e^{-ik\theta_{64} \cdot z})^\top \in \mathbb{C}^{64}$. Although, the sum is finite, we expect the value of $W(z)$ to be much larger for the points belonging to D than for those lying outside of the domain.

Figure 1 represents a reconstruction for the kite-shaped inhomogeneity D with $n(x, y) = 1.1 + i(x^4 + y^4)$, $(x, y) \in D$, the wave number $k = 4$, $\mu(x, y) = i - 2x^2$, $(x, y) \in \partial D$ and $\lambda = 1$. For the reconstruction, we choose $t = \pi/4$.

In figure 2, $k = 3$, $n(x, y) = 1+x^2+i(x^2+y^2+2)$, $\lambda = 1-i$ and $\mu(x, y) = -2+i(x^2+y^2)$.

Figure 3 corresponds to $k = 4$, $n(x, y) = 1 + 10i|\sin(y)| + x^2 + y^2$, $\lambda = 1$ and $\mu(x, y) = i - 2x^2$. In this last example, the condition (3.1) on q is violated. Although we

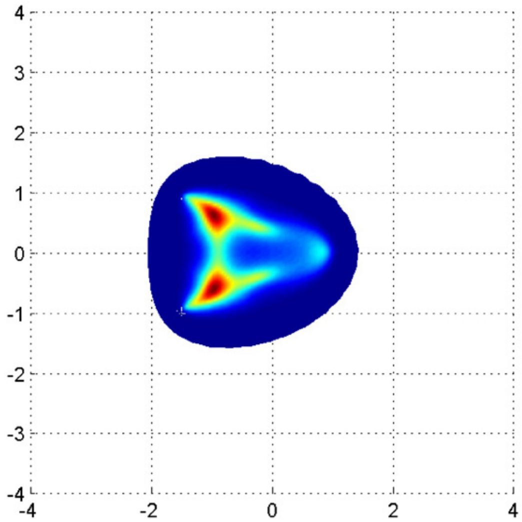

Figure 3. Reconstruction of the inhomogeneity with $n(x, y) = 1 + 10i|\sin(y)| + x^2 + y^2$, $k = 4$, $\mu(x, y) = i - 2x^2$ and $\lambda = 1$.

do not have a theoretical justification for this case, figure 3 demonstrates that $F_\sharp$ still enables us to reconstruct the inhomogeneity.

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References

[1] Angell T S and Kirsch A 1992 The conductive boundary condition for Maxwell's equations *SIAM J. Appl. Math.* **52** 1597–610

[2] Angell T S, Kleinman R E and Hettlich F 1990 The resistive and conductive problems for the exterior Helmholtz equation *SIAM J. Appl. Math.* **50** 1607–22

[3] Bourgeois L, Chaulet N and Haddar H 2012 On simultaneous identification of a scatterer and its generalized impedance boundary condition *SIAM J. Sci. Comput.* **34** A1824–48

[4] Cakoni F and Colton D 2006 *Qualitative Methods in Inverse Scattering Theory: An Introduction* (Berlin: Springer)

[5] Cakoni F, Colton D and Haddar H 2010 The interior transmission problem for regions with cavities *SIAM J. Math. Anal.* **42** 145–62

[6] Cakoni F and Gintides D 2010 New results on transmission eigenvalues *Inverse Problems Imaging* **4** 39–48

[7] Cakoni F, Gintides D and Haddar H 2010 The existence of an infinite discrete set of transmission eigenvalues *SIAM J. Math Anal.* **42** 237–55

[8] Cakoni F and Haddar H 2009 On the existence of transmission eigenvalues in an inhomogeneous medium *Appl. Anal.* **88** 475–93

[9] Cakoni F and Kirsch A 2010 On the interior transmission eigenvalue problem *Int. J. Comput. Sci. Math.* **3** 142–67

[10] Charalambopoulus A, Kirsch A, Anagnostopoulus K A, Gintides D and Kiriaki K 2007 The factorization method in inverse elastic scattering from penetrable bodies *Inverse Problems* **23** 27–51

[11] Colton D and Kress R 1998 *Inverse Acoustic and Electromagnetic Scattering Theory* (*Applied Mathematical Sciences* vol 93) 2nd edn (Berlin: Springer)

[12] Gerlach T and Kress R 1996 Uniqueness in inverse obstacle scattering with conductive boundary condition *Inverse Problems* **12** 619–25 (Corrigendum)
Gerlach T and Kress R 1996 *Inverse Problems* **12** 1075
[13] Hettlich F 1994 On the uniqueness of the inverse conductive scattering problem for the Helmholtz equation *Inverse Problems* **10** 129–44
[14] Hettlich F 1994 Two methods for solving an inverse conductive scattering problem *Inverse Problems* **10** 375–85
[15] Hettlich F 1998 The Landweber iteration applied to inverse conductive scattering problems *Inverse Problems* **14** 931–47
[16] Kirsch A 1998 Charaterization of the shape of a scattering obstacle using the spectral data of the far field operator *Inverse Problems* **14** 1489–512
[17] Kirsch A 1999 Factorization of the far field operator for the inhomogeneous medium case and an application in inverse scattering theory *Inverse Problems* **15** 413–29
[18] Kirsch A 2002 The MUSIC-algorithm and the factorization method in inverse scattering theory for inhomogeneous media *Inverse Problems* **18** 1025–40
[19] Kirsch A 2005 The factorization method for a class of inverse elliptic problems *Math. Nachr.* **278** 258–77
[20] Kirsch A 2009 On the existence of transmission eigenvalues *Inverse Problems Imaging* **3** 155–72
[21] Kirsch A 2011 *An Introduction to the Mathematical Theory of Inverse Problems* 2nd edn (Berlin: Springer)
[22] Kirsch A and Grinberg N 2008 *The Factorization Method for Inverse Problems* (Oxford: Oxford University Press)
[23] Kirsch A and Kleefeld A 2012 The factorization method for a conductive boundary condition *J. Integral Eqns. Appl.* **24** 575–601
[24] Kirsch A and Liu X 2013 The factorization method for inverse acoustic scattering by a penetrable anisotropic obstacle *Math. Methods Appl. Sci.* DOI: 10.1002/mma.2877
[25] Kirsch A and Monk P 1994 An analysis of the coupling of finite-element and Nystrom methods in acoustic scattering *IMA J. Numer. Anal.* **14** 523–44
[26] Kress R 1999 *Linear Integral Equation* 2nd edn (Berlin: Springer)
[27] Lechleiter A 2009 The factorization method is independent of transmission eigenvalues *Inverse Problems Imaging* **3** 123–38
[28] Mclean W 2000 *Strongly Elliptic Systems and Boundary Integral Equation* (Cambridge: Cambridge University Press)
[29] Päivärinta L and Sylvester J 2008 Transmission eigenvalues *SIAM J. Math. Anal.* **40** 738–53
[30] Schmucker U 2008 Interpretation of induction anomalies above nonuniform surface layers *Geophysics* **36** 156–65
[31] Vasseur G and Weidelt P 1977 Bimodal electromagnetic induction in nonuniform thin sheets with an application to the northern Pyrenean induction anomaly *Geophys. J. R. Astron. Soc.* **51** 669–90