The Factorization Method for Inverse Scattering by a Penetrable Anisotropic Obstacle with Conductive Boundary Condition

Oleksandr Bondarenko*[1] and Andreas Kirsch†[2]

[1,2]Department of Mathematics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany.

Abstract

In this paper we consider a problem of scattering by a coated anisotropic dielectric. Our main focus is to show that the factorization method can be applied for reconstruction of the scattering object from the far field data. At the end we present some numerical results to demonstrate the feasibility of the method.

Keywords: Inverse scattering, inhomogeneous medium, conductive boundary condition, factorization method.

1 Introduction

In this paper we show that the factorization method [10] can be applied to solve a scalar inverse scattering problem with conductive transmission conditions. The problem we consider is derived as the TE-mode from the time-harmonic Maxwell system where the scattering medium is coated by a thin highly conductive layer. The TE-mode, the case when the magnetic field has only one non-zero component [11], corresponds to the scattering of electromagnetic waves by an infinitely long cylinder. The appearance of the thin highly conductive layer leads to conductive transmission conditions [7] which has been known for a long time in the study of electromagnetic induction in the earth [15], [17].

Our work is motivated by the paper of Cakoni, Colton and Monk [2] in which the authors consider the scattering of time-harmonic electromagnetic waves by an orthotropic dielectric partially coated with a thin highly conductive layer and suggest a method for determining the physical properties of the coating. The first step of the method is to reconstruct the boundary of the scattering medium from the far field measurements. This is done by the linear sampling method. Though the linear sampling method provides good numerical

*bondarenko@kit.edu (OB)
†kirsch@kit.edu (AK)

reconstructions, its mathematical theory is incomplete. In this work we show that in the special case of completely coated inhomogeneity, the (mathematically rigorous) factorization method can be applied for reconstruction of the shape and location of the scatterer.

Throughout this paper, we try to keep the same notation as in [2]. We also refer to the original paper for the detailed derivation of the mathematical model of the problem. To start with, let $D \subset \mathbb{R}^2$ be a finite union of bounded domains with Lipschitz boundary such that the exterior $\mathbb{R}^2 \backslash \overline{D}$ is connected and let ν denote the unit normal to ∂D directed into the exterior of D. Let η represent the (real-valued) surface conductivity on ∂D and A be a matrix-valued function defined on $\overline{D}$. Further, let $u^i(x) := e^{ikx\cdot d}, x \in \mathbb{R}^2$, denote the incident plane wave with wave number $k > 0$ and incidence direction $d \in S^1 := \{x \in \mathbb{R}^2 : |x| = 1\}$, and let v and u represent the total fields inside and outside the inhomogeneity, respectively. The mathematical model for the conductive scattering problem under consideration is given by the following set of equations:

$$\nabla \cdot A \nabla v + k^2 v = 0 \qquad \text{in } D, \tag{1.1}$$

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2 \backslash \overline{D}, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = 0 \qquad \text{on } \partial D, \tag{1.3}$$

$$u - v - i\eta \frac{\partial v}{\partial \nu_A} = 0 \qquad \text{on } \partial D, \tag{1.4}$$

$$u = u^s + u^i, \tag{1.5}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial \nu} - iku^s\right) = 0, \qquad r = |x|. \tag{1.6}$$

The radiation condition [16] for the scattered field u^s (1.6) is assumed to hold uniformly in all directions $\hat{x} = x/|x|$. For now, for smooth v and u,

$$\frac{\partial u}{\partial \nu}(x) = \nu(x) \cdot \nabla u(x), \qquad \frac{\partial v}{\partial \nu_A}(x) = \nu(x) \cdot A(x) \nabla v(x), \quad x \in \partial D,$$

denote the normal and co-normal derivatives, respectively.

We assume $A \in C^1(\overline{D}, \mathbb{C}^{2\times 2})$ and denote by $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ the matrices with the real and the imaginary parts of the entries of A, respectively. By the physics of the problem holds $\mathrm{Re}(A(x))$ and $\mathrm{Im}(A(x))$ are symmetric for all $x \in \overline{D}$, and $\mathrm{Re}(\overline{\xi} \cdot A(x)\xi) \geq \gamma |\xi|^2$ and $\mathrm{Im}(\overline{\xi} \cdot A(x)\xi) \leq 0$ for all $\xi \in \mathbb{C}^2$ and $x \in \overline{D}$, where γ is a positive constant. Due to the symmetry of A it holds that $\mathrm{Im}(\overline{\xi} \cdot A\xi) = \overline{\xi} \cdot \mathrm{Im}(A)\xi$ and $\mathrm{Re}(\overline{\xi} \cdot A\xi) = \overline{\xi} \cdot \mathrm{Re}(A)\xi$. We also want to account for η having discontinuities and assume $\eta \in L^\infty(\partial D)$ with $\eta \geq \eta_0 > 0$ a.e. on ∂D. In the following we specify in which space setting we will be considering the scattering problem (1.1)–(1.6).

Let $H^1(D)$ denote the Sobolev space and $H^1_{loc}(\mathbb{R}^2 \backslash D)$ the local Sobolev space defined in the usual way as

$$H^1(D) := \{u : u \in L^2(D), |\nabla u| \in L^2(D)\}, \qquad \text{and}$$

$$H^1_{loc}(\mathbb{R}^2 \backslash \overline{D}) := \{u : u \in H^1(B_R \backslash \overline{D}), \text{ for every } R, \text{ such that } D \subset B_R\},$$

where B_R is a ball of radius $R > 0$ centered at the origin $B_R := \{x \in \mathbb{R}^2 : |x| < R\}$. Let $H^{1/2}(\partial D)$ denote the trace space of $H^1(D)$ and $H^{-1/2}(\partial D)$ its dual. For $v \in H^1(D)$ with $\nabla \cdot A\nabla v \in L^2(D)$ the trace $\partial v/\partial \nu_A \in H^{-1/2}(\partial D)$ is well defined (see e.g. Theorem 5.7 in [1]).

From now on, we assume that the total fields $v \in H^1(D)$ and $u \in H^1_{loc}(\mathbb{R}^2\backslash\overline{D})$ satisfy (1.1) and (1.2), respectively, in the distributional sense. The boundary conditions (1.3) and (1.4) are assumed in terms of the trace operator. By the trace theorem, v and u possess traces in $H^{1/2}(\partial D)$. The traces of the conormal and the normal derivative, $\partial v/\partial \nu_A$ and $\partial u/\partial \nu$, respectively, are in $H^{-1/2}(\partial D)$. From the regularity theory for elliptic differential equations [4] it is known that u^s is analytic in $\mathbb{R}^2\backslash\overline{D}$. In particular, the radiation condition (1.6) makes sense.

The radiation condition implies the following asymptotic behavior of the scattered field u^s (see e.g. [3])

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} u^\infty(\hat{x}) + O(|x|^{-3/2}),\ |x| \to \infty,$$

uniformly with respect to $\hat{x} = x/|x| \in S^1$. To indicate that the far field corresponds to the scattered field due to the incident plane field with direction $d \in S^1$ we write $u^\infty = u^\infty(\cdot, d)$. The *inverse problem* we consider, is to determine the scatterer D from the knowledge of the far field patterns $u^\infty(\hat{x}, d)$ for all $\hat{x}, d \in S^1$.

Before we turn to solving the inverse problem we first show that the direct problem is well posed. The well posedness will play an important role in proving the applicability of the factorization method.

2 The Direct Problem

Let D, A, k and η be given and satisfy the assumptions described in the introduction. The direct problem reads as follows: For $f \in H^{1/2}(\partial D)$ and $h \in H^{-1/2}(\partial D)$ find $u \in H^1_{loc}(\mathbb{R}^2\backslash\overline{D})$ and $v \in H^1(D)$ such that

$$\nabla \cdot A\nabla v + k^2 v = 0 \qquad \text{in } D, \tag{2.1}$$

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2\backslash\overline{D}, \tag{2.2}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = h \qquad \text{on } \partial D, \tag{2.3}$$

$$u - v - i\eta\frac{\partial v}{\partial \nu_A} = f \qquad \text{on } \partial D, \tag{2.4}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial \nu} - iku\right) = 0, \qquad r = |x|. \tag{2.5}$$

This is a general version of (1.1)–(1.6). Since the incident field u^i satisfies the Helmholtz equation in $\mathbb{R}^2\backslash\overline{D}$, for $f = -u^i$ and $h = -\partial u^i/\partial \nu$, (2.1)–(2.5) is equivalent to (1.1)–(1.6), where u in (2.2) denotes the scattered field in $\mathbb{R}^2\backslash\overline{D}$, and v in (2.1) is the total field in D.

The problem (2.1)–(2.5) has at most one solution, i.e., for $f = 0$ and $h = 0$ the only solution to (2.1)–(2.5) is given by $v = 0$ and $u = 0$. For a proof we refer to Lemma 3.1 in [2].

To show the existence we follow the approach introduced by P.Hähner in [5], the main idea of which is to consider an equivalent form of (2.1)–(2.5) in a bounded domain B_R.
We define the following Dirichlet-to-Neumann mapping $\Lambda_k : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ by

$$\Lambda_k : h \mapsto \frac{\partial \tilde{u}}{\partial \nu}, \tag{2.6}$$

where $\tilde{u} \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{B_R})$ is the solution of the exterior Dirichlet problem

$$\Delta \tilde{u} + k^2 \tilde{u} = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{B_R}, \tag{2.7}$$

$$\tilde{u} = h \quad \text{on } \partial B_R, \tag{2.8}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial \nu} - iku\right) = 0, \quad r = |x|. \tag{2.9}$$

Let $R > 0$ be big enough such that $\overline{D} \subset B_R$. Then (3.29)–(3.33) is equivalent (for the justification see e.g. Lemma 5.24 in [1]) to the following problem in the bounded domain B_R:

$$\nabla \cdot A \nabla v + k^2 v = 0 \qquad \text{in} \quad D, \tag{2.10}$$

$$\Delta u + k^2 u = 0 \qquad \text{in} \quad \mathbb{R}^2 \backslash \overline{D}, \tag{2.11}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = h \qquad \text{on} \quad \partial D, \tag{2.12}$$

$$u - v - i\eta \frac{\partial v}{\partial \nu_A} = f \qquad \text{on} \quad \partial D, \tag{2.13}$$

$$\frac{\partial u}{\partial \nu} = \Lambda_k u \quad \text{on} \quad \partial B_R. \tag{2.14}$$

An equivalent variational formulation of (2.10)–(2.14) reads as follows: find $w \in H^1(B_R \backslash \partial D)$ such that

$$\iint\limits_{D} \overline{\nabla \varphi} \cdot A \nabla w - k^2 \overline{\varphi} w \, \mathrm{d}x + \iint\limits_{B_R \backslash D} \overline{\nabla \varphi} \cdot \nabla w - k^2 \overline{\varphi} w \, \mathrm{d}x - \int\limits_{\partial D} \frac{i}{\eta} [\overline{\varphi}][w] \, \mathrm{d}s - \langle \Lambda_k w, \varphi \rangle$$

$$= -i \int\limits_{\partial D} \frac{1}{\eta} f [\overline{\varphi}] \, \mathrm{d}s - \langle h, \varphi_+ \rangle. \tag{2.15}$$

for all $\varphi \in H^1(B_R \backslash \partial D)$, where $[\varphi]$ and $[w]$ denote the jumps $\varphi_+ - \varphi_-$ or $w_+ - w_-$, respectively, across ∂D. Here and in the following, we denote by $\langle \cdot, \cdot \rangle$ the dual form in the dual system $\langle H^{-1/2}(\partial U), H^{1/2}(\partial U) \rangle$ with $U = D$ or $U = B_R$, depending on the context.

One readily sees that, if v and u solve (2.10)–(2.14) then $w|_D := v$ and $w|_{B_R \backslash \overline{D}} := u|_{B_R \backslash \overline{D}}$ satisfy (2.15). And vice versa, if w is a solution of (2.15) then $v := w|_D$ and $u|_{B_R \backslash \overline{D}} := w|_{B_R \backslash \overline{D}}$ satisfy (2.10)–(2.13) and $\partial u / \partial \nu = \Lambda_k u$ on ∂B_R.

Theorem 2.1. *For every $f \in H^{1/2}(\partial D)$ and $h \in H^{-1/2}(\partial D)$ the conductive transmission problem (2.1)–(2.5), or, equivalently, (2.15) is uniquely solvable. Moreover, the solution $w \in H^1(B_R \backslash \partial D)$ depends continuously on the boundary data, i.e., there exists a constant $C_R > 0$, independent of h and f, such that*

$$||w||_{H^1(B_R \backslash \partial D)} \leq C_R \big(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)} \big).$$

Proof. It is well-known (see e.g. Theorem 5.22 in [1]) that there exists an operator $\Lambda_0 : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ such that $\Lambda_k - \Lambda_0$ is compact and $\langle \Lambda_0 \varphi, \varphi \rangle \leq 0$ for all $\varphi \in H^{1/2}(\partial B_R)$. We define the following continuous sesquilinear forms on $H^1(B_R \backslash \partial D) \times H^1(B_R \backslash \partial D)$:

$$a_1(w,\varphi) = \iint\limits_D \overline{\nabla\varphi} \cdot A w + \overline{\varphi} w \,\mathrm{d}x + \iint\limits_{B_R \backslash D} \overline{\nabla\varphi} \cdot \nabla w + \overline{\varphi} w \,\mathrm{d}x - \int\limits_{\partial D} \frac{i}{\eta} [\overline{\varphi}][w] \,\mathrm{d}s - \langle \Lambda_0 w, \varphi \rangle$$

and

$$a_2(w,\varphi) = -(k^2+1) \iint\limits_{B_R} \overline{\varphi} w \,\mathrm{d}x - \langle (\Lambda_k - \Lambda_0) w, \varphi \rangle.$$

The right-hand-side of (2.15) defines a bounded conjugate linear functional L on $H^1(B_R \backslash \partial D)$:

$$L(\varphi) = -i \int\limits_{\partial D} \frac{1}{\eta} f [\overline{\varphi}] \,\mathrm{d}s - \langle h, \varphi_+ \rangle.$$

Let $\eta_* = \text{ess inf}_{\partial D}\, \eta$. By the Cauchy-Schwarz inequality and the trace theorem there exist positive constant c and a positive constant C, dependent on η, such that

$$\begin{aligned} |L\varphi| &\leq \frac{c}{\eta_*} ||f||_{H^{1/2}(\partial D)} ||\varphi||_{H^1(B_R \backslash \partial D)} + c||h||_{H^{-1/2}(\partial D)} ||\varphi||_{H^1(B_R \backslash \partial D)} \\ &\leq C\big(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}\big) ||\varphi||_{H^1(B_R \backslash \partial D)} \quad \text{for all} \quad \varphi \in H^1(B_R \backslash \partial D). \end{aligned}$$

Thus,

$$||L|| \leq C(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}).$$

We write (2.15) as the problem of determining $w \in H^1(B_R \backslash \partial D)$ such that

$$a_1(w,\varphi) + a_2(w,\varphi) = L(\varphi) \quad \text{for all } \varphi \in H^1(B_R \backslash \partial D). \tag{2.16}$$

By assumption, for the matrix A we have $\text{Re}\ \overline{\xi} \cdot A(x)\xi \geq \gamma |\xi|^2$ for all $\xi \in \mathbb{C}^2$ and all $x \in D$ and some $\gamma > 0$. Thus,

$$\begin{aligned} \text{Re}\ a_1(w,w) &= \text{Re} \iint\limits_D \overline{\nabla w} \cdot A \nabla w + |w|^2 \,\mathrm{d}x + \iint\limits_{B_R \backslash D} |\nabla w|^2 + |w|^2 \,\mathrm{d}x - \langle \Lambda_0 w, \varphi \rangle \\ &\geq \text{Re} \iint\limits_D \overline{\nabla w} \cdot A \nabla w + |w|^2 \,\mathrm{d}x + \iint\limits_{B_R \backslash D} |\nabla w|^2 + |w|^2 \,\mathrm{d}x \\ &\geq \min\{\gamma, 1\} ||w||^2_{H^1(D)} + ||w||^2_{H^1(B_R \backslash \overline{D})} \geq \min\{1,\gamma\} ||w||^2_{H^1(B_R \backslash \partial D)}. \end{aligned}$$

By the Riesz representation theorem we define the bounded linear operators $\mathcal{A}_1 : H^1(B_R \backslash \partial D) \to H^1(B_R \backslash \partial D)$ and $\mathcal{A}_2 : H^1(B_R \backslash \partial D) \to H^1(B_R \backslash \partial D)$ by

$$(\mathcal{A}_1 w, \varphi)_{H^1(B_R \backslash \partial D)} = a_1(w,\varphi) \qquad \text{and} \qquad (\mathcal{A}_2 w, \varphi)_{H^1(B_R \backslash \partial D)} = a_2(w,\varphi).$$

Then, in terms of the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ (2.16) can be written as

$$\mathcal{A}_1 w + \mathcal{A}_2 w = F \tag{2.17}$$

with $F \in H^1(B_R\backslash\partial D)$ also defined by the Riesz representation theorem through $(F,\varphi)_{H^1(B_R\backslash\partial D)} = L(\varphi)$. In particular, $\|F\|_{H^1(B_R\backslash\partial D)} = \|L\| \leq C(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)})$.

Since Re $a_1(w,w) \geq c\|w\|^2_{H^1(B_R\backslash\partial D)}$, by the Lax-Milgram Lemma, the operator $\mathcal{A}_1$ is boundedly invertible on $H^1(B_R\backslash\partial D)$. By compactness of $\Lambda_k - \Lambda_0$ and the compact embedding of $H^1(B_R\backslash\partial D)$ into $L^2(B_R)$ we conclude that $\mathcal{A}_2$ is compact. Riesz-Fredholm theory yields that for all $F \in H^1(B_R\backslash)$ the solution of (2.17) exists, provided $\mathcal{A}_1 + \mathcal{A}_2$ is injective.

Assume, $\mathcal{A}_1 w + \mathcal{A}_2 w = 0$. This is equivalent to

$$a_1(w,\varphi) + a_2(w,\varphi) = 0 \quad \text{for all } \varphi \in H^1(B_R\backslash\partial D),$$

or to (2.1)–(2.5) with $h = 0$ and $f = 0$. By Lemma 3.1 in [2] the problem (2.1)–(2.5) has at most one solution, and therefore $w = 0$. Thus, (2.17) is uniquely solvable and for the solution w holds

$$||w||_{H^1(B_R\backslash\partial D)} \leq ||\mathcal{A}_1 + \mathcal{A}_2||^{-1} C\big(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}\big), \tag{2.18}$$

or

$$||w||_{H^1(B_R\backslash\partial D)} \leq C_R\big(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}\big), \tag{2.19}$$

where $C_R > 0$ depends on η, R, D and the matrix A, and does not depend on f and h.

□

Remark 2.2. *Since $\eta \in L^\infty(\partial D)$ and $f,u,v \in H^{1/2}(\partial D) \subset L^2(\partial D)$ the boundary condition (2.4)*

$$\frac{\partial v}{\partial \nu_A} = \frac{i}{\eta}(f + v - u)$$

implies that $\partial v/\partial \nu_A \in L^2(\partial D)$. From the trace theorem and Theorem 2.1 we have the following estimate on the norm of $\partial v/\partial \nu_A$:

$$\left\| \frac{\partial v}{\partial \nu_A} \right\|_{L^2(\partial D)} \leq c(||f||_{H^{1/2}(\partial D)} + ||h||_{H^{-1/2}(\partial D)}),$$

with $c > 0$ independent of f and h. As we will see in the next section the regularity of $\partial v/\partial \nu_A$ will play an important role in proving the factorization method.

3 The Inverse Problem

In this section we show that the scatterer D can be determined via the factorization method. The section is organized as follows. We introduce the far field operator $F : L^2(S^1) \to L^2(S^1)$ (an operator which contains the given data u^∞), and show that F is injective for all $k \in \mathbb{R}_{>0}\backslash S$, where S is at most a discrete subset of $\mathbb{R}_{>0}$. Then we derive a symmetric factorization of F of the form $F = H^*TH$. We show that there is a link between the range of H^* and the scattering domain D. Precisely, a given point $z \in \mathbb{R}^2$ belongs to D if, and only if, the function $\phi_z(\hat{x}) = e^{-ikz\cdot\hat{x}}$, $x \in S^1$, is the range of H^*. Finally, we study properties of the middle operator T and show the following range identity $\mathcal{R}(H^*) = \mathcal{R}\big((|\text{Re } F|+|\text{Im } F|)^{1/2}\big)$ with Re $F = (F + F^*)/2$ and Im $F = (F - F^*)/2i$. That is, the inhomogeneity D can be characterized by the range of an operator which can be computed from the data operator F. Finally, by an application of the Picard criterion [9], we construct a characteristic function of the scatterer D.

3.1 Far Field Operator. Interior eigenvalue problem

Let $u^\infty(\hat{x},\hat{\theta})$ denote the far field pattern corresponding to the incidence direction $\hat{\theta}\in S^1$ and the observation direction $\hat{x}\in S^1$. Our aim is to determine the location and the shape of the scatterer from the knowledge of the far field operator $F: L^2(S^1)\to L^2(S^1)$, defined by

$$Fg(\hat{x}) = \int\limits_{S^1} g(\hat{\theta})u^\infty(\hat{x},\hat{\theta})\,\mathrm{d}s(\hat{\theta}).$$

With respect to injectivity of F we have the following result.

Theorem 3.1. *Assume that k^2 is not an eigenvalue of the following interior eigenvalue problem*

$$\Delta w + k^2 w = 0 \text{ in } D, \qquad \nabla\cdot(A\nabla v) + k^2 v = 0 \text{ in } D, \tag{3.20}$$

$$\frac{\partial w}{\partial \nu} = 0 \text{ on } D, \qquad \frac{\partial v}{\partial \nu_A} = 0 \text{ on } \partial D, \tag{3.21}$$

$$w = v \text{ on } \partial D, \tag{3.22}$$

i.e., the only solution $(w,v)\in H^1(D)\times H^1(D)$ of is the trivial one $(w,v)=(0,0)$. Then the far field operator F is injective.

Proof. Let $g\in L^2(S^1)$ be such that $Fg=0$ on S^1. By the superposition principle $Fg=u^\infty$, where u^∞ is the far field pattern corresponding to the incident field given by the Herglotz function

$$v_g(x) = \int\limits_{S^1} e^{ikx\cdot d} g(d)\,\mathrm{d}s(d),\ x\in\mathbb{R}^2. \tag{3.23}$$

Thus, u^∞ is the far field pattern of the function u which satisfies:

$$\begin{aligned}
\nabla\cdot A\nabla u + k^2 u &= 0 \quad \text{in } D,\\
\Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^2\backslash\overline{D},\\
\frac{\partial u}{\partial \nu} - \frac{\partial u}{\partial \nu_A} &= -\frac{\partial v_g}{\partial \nu} \quad \text{on } \partial D,\\
u_+ - u_- - i\eta\frac{\partial u}{\partial \nu_A} &= -v_g \quad \text{on } \partial D,\\
\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u}{\partial \nu} - iku\right) &= 0, \qquad r=|x|.
\end{aligned}$$

Here, u_+ and u_- denote the traces of u taken from the exterior and interior of the domain D, respectively. By assumption, $u^\infty=0$. Rellich's Lemma and the unique continuation principle imply that u vanishes in $\mathbb{R}^2\backslash\overline{D}$. Therefore, the pair $(w,v):=(v_g|_D, u|_D)$ is a solution of the following problem:

$$\nabla\cdot A\nabla v + k^2 v = 0 \text{ in } D, \tag{3.24}$$

$$\Delta w + k^2 w = 0 \text{ in } D, \tag{3.25}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = 0 \text{ on } \partial D, \tag{3.26}$$

$$w - v = i\eta\frac{\partial v}{\partial \nu_A} \text{ on } \partial D. \tag{3.27}$$

We show that for a $(w,v) \in H^1(D) \times H^1(D)$ which solves (3.24)–(3.27), the traces of w and v on ∂D coincide. Indeed, let $(w,v) \in H^1(D) \times H^1(D)$ be a solution of (3.24)–(3.27). By Green's first theorem we have

$$\left\langle \frac{\partial v}{\partial \nu_A}, \varphi \right\rangle = \iint\limits_D A\nabla v \cdot \overline{\nabla \varphi} - k^2 v\overline{\varphi}\, \mathrm{d}x$$

for all $\varphi \in H^1(D)$. We set $\varphi := w - v$. Furthermore, using the boundary conditions (3.26)–(3.27) and the Green's first theorem we get

$$\begin{aligned}
\int\limits_{\partial D} \frac{1}{i\eta}(w-v)\overline{(w-v)}\mathrm{d}s &= \left\langle \frac{\partial v}{\partial \nu_A}, w - v \right\rangle \\
&= \left\langle \frac{\partial v}{\partial \nu_A}, w \right\rangle - \iint\limits_D A\nabla v \cdot \overline{\nabla v} - k^2 |v|^2 \,\mathrm{d}x \\
&= \left\langle \frac{\partial w}{\partial \nu}, w \right\rangle - \iint\limits_D A\nabla v \cdot \overline{\nabla v} - k^2 |v|^2 \,\mathrm{d}x \\
&= \iint\limits_D |\nabla w|^2 - k^2|w|^2 \,\mathrm{d}x - \iint\limits_D A\nabla v \cdot \overline{\nabla v} - k^2|v|^2 \mathrm{d}x.
\end{aligned}$$

This implies that

$$\mathrm{Im} \int\limits_{\partial D} \frac{1}{i\eta}|w-v|^2 \,\mathrm{d}s = -\mathrm{Im} \iint\limits_D \overline{\nabla v} \cdot A\nabla v \,\mathrm{d}x \tag{3.28}$$

Since $\mathrm{Im}\ \overline{\xi} \cdot A(x)\xi = \overline{\xi} \cdot \mathrm{Im}\ (A(x))\xi \leq 0$ for all $\xi \in \mathbb{C}$ and all $x \in D$, the equality (3.28) is possible only if $\int_{\partial D} |w-v|^2 \mathrm{d}s = 0$. That is, the traces of u and w coincide on ∂D. The boundary conditions (3.26)–(3.27) imply $\partial v/\partial \nu_A = \partial w/\partial \nu = 0$ on ∂D. Thus, (3.20)–(3.22) is an equivalent formulation of (3.24)–(3.27).
If k^2 is not an eigenvalue of the interior eigenvalue problem then $(w,v) = (0,0)$ is the only solution of (3.20)–(3.21). In particular, $v_g = 0$ in D and, by analyticity, in all of $\mathbb{R}^2$. This implies (see e.g. [1], Section 3.2) that $g = 0$.

□

Remark 3.2. *The interior eigenvalues form at most a discrete countable set with infinity as the only accumulation point.*

By the definition of the problem (3.24)–(3.27), the interior eigenvalues belong to a subset of the intersection of Neumann eigenvalues of $-\nabla \cdot A\nabla$ and $-\Delta$ in D. It can be shown that if $\overline{\xi} \cdot \mathrm{Im}(A(x_0))\xi < 0$ for all $\xi \in \mathbb{C}\backslash\{0\}$ at a point $x_0 \in D$ then there are no eigenvalues of $-\nabla \cdot A\nabla$, and, therefore, no interior eigenvalues. However, as we show below, if $\mathrm{Im}\ A = 0$, the interior eigenvalues can exist.

Assume $D = B_1$ is a unit disk. Let $A = \mathrm{diag}(\frac{1}{a}, \frac{1}{a})$ be a real-valued diagonal matrix with $a \in \mathbb{R}_{>0}$. Then the problem (3.20)–(3.22) reads as

$$\begin{aligned} \Delta w + k^2 w &= 0 \text{ in } B_1, & \frac{1}{a}\Delta v + k^2 v &= 0 \text{ in } B_1, \\ \frac{\partial w}{\partial \nu} &= 0 \text{ on } \partial B_1, & \frac{\partial v}{\partial \nu} &= 0 \text{ on } \partial B_1, \end{aligned}$$
$$w = v \text{ on } \partial B_1,$$

Let w be given in polar coordinates as $w(r,\varphi) := J_n(kr)e^{in\varphi}$ for $r \in [0,1]$, $\varphi \in [0,2\pi)$ and some $n \in \mathbb{Z}$, with J_n being the $n-$th Bessel function. Then w solves the Helmholtz equation in B_1. We choose $k \in \mathbb{R}_{>0}$ such that $J_n'(kr)|_{r=1} = 0$. In this way, w is a Neumann eigenfunction of $-\Delta$ in B_1 corresponding to the eigenvalue k^2. Let $k_D = k\sqrt{a}$ and let $v(r,\varphi) = \frac{J_n(k)}{J_n(k_D)} J_n(k_D r)e^{in\varphi}$. We choose a so, that $J_n'(k_D r)|_{r=1} = 0$. Then v is a Neumann eigenfunction of $-\Delta$ in B_1 corresponding to the eigenvalue k_D^2. Moreover, on the boundary $r = 1$ holds:

$$w(1,\varphi) = J_n(k)e^{in\varphi} = \frac{J_n(k)}{J_n(k_D)} J_n(k_D)e^{in\varphi} = v(1,\varphi) \quad \text{for all} \quad \varphi \in [0,2\pi).$$

Thus, k^2 is an interior eigenvalue.

3.2 Factorization of F

Throughout this section we assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D.

Let $\Lambda_k^- : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ denote the interior Dirichlet-to-Neumann operator:

$$\Lambda_k^- : f \mapsto \frac{\partial v}{\partial \nu},$$

with v being a solution of the Helmholtz equation in D: $\Delta v + k^2 v = 0$ in D with $v = f$ on ∂D. Under the assumption on k, the operator Λ_k^- is well-defined and bounded. Further, we define the exterior Dirichlet-to-Neumann operator $\Lambda_k^+ : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$,

$$\Lambda_k^+ : f \mapsto \frac{\partial v}{\partial \nu},$$

where v is the radiating solution of the exterior Dirichlet problem

$$\begin{aligned} \Delta v + k^2 v &= 0 \text{ in } \mathbb{R}^2 \setminus \overline{D}. \\ v &= f \text{ on } \partial D. \end{aligned}$$

Since the exterior Dirichlet and Neumann problems are well-posed, the operator Λ_k^+ is well-defined, bounded and has a bounded inverse, which we denote by $(\Lambda_k^+)^{-1} : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$. The subscript k indicates that the wave number is given by k. In the following we will also use operators $\Lambda_i^\pm$, which correspond to the wave number $k = i$.

The first step in the derivation of a symmetric factorization of the far field operator F is to define the data-to-pattern operator $G : H^{1/2}(\partial D) \to L^2(S^1)$ by $G\varphi = u^\infty$, where u^∞ is the far field pattern of the solution to (2.1)–(2.5) with the right hand side given by $f = \varphi$ and $h = \Lambda_k^- \varphi$:

$$\nabla \cdot A \nabla v + k^2 v = 0 \qquad \text{in } D, \tag{3.29}$$

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2 \backslash \overline{D}, \tag{3.30}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu_A} = \Lambda_k^- \varphi \ \text{ on } \partial D, \tag{3.31}$$

$$u - v - i\eta \frac{\partial v}{\partial \nu_A} = \varphi \qquad \text{on } \partial D, \tag{3.32}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial \nu} - iku \right) = 0, \qquad r = |x|. \tag{3.33}$$

Let $H : L^2(S^1) \to H^{1/2}(\partial D)$ denote the Herglotz operator

$$Hg(x) = \int_{S^1} e^{ikx \cdot d} g(d) \, \mathrm{d}s(d), \quad x \in \partial D.$$

Since $Hg = v_g|_{\partial D}$ and $\Lambda_k^- Hg = \partial v_g / \partial \nu|_{\partial D}$, by the superposition principle it follows that $F = -GH$.

The adjoint operator $H^* : H^{-1/2}(\partial D) \to L^2(S^1)$ is given by

$$H^* \psi(\hat{x}) = \int_{\partial D} \psi(y) e^{-ik\hat{x} \cdot y} \mathrm{d}s(y), \quad \hat{x} \in S^1. \tag{3.34}$$

For simplicity of notation, in (3.34) we use the integral instead of the the dual form.

By the asymptotic behavior of the fundamental solution of the Helmholtz equation

$$\Phi_k(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|), \tag{3.35}$$

where $H_0^{(1)}$ being the Hankel function of the first kind of order zero, $\frac{e^{i\pi/4}}{\sqrt{8\pi k}} H^* \psi$ is the far field of the single layer potential $S_L \psi$,

$$(S_L \psi)(x) = \int_{\partial D} \psi(y) \Phi_k(x, y) \, \mathrm{d}s(y), \quad x \in \mathbb{R}^2 \backslash \partial D,$$

It is well-known [14] that the single layer potential can be continuously extended to the boundary ∂D, i.e.,

$$S_L \psi|_{\pm} = S\psi \quad \text{ on } \partial D,$$

where $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is given by

$$S\psi(x) = \int_{\partial D} \psi(y) \Phi(x, y) \, \mathrm{d}s(y), \quad x \in \partial D.$$

Furthermore, the following jump condition hold on ∂D [14]:

$$\Lambda_k^+ S\psi - \Lambda_k^- S\psi = -\psi.$$

The single layer potential $S_L\psi$, with a given $\psi \in H^{-1/2}(\partial D)$, solves the Helmholtz equation in $\mathbb{R}^2 \backslash \partial D$ and satisfies the radiation condition. Therefore, we can view H^* as the data-to-pattern operator $H^* : \psi \mapsto w^\infty$, where w^∞ is the far field operator of the solution of the following transmission problem:

$$\Delta w + k^2 w = 0 \qquad \text{in } D, \tag{3.36}$$

$$\Delta w + k^2 w = 0 \qquad \text{in } \mathbb{R}^2 \backslash \overline{D}, \tag{3.37}$$

$$\Lambda_k^+ w_+ - \Lambda_k^- w_- = -\psi \quad \text{on } \partial D, \tag{3.38}$$

$$w_+ - w_- = 0 \qquad \text{on } \partial D, \tag{3.39}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial \nu} - ikw\right) = 0, \qquad r = |x|. \tag{3.40}$$

In particular, since w^∞ is analytic, $H^* : H^{-1/2}(\partial D) \to (C^\infty(S^1) \hookrightarrow) L^2(S^1)$ is compact.

Now, let $\varphi \in H^{1/2}(\partial D)$ be given and let $u^\infty = G\varphi$ be the far-field pattern of the solution u to (3.29)–(3.33) with the data φ. We define $w \in H^1_{loc}(\mathbb{R}^2)$ by

$$w = \begin{cases} \tilde{u} & \text{in } D, \\ u|_{\mathbb{R}^2 \backslash \overline{D}} & \text{in } \mathbb{R}^2 \backslash \overline{D}, \end{cases}$$

where $\tilde{u} \in H^1(D)$ is the solution of the interior Dirichlet problem with boundary data given by the trace of u from outside:

$$\Delta \tilde{u} + k^2 \tilde{u} = 0 \qquad \text{in } D,$$

$$\tilde{u} = u \quad \text{on } \partial D.$$

Then w solves (3.36)–(3.40) with $\psi = -(\Lambda_k^- - \Lambda_k^+)u$.

Since $u = w$ in $\mathbb{R}^2 \backslash \overline{D}$, the far fields of u and w coincide, i.e., $G\varphi = u^\infty = w^\infty = H^*(\Lambda_k^- - \Lambda_k^+)u$. This holds for all $\varphi \in H^{1/2}(\partial D)$. Thus, for each $\varphi \in H^{1/2}(\partial D)$, $G\varphi = H^* T\varphi$ with $T : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ given by

$$T : \varphi \mapsto (\Lambda_k^- - \Lambda_k^+)u, \tag{3.41}$$

where u is the trace of the solution to (3.29)–(3.33). We have just derived the factorization of the far field operator in the form $F = -H^* T H$.

In the next theorem we show that the scattering domain D and can be characterized the range of H^*.

Theorem 3.3. *For any $z \in \mathbb{R}^2$, let $\phi_z \in L^2(S^1)$ be defined as*

$$\phi_z(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}, \quad \hat{x} \in S^1. \tag{3.42}$$

Then $z \in D$ if, and only if, $\phi_z \in \mathcal{R}(H^*)$.

Proof. We first show that for $z \in D$ holds $\phi_z \in \mathcal{R}(H^*)$. Let $f \in H^{1/2}(\partial D)$ be given by $f = \Phi_k(\cdot, z)$ on ∂D, where Φ_k is the fundamental solution (3.35). Let $u \in H^1_{loc}(\mathbb{R}^2 \backslash \overline{D})$ be the radiating solution of the exterior Dirichlet problem

$$\begin{aligned} \Delta u + k^2 u = 0 \quad & \text{in } \mathbb{R}^2 \backslash \overline{D}, \\ u = f \quad & \text{on } \partial D. \end{aligned}$$

The exterior Dirichlet problem is uniquely solvable and therefore, the far fields u^∞ and $\Phi_k^\infty(\cdot, z)$ coincide, i.e., $u^\infty(\hat{x}) = \Phi_k^\infty(\hat{x}, z) = \phi_z(\hat{x})$ for all $\hat{x} \in S^1$.

We define $w \in H^1_{loc}(\mathbb{R}^2)$ by

$$w = \begin{cases} \tilde{u} & \text{in } D, \\ u|_{\mathbb{R}^2 \backslash \overline{D}} & \text{in } \mathbb{R}^2 \backslash \overline{D}, \end{cases}$$

where $\tilde{u} \in H^1(D)$ is the solution of the interior Dirichlet problem:

$$\begin{aligned} \Delta \tilde{u} + k^2 \tilde{u} = 0 \quad & \text{in } D, \\ u = f \quad & \text{on } \partial D. \end{aligned}$$

Then w is the radiating solution of the following transmission problem:

$$\begin{aligned} \Delta w + k^2 w = 0 \qquad & \text{in } \mathbb{R}^2 \backslash \overline{D}, \\ \Delta w + k^2 w = 0 \qquad & \text{in } D, \\ w_+ - w_- = 0 \qquad & \text{on } \partial D \\ \Lambda_k^+ w_+ - \Lambda_k^- w_- = -(\Lambda_k^- - \Lambda_k^+) f \qquad & \text{on } \partial D, \end{aligned}$$

with the far field pattern $w^\infty = u^\infty = \phi_z$. Since $w^\infty = H^*(\Lambda_k^- - \Lambda_k^+) f$, we get $\phi_z \in \mathcal{R}(H^*)$.

We prove the other direction by contradiction. Let $z \in \mathbb{R}^2 \backslash D$, and assume there is $\psi \in H^{-1/2}(\partial D)$ such that $H^* \psi = \phi_z$. That is, we assume that the far field of the solution $w \in H^1_{loc}(\mathbb{R}^2)$ to (3.36)–(3.40) with boundary data ψ coincide with the far field of the fundamental solution $\Phi_k(\cdot, z)$. By Rellich's Lemma and the unique continuation principle, w and $\Phi_k(\cdot, z)$ coincide in $\mathbb{R}^2 \backslash (D \cup \{z\})$. But for any disk B_z containing z in its interior, by assumption, $w \in H^1(B_z)$. At the same time, for any disk B_z containing z for the fundamental solution Φ_k we have $\Phi_k(\cdot, z) \notin H^1(B_z)$. We arrive at a contradiction.

□

In the following theorem, we collect properties of the operators H and T in order to establish a range identity between the operator H^* and operator $F_\sharp := |\text{Re } F| + |\text{Im } F|$. The range identity was first stated by Kirsch in [8] and further refined by Lechleiter in [13].

Theorem 3.4. *Assume k^2 is not an interior eigenvalue and not an interior Dirichlet eigenvalue of $-\Delta$ in D. Then*

(a) H is compact and injective.

(b) $(-T)$ *has the form* $(-T) = T_0 + T_1$*, where* T_0 *is a coercive self-adjoint operator and* $T_1 : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ *is compact. By coercivity we mean that there exists a constant* $c > 0$ *such that*

$$\langle T_0\varphi, \varphi\rangle \geq c||\varphi||^2_{H^{1/2}(\partial D)} \quad \textit{for all } \varphi \in H^{1/2}(\partial D).$$

(c) $\langle Im\ (-T)\varphi, \varphi\rangle > 0$ *for all* $\varphi \in H^{1/2}(\partial D), \varphi \neq 0$.

Proof. (a) The injectivity of the Herglotz operator for the three dimensional case was shown e.g. in Corollary 5.13 in [3]. The same arguments apply for the two dimensional case. The operator H is compact as the adjoint of the compact operator $H^* : H^{-1/2}(\partial D) \to L^2(S^1)$.

(b) First, we write $T\varphi = (\Lambda_k^- - \Lambda_k^+)u$ as

$$T\varphi = (\Lambda_k^- - \Lambda_i^-)u + (\Lambda_i^+ - \Lambda_k^+)u + (\Lambda_i^- - \Lambda_i^+)u.$$

The differences $(\Lambda_k^- - \Lambda_i^-)$ and $(\Lambda_i^+ - \Lambda_k^+) : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ are compact. The compactness of $(\Lambda_k^- - \Lambda_i^-)$ was shown e.g. in [12]. The compact embedding of $H^1(B_R \backslash \overline{D})$ into $L^2(B_R \backslash \overline{D})$ and the compactness of the operators $(\Lambda_k - \Lambda_0)$ and $(\Lambda_i - \Lambda_0) : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ defined in (2.6) imply $(\Lambda_i^+ - \Lambda_k^+)$ is compact. Let $A : H^{1/2}(\partial D) \to H^{1/2}(\partial D)$ define the mapping $\varphi \mapsto u$. By well-posedness of the direct problem and the trace theorem A is bounded. Thus, T can be written as $T = \tilde{T}_1 + (\Lambda_i^- - \Lambda_i^+)A$ with compact operator $\tilde{T}_1 := (\Lambda_k^- - \Lambda_i^-)A + (\Lambda_i^+ - \Lambda_k^+)A$.

We write the equation (3.31)

$$\Lambda_k^+ u - \frac{\partial v}{\partial \nu_A} = \Lambda_k^- \varphi$$

as

$$\Lambda_i^+ u = (\Lambda_i^+ - \Lambda_k^+)u + \frac{\partial v}{\partial \nu_A} + (\Lambda_k^- - \Lambda_i^-)\varphi + \Lambda_i^- \varphi.$$

Thus

$$u = (\Lambda_i^+)^{-1}(\Lambda_i^+ - \Lambda_k^+)u + (\Lambda_i^+)^{-1}\frac{\partial v}{\partial \nu_A} + (\Lambda_i^+)^{-1}(\Lambda_k^- - \Lambda_i^-)\varphi + (\Lambda_i^+)^{-1}\Lambda_i^- \varphi. \tag{3.43}$$

Let $B : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ represent the mapping $\varphi \mapsto \partial v/\partial \nu_A$. By the well-posedness of the problem we get that $\varphi \mapsto (u - v - \varphi)$ is bounded from $H^{1/2}(\partial D)$ to $H^{1/2}(\partial D)$. The boundary condition (3.32)

$$\frac{\partial v}{\partial \nu_A} = \frac{1}{i\eta}(u - v - \varphi)$$

yields that $\varphi \mapsto \partial v/\partial \nu_A$ is bounded as a mapping from $H^{1/2}(\partial D)$ to $L^2(\partial D)$. Parameterizing ∂D and using Rellich's embedding theorem [1] we conclude that $L^2(\partial D)$ is compactly embedded in $H^{-1/2}(\partial D)$. Thus, the operator B is compact from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$. Now, using (3.43) we can write $(-T)$ as the sum $(-T) = T_0 + T_1$, where

$$T_1 := -\tilde{T}_1 - (\Lambda_i^- - \Lambda_i^+)(\Lambda_i^+)^{-1}\Big((\Lambda_i^+ - \Lambda_k^+)A + B + (\Lambda_k^- - \Lambda_i^-)\Big)$$

is compact, and

$$T_0 := (\Lambda_i^+ - \Lambda_i^-)(\Lambda_i^+)^{-1}\Lambda_i^-.$$

Since Λ_i^- is self-adjoing and coercive and $-(\Lambda_i^+)^{-1}$ is positive [12], the coercivity of $T_0 : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ follows immediately:

$$\begin{aligned}\langle T_0\varphi, \varphi\rangle &= \langle -\Lambda_i^-(\Lambda_i^+)^{-1}\Lambda_i^-\varphi, \varphi\rangle + \langle \Lambda_i^-\varphi, \varphi\rangle \\ &\geq \langle -(\Lambda_i^+)^{-1}\Lambda_i^-\varphi, \Lambda_i^-\varphi\rangle + c||\varphi||^2_{H^{1/2}(\partial D)} \\ &\geq c||\varphi||^2_{H^{1/2}(\partial D)}\end{aligned}$$

for all $\varphi \in H^{\frac{1}{2}}(\partial D)$.

(c) It is easy to see that $\Lambda_k^- : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is self-adjoint. To show that $\mathrm{Im}\langle(-T)\varphi, \varphi\rangle > 0$ for all $\varphi \in H^{1/2}(\partial D), \varphi \neq 0$ we will use of the boundary condition (3.31), which in terms of Dirichlet-to-Neumann operators $\Lambda_k^{\pm}$ has the form

$$\Lambda_k^+ u - \frac{\partial v}{\partial \nu_A} = \Lambda_k^- \varphi.$$

We write $\langle(-T)\varphi, \varphi\rangle$ as

$$\begin{aligned}\langle(-T)\varphi, \varphi\rangle &= \langle(\Lambda_k^+ - \Lambda_k^-)u, \varphi\rangle = \langle \Lambda_k^+ u, \varphi\rangle - \langle u, \Lambda_k^-\varphi\rangle \\ &= \langle \frac{\partial v}{\partial \nu_A}, \varphi\rangle + \langle \Lambda_k^-\varphi, \varphi\rangle - \langle u, \Lambda_k^+ u\rangle + \langle u, \frac{\partial v}{\partial \nu_A}\rangle.\end{aligned}$$

Then

$$\begin{aligned}2i\ \mathrm{Im}\langle(-T)\varphi, \varphi\rangle &= \langle(-T)\varphi, \varphi\rangle - \overline{\langle(-T)\varphi, \varphi\rangle} \\ &= \langle \frac{\partial v}{\partial \nu_A}, \varphi\rangle - \langle u, \Lambda_k^+ u\rangle + \langle u, \frac{\partial v}{\partial \nu_A}\rangle - \left(\langle \varphi, \frac{\partial v}{\partial \nu_A}\rangle - \langle \Lambda_k^+ u, u\rangle + \langle \frac{\partial v}{\partial \nu_A}, u\rangle\right) \\ &= 2i\ \mathrm{Im}\langle \Lambda_k^+ u, u\rangle + \langle \frac{\partial v}{\partial \nu_A}, \varphi - u\rangle - \langle \varphi - u, \frac{\partial v}{\partial \nu_A}\rangle \\ &= 2i\ \mathrm{Im}\langle \Lambda_k^+ u, u\rangle + 2i\ \mathrm{Im}\langle \frac{\partial v}{\partial \nu_A}, \varphi - u\rangle \\ &= 2i\ \mathrm{Im}\langle \Lambda_k^+ u, u\rangle + 2i\ \mathrm{Im}\left(\frac{\partial v}{\partial \nu_A}, -i\eta\frac{\partial v}{\partial \nu_A}\right)_{L^2(\partial D)} - 2i\ \mathrm{Im}\langle \frac{\partial v}{\partial \nu_A}, v\rangle.\end{aligned}$$

In the last step we the boundary condition (3.32). Thus,

$$\begin{aligned}\mathrm{Im}\langle(-T)\varphi, \varphi\rangle &= \ \mathrm{Im}\langle \Lambda_k^+ u, u\rangle + \left(\frac{\partial v}{\partial \nu_A}, \eta\frac{\partial v}{\partial \nu_A}\right)_{L^2(\partial D)} -\ \mathrm{Im}\langle \frac{\partial v}{\partial \nu_A}, v\rangle \\ &\geq \eta_0\left\|\frac{\partial v}{\partial \nu_A}\right\|_{L^2(\partial D)} +\ \mathrm{Im}\langle \Lambda_k^+ u, u\rangle -\ \mathrm{Im}\langle \frac{\partial v}{\partial \nu_A}, v\rangle,\end{aligned}$$

with η_0 =ess $\inf_{\partial D} \eta$. We compute the imaginary parts of $\langle \Lambda_k^+ u, u\rangle$ and $\langle \frac{\partial v}{\partial \nu_A}, v\rangle$. The Green's first theorem and the Sommerfeld radiation condition yield

$$\mathrm{Im}\left\langle \Lambda_k^+ u, u\right\rangle = \mathrm{Im}\left\langle \frac{\partial u}{\partial \nu}, u\right\rangle = \mathrm{Im}\left(-\iint\limits_{B_R\setminus\overline{D}} |\nabla u|^2 - k^2|u|^2\,\mathrm{d}x\right) + \mathrm{Im}\left(\int\limits_{|x|=R} \frac{\partial u}{\partial \nu}\overline{u}\,\mathrm{d}s\right)$$

$$= \mathrm{Im}\left(ik\int\limits_{|x|=R} |u|^2\mathrm{d}s + o(1)\right) \quad \text{as } R\to\infty,$$

and, by assumption on A,

$$\mathrm{Im}\left\langle \frac{\partial v}{\partial \nu_A}, v\right\rangle = \mathrm{Im}\iint\limits_{D} A\nabla v\cdot\overline{\nabla v} - k^2|v|^2\,\mathrm{d}x = \iint\limits_{D}(\mathrm{Im}\ A)\nabla v\cdot\overline{\nabla v}\,\mathrm{d}x \le 0.$$

Thus, Im $\langle(-T)\varphi,\varphi\rangle \ge 0$. Assume there exists $\varphi\in H^{1/2}(\partial D)$ such that Im $\langle(-T)\varphi,\varphi\rangle = 0$. Then $\|\partial v/\partial\nu_A\|_{L^2(\partial D)} = 0$ and $\lim_{R\to\infty}\int_{|x|=R}|u|^2\,\mathrm{d}s = 0$. Rellich's Lemma and the unique continuation principle imply $u = 0$ in $\mathbb{R}^2\setminus\overline{D}$. Thus, $\partial u/\partial\nu = 0$. The boundary condition (3.31) yields $\Lambda_k^-\varphi = 0$. Since k^2 is not an interior eigenvalue, we conclude $\varphi = 0$. Thus, Im $\langle(-T)\varphi,\varphi\rangle > 0$ for all $\varphi\neq 0$.

□

The range identity theorem (Theorem 2.1 in [13]) yields $\mathcal{R}(H^*) = F_\sharp^{1/2}$ with

$$F_\sharp = \mathcal{R}(|\mathrm{Re}\ F| + |\mathrm{Im}\ F|). \tag{3.44}$$

Therefore, by Theorem 3.3 the domain D can be characterized by the range of $F_\sharp$. Now we can state the main result of this paper.

Theorem 3.5. *Assume that k^2 is not a Dirichlet eigenvalue and not an eigenvalue of the interior eigenvalue problem (3.20)–(3.22).*

For $z\in\mathbb{R}^2$ we define $\phi_z\in L^2(S^1)$ by (3.42). Then

$$z\in D \Longleftrightarrow \phi_z\in\mathscr{R}(F_\sharp^{1/2}),$$

and consequently

$$z\in D \Longleftrightarrow \sum_{j=1}^{\infty}\frac{|\langle\phi_z,\psi_j\rangle_{L^2(S^2)}|^2}{|\lambda_j|} < \infty. \tag{3.45}$$

where (λ_j,ψ_j) is an eigensystem of the operator $F_\sharp : L^2(S^2)\to L^2(S^2)$ given by

$$F_\sharp = Re\ |F| + Im\ |F|. \tag{3.46}$$

The the sign of the function

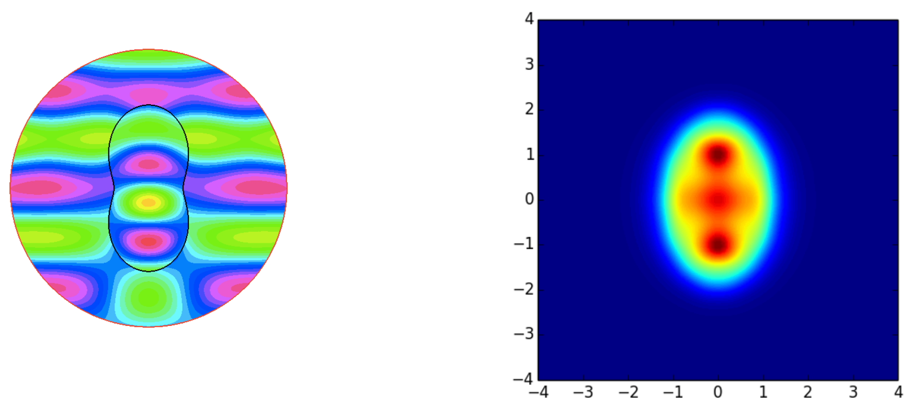
$$W(z) = \left[\sum_{j=1}^{\infty}\frac{|\langle\phi_z,\psi_j\rangle_{L^2(S^2)}|^2}{|\lambda_j|}\right]^{-1} \tag{3.47}$$

is the characteristic function of D.

4 Numerical Results

In this section we present a numerical example to demonstrate the applicability of the factorization method. We compute the forward problem for a peanut-shaped scatterer with ∂D parametrized by $\gamma(t) = (-1.5\sqrt{\cos^2(t) + .25\sin^2(t)}\sin(t), 1.8\sqrt{\cos^2(t) + .25\sin^2(t)}\cos(t))$, $t \in [0, 2\pi]$. Further, $k = 3, \eta = 0.5$ and $n = 0.5$. For the solution I used P^1 finite elements discretization. For the matrix A we set A =diag$(2, 2)$, the wave number is $k = 3$ and the conductivity $\eta = 0.5$. We reduce the scattering problem over $\mathbb{R}^2$ to a problem over a bounded domain with the help of Neumann-to-Dirichlet mapping [5] (see Section 2) and solve the forward problem using a P^1 finite elements discretization with the help of FreeFem++ package [6]. Figure 1 (a) represents the real part of the total field corresponding to the incident field with incident direction $d = [\cos(1.5\pi)\ \sin(1.5\pi)]^\top$.

Our data set is represented by a matrix $F \in \mathbb{C}^{64\times 64}$, where $F_{jl} = u^\infty(\theta_j, \theta_l), j, l \in \{1, \ldots 64\}$, and $u^\infty(\theta_j, \theta_l), j, l \in \{1, \ldots 64\}$ are the far fields corresponding to the incident direction of the plane wave $\theta_j = 2\pi j/64$ and the observation point $\theta_l = 2\pi l/64$.



(a) (b)

Figure 1: *(a) Total field of a peanut-shaped obstacle corresponding to the incident direction* $d = [\cos(1.5\pi)\ \sin(1.5\pi)]^\top$. *(b) Reconstruction by the Factorization Method.*

The implementation of the factorization method is done as follows. First we compute a discretized version of the operator $F_\sharp$ defined in (3.44):

$$F_\sharp = \text{Re } |F| + \text{Im } |F|.$$

Here, $F_\sharp \in \mathbb{C}^{64\times 64}$ and the real and imaginary parts of the matrix F are given by

$$\text{Re } F = \frac{F + F^*}{2} \qquad \text{and} \qquad \text{Im } F = \frac{F - F^*}{2i},$$

where F^* is the conjugate transpose of F. The absolute value of a matrix $A \in C^{N\times N}$ with a singular value decomposition $A = U\Lambda V^*$ is defined by

$$|A| = U|\Lambda|V^*,$$

with $|\Lambda|$ =diag$|\lambda_j|$, $j = 1, \dots, N$.

For the reconstruction we used a grid $\mathcal{G}$ of 200×200 equally spaced sampling points on the rectangle $[-4, 4] \times [-4, 4]$. Let $\{(\sigma_n, \psi_n) : n = 1, ..., 64\}$ represent the eigensystem of the matrix $F_\sharp$. Then the analogous W of the indicator function in (3.47) is given by

$$W(z) := \left[\sum_{j=1}^{64} \frac{|\phi_z^* \psi_n|^2}{|\sigma_n|} \right]^{-1}, \; z \in \mathcal{G},$$

where $\phi_z = (e^{-ik\theta_1 \cdot z}, e^{-ik\theta_2 \cdot z}, \dots, e^{-ik\theta_{64} \cdot z})^\top \in \mathbb{C}^{64}$. Although, the sum is finite we expect the value of $W(z)$ to be much larger for the points belonging to D then for those lying outside of the domain. Figure 1 (b) represents the plot of the the indicator function.

References

[1] F. Cakoni and D. Colton, *A Qualitative Approach to Inverse Scattering Theory*, Springer, New York, 2014.

[2] F. Cakoni, D. Colton, P. Monk, The determination of the surface conductivity of a partially coated dielectric *SIAM J. Appl. Math.,* **65 (3)**, 2004, 767-789

[3] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, vol. 93 of Applied Mathematical Sciences, 3rd edition, Springer, New York, 2013.

[4] D.Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin Heidelberg New York 1977.

[5] P. Hähner, On the uniqueness of the shape of a penetrable, anisotropic obstacle, *J. Comput. Appl. Math.* **116**, (2000), 167-180.

[6] Hecht, F. New development in FreeFem++. J. Numer. Math. 20 (2012), no. 3-4, 251–265. 65Y15

[7] T.S. Angell and A. Kirsch, The conductive boundary condition for Maxwell's equations, *SIAM J. Appl. Math.*, (1992), 52:1597-1610.

[8] A. Kirsch, Charaterization of the shape of a scattering obstacle using the spectral data of the far field operator, *Inverse Problems* **14** (1998), 1489-122.

[9] A. Kirsch, *An introduction to the mathematical theory of inverse problems*, (Second Edition), Springer, 2011.

[10] A. Kirsch and N. Grinberg, The Factorization Method for Inverse Problems, Oxford University Press, 2008.

[11] A. Kirsch and F. Hettlich, *The Mathematical Theory of Time-Harmonic Maxwell's Equations*, vol. 190 of Applied Mathematical Sciences, Springer, Switzerland, 2015.

[12] A. Kirsch and X. Liu, The Factorization method for inverse acoustic scattering by a penetrable anisotropic obstacle, *Math. Meth. Appl. Sci.* 37 (8), 2014, 1159-1170.

[13] A. Lechleiter, The factorization method is independent of transmission eigenvalues, *Inverse Problems Imaging* **3**, (2009), 123-138.

[14] W. Mclean, *Strongly Elliptic Systems and Boundary Integral Equation*, Cambridge University Press, Cambridge, 2000.

[15] U. Schmucker, Interpretation of induction anomalies above nonuniform surface layers, *Geophys.*, (2008), 36:156-165.

[16] A. Sommerfeld, *Partial differential equations in physics*, Academic Press, New York, 1949.

[17] G. Vasseur and P. Weidelt, Bimodal electromagnetic induction in nonuniform thin sheets with an application to the northern Pyrenean induction anomaly, *Geophys. J. Roy. Astronom. Soc.*, (1977), 2:669-690.