Karlsruhe Institute of Technology

Concavity Properties of Solutions of a Class of Singular Nonlinear Elliptic Boundary Value Problems

Master Thesis of

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1. Introduction

In this thesis, we are interested in the concavity properties of solutions of a class of singular nonlinear elliptic boundary value problems. First, we consider the following isotropic boundary value problem:

$$\begin{cases} \Delta u + c(x)u^{-\gamma} = 0 & \text{in} \quad \Omega \\ u(x) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

If $\Omega \subset \mathbb{R}^N$ is a bounded domain and if it has a sufficiently regular boundary $\partial\Omega$, if c is positive and sufficiently regular on $\overline{\Omega}$, and γ is any positive number, then there is a unique solution of (1.1), positive on Ω, which is in $C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$. This was shown by Lazer-McKenna in [LM91].

In this thesis, we prove that, if in addition Ω is convex, c is a concave C^2 function, and $\gamma \geq 2$, then the unique solution of (1.1) is concave.

We also prove that, in the special case when $c \equiv m$, $m > 0$, the unique solution of (1.1) is concave if $\gamma > 1$.

Now if $\Omega \subset \mathbb{R}^2$, we can rewrite (1.1) as follows

$$\begin{cases} u^{\gamma} u_{xx} + u^{\gamma} u_{yy} + c(x, y) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

If we change the above problem slightly, we get the following anisotropic boundary value problem:

$$\begin{cases} u^{\alpha} u_{xx} + u^{\beta} u_{yy} + c(x, y) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

Existence of a positive $C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ solution is proved by Choi-Lazer-McKenna in [CLM95] if $\Omega \subset \mathbb{R}^N$ is a bounded, convex domain and if it has a sufficiently regular boundary $\partial\Omega$, if c is positive and sufficiently regular on $\overline{\Omega}$, and if $\alpha > \beta \geq 0$.

In this thesis, we obtain a lower bound for solutions of (1.2) if $\beta > 1$, and $c \in C(\Omega)$, in addition to the above conditions. Namely, if u is a solution, then there exists a positive constant K such that

$$u(x) \geq \ K \left[d_{\partial\Omega}(x)\right]^{\frac{2}{1+\beta}} \qquad x \in \Omega,$$

where $d_{\partial\Omega}(x) : \Omega \to \mathbb{R}$ is the distance of x from the boundary of Ω, $\partial\Omega$:

$$d_{\partial\Omega}(x) := \inf_{y \in \partial\Omega} \|x - y\| \, .$$

Naturally, we are also interested in the concavity properties of the solutions of (1.2). We were able to obtain the following result, albeit only on certain types of "rectangular" domains: If $\alpha > \beta = 0$ and $c \in C^2(\Omega),\ c \geq m > 0$ is concave, then if a positive solution of (1.2) exists, it is concave in x-direction.

Choi-Lazer-McKenna also proved in [CLM95] that, if $M \geq c \geq m > 0$, uniqueness holds in the class of positive solutions with $u_{xx} \leq 0$.

Consequently, if $\alpha > \beta = 0$ and $c \in C^2(\Omega),\ c \geq m > 0$ is concave and bounded, we obtain uniqueness of a positive solution. We see that this uniqueness result is a new one.

To prove concavity (or partial concavity) of solutions, we use a method presented by Alvarez-Lasry-Lions in [ALL97]. We first present this method in the form useful to us.

2. Convexity of Solutions of Second Order Boundary Value Problems

The method for proving convexity of solutions of second order elliptic boundary value problems presented by Alvarez-Lasry-Lions in [ALL97] goes as follows:

Suppose we have a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the boundary value problem:

$$\begin{cases} F(x, u, Du, D^2u) & = 0 \qquad \text{in } \Omega \\ \qquad\qquad\quad u & = \phi \qquad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a convex, bounded domain, $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R}$ and S^N is the set of symmetric $N \times N$ matrices. We want to prove that u is convex.

We define the *convex envelope* $u_* : \overline{\Omega} \to \mathbb{R}$ of u:

$$\begin{aligned} u_*(x) \quad := \quad \inf \Bigg\{ & \sum_{i=1}^{k} \lambda_i u(x_i) \mid x = \sum_{i=1}^{k} \lambda_i x_i, \text{ with } x_i \in \overline{\Omega}, \\ & \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i > 0,\ k \leq N+1 \Bigg\} \end{aligned} \tag{2.2}$$

Remark 2.0.1
This definition takes into account Carathéodory's Theorem, which states the following:

If a point $x \in \mathbb{R}^N$ lies in the convex hull of a set P, then there is a subset P' of P with cardinality less or equal than $N + 1$, such that x lies in the convex hull of P'.

Obviously,

$$u_* \leq u \quad \text{on } \overline{\Omega}. \tag{2.3}$$

It is also easy to see that u_* really is convex. Namely, if $x, y \in \overline{\Omega}$, and $1 > \lambda > 0$, we have:

$$\begin{aligned} u_*(\lambda x + (1-\lambda)y) &= \inf\left\{ \sum_{i=1}^{M} \xi_i u(z_i) \mid \lambda x + (1-\lambda)y = \sum_{i=1}^{M} \xi_i z_i, \right. \\ &\qquad \left. z_i \in \overline{\Omega},\ \xi_i > 0,\ \sum_{i=1}^{M} \xi_i = 1 \right\} \\ &\le \inf\left\{ \lambda \sum_{i=1}^{m} \mu_i u(x_i) + (1-\lambda) \sum_{i=m}^{M} \eta_i u(y_i) \mid \sum_{i=1}^{m} \mu_i x_i = x, \right. \\ &\qquad \left. \sum_{i=m}^{M} \eta_i y_i = y,\ \sum_{i=1}^{m} \mu_i = 1,\ \sum_{i=m}^{M} \eta_i = 1 \right\} \\ &= \lambda \inf\left\{ \sum_{i=1}^{m} \mu_i u(x_i) \mid \sum_{i=1}^{m} \mu_i x_i = x,\ \sum_{i=1}^{m} \mu_i = 1 \right\} \\ &\quad + (1-\lambda) \inf\left\{ \sum_{i=m}^{M} \eta_i u(y_i) \mid \sum_{i=m}^{M} \eta_i y_i = y,\ \sum_{i=m}^{M} \eta_i = 1 \right\} \\ &= \lambda u_*(x) + (1-\lambda) u_*(y). \end{aligned}$$

If we can prove that:

i) $u_* = u$ on $\partial\Omega$,

ii) u_* is a supersolution of (2.1) in Ω (in some sense),

iii) a (suitable) comparison principle holds,

then $u_* \ge u$ on $\overline{\Omega}$, that is, because of (2.3), $u_* = u$. Hence, u is convex.

We will prove that u_* is a viscosity supersolution, so we start this chapter by introducing the viscosity theory.

Next, since u_* is convex, we explore how convexity properties translate into viscosity sense. (We do the same for concavity, since we will need those results later in chapter 3.)

In the final section, we prove some properties of u_*. What we are able to show for u_*, is that it is a lower semicontinuous function. (If i) holds, then it is continuous.) This justifies the choice of viscosity theory as a tool. We also give sufficient conditions so that i) and ii) hold.

2.1. Viscosity Solutions

The theory of viscosity solutions applies to partial differential equations of the form (2.1) where F is *degenerate elliptic*, i.e. such that for every $(x, r, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ it holds

$$F(x, r, p, A) \le F(x, r, p, B) \quad \text{for} \quad B \le A. \tag{2.4}$$

Remark 2.1.1
For arbitrary square matrices A, B we write $A \ge B$ if $A - B \ge 0$, i.e. if $A - B$ is positive semidefinite. This defines a partial ordering on the set of all square matrices.

Before we can define viscosity solutions, we need the notions of second order sub- and superjets. First, let us introduce the notations:

$$\begin{aligned} \mathrm{USC}(\Omega) &= \{\text{upper semicontinuous functions } u : \Omega \to \mathbb{R}\} \\ \mathrm{LSC}(\Omega) &= \{\text{lower semicontinuous functions } u : \Omega \to \mathbb{R}\} \end{aligned}$$

Definition 2.1.2

(a) Let $u \in USC(\Omega)$ and $x \in \Omega$. The ***second order superjet*** *$J^{2,+}_\Omega u(x)$ is the set of all $(p, A) \in \mathbb{R}^N \times S^N$ such that*

$$u(y) \le u(x) + \langle p, y-x\rangle + \frac{1}{2}\langle A(y-x), y-x\rangle + o(|y-x|^2), \quad \text{as } \Omega \ni y \to x.$$

Its closure we define as follows:

$$\overline{J}^{2,+}_\Omega u(x) := \left\{(p, A) \mid \exists\, (p_n, A_n) \in J^{2,+}_\Omega u(x_n) \ s.t.\ (x_n, u(x_n), p_n, A_n) \to (x, u(x), p, A)\right\}$$

(b) Let $u \in LSC(\Omega)$ and $x \in \Omega$. The ***second order subjet*** *$J^{2,-}_\Omega u(x)$ is the set of all $(p, A) \in \mathbb{R}^N \times S^N$ such that*

$$u(y) \ge u(x) + \langle p, y-x\rangle + \frac{1}{2}\langle A(y-x), y-x\rangle + o(|y-x|^2), \quad \text{as } \Omega \ni y \to x.$$

Its closure we define as follows:

$$\overline{J}^{2,-}_\Omega u(x) := \left\{(p, A) \mid \exists\, (p_n, A_n) \in J^{2,-}_\Omega u(x_n) \ s.t.\ (x_n, u(x_n), p_n, A_n) \to (x, u(x), p, A)\right\}$$

Remark 2.1.3

It holds (see [CIL92]):

$$\begin{aligned} J^{2,-}_\Omega u(x) &= \{(D\phi, D^2\phi) \mid \phi \in C^2 \text{ and } u - \phi \text{ has a local minimum in } x\} \\ J^{2,+}_\Omega u(x) &= \{(D\phi, D^2\phi) \mid \phi \in C^2 \text{ and } u - \phi \text{ has a local maximum in } x\} \end{aligned}$$

See figure 2.1 for an example of an element in subjet.

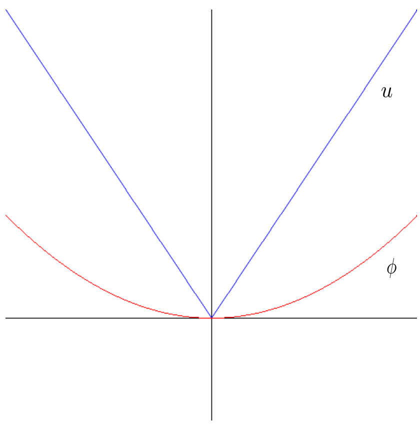


Figure 2.1.: Now, $(D\phi, D^2\phi) \in J^{2,-}_\Omega u(0)$. Note that $J^{2,+}_\Omega u(0) = \emptyset$.

Remark 2.1.4

Let U, $\tilde{U}$ be open. It is clear that if $x \in U \cap \tilde{U}$, then $J^{2,-}_U u(x) = J^{2,-}_{\tilde{U}} u(x)$. The same holds of course for superjets.

Finally, we can define viscosity solutions:

Definition 2.1.5
Let F be elliptic degenerate in the sense of (2.4)*, and $\Omega \subset \mathbb{R}^N$.*

*(a) A **viscosity subsolution** of $F = 0$ on Ω is a function $u \in$USC(Ω) such that*

$$F(x, u(x), p, A) \leq 0 \quad \text{for all} \quad x \in \Omega \text{ and } (p, A) \in J_\Omega^{2,+}u(x). \tag{2.5}$$

*(b) A **viscosity supersolution** of $F = 0$ on Ω is a function $u \in$LSC(Ω) such that*

$$F(x, u(x), p, A) \geq 0 \quad \text{for all} \quad x \in \Omega \text{ and } (p, A) \in J_\Omega^{2,-}u(x). \tag{2.6}$$

*(c) A continuous function u is a **viscosity solution** of $F = 0$ on Ω if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ on Ω.*

Remark 2.1.6
If F is continuous, and u is a supersolution of $F = 0$ on Ω, then

$$F(x, u(x), p, A) \geq 0 \quad \text{for all} \quad x \in \Omega \text{ and } (p, A) \in \overline{J}_\Omega^{2,-}u(x).$$

Similar remarks apply for subsolutions and solutions.

Remark 2.1.7
Clearly, every classical solution of $F = 0$ is also a viscosity solution.

2.1.1. Convexity and Subjets

Let us see how convexity relates to subjets. We will need these results in proving that the convex hull u_* is a viscosity supersolution.

When $u \in$LSC($\overline{\Omega}$) is a convex function in a convex domain Ω, we can expect some properties of smooth convex functions to translate to special features of superjets and subjets.

Remark 2.1.8
If u is convex in Ω, $J_\Omega^{2,-}u(x) \neq \emptyset$ for every $x \in \Omega$.

This is a consequence of the Supporting Hyperplane Theorem: if S is a closed convex set in R^N, and x is a point on the boundary, then there exists a supporting hyperplane of S containing x. Since u is convex, its epigraph epi(u) is a convex set. Let $x \in \Omega$. Then there exists a supporting hyperplane of epi(u) which contains $(x, u(x))$, and separates the Euclidean space $\mathbb{R}^N$ into two half spaces, one containing epi(u). We choose a C^2 function ϕ such that the graph of ϕ lies in the other half space, and such that $\phi(x) = u(x)$ holds. Now $(D\phi(x), D^2\phi(x)) \in J_\Omega^{2,-}u(x)$.

It can be shown that there is a connection between $J_\Omega^{2,-}u(x)$ and the so called first order subdifferential $D_\Omega^- u(x)$ of u, which is defined in the following way for $x \in \overline{\Omega}$:

$$D_\Omega^- u(x) := \left\{ p \in \mathbb{R}^N \mid \forall y \in \overline{\Omega} \quad u(y) \geq u(x) + \langle p, y - x \rangle \right\} \tag{2.7}$$

Lemma 2.1.9
Let Ω be a convex open set, and let $u \in$LSC$(\overline{\Omega})$ be convex. If $p \in D^-_\Omega u(x)$, then $(p,0) \in J^{2,-}_\Omega u(x)$; conversely, if $(p,A) \in \overline{J}^{2,-}_\Omega u(x)$, then $p \in D^-_\Omega u(x)$.

For proof, see [FS93]. This result will be important for us soon.

We introduce the notations

$$S^N_+ = \left\{ A \in S^N \mid A \geq 0 \right\}, \quad S^N_{++} = \left\{ A \in S^N \mid A > 0 \right\}.$$

Definition 2.1.10
We will call a function u seminconcave if there exists a constant C such that the mapping

$$x \mapsto u(x) - C \left\| x \right\|^2$$

is concave.

Lemma 2.1.11
Let Ω be a convex open set, and let $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ be elliptic degenerate in the sense of (2.4). Then, a convex function $u \in$LSC$(\overline{\Omega})$ is a viscosity supersolution of $F = 0$ in Ω if and only if, for every $x \in \Omega$

$$F(x, u(x), p, A) \geq 0 \quad \text{for all } (p, A) \in J^{2,-}_\Omega u(x) \text{ such that } A \in S^N_+. \tag{2.8}$$

Proof We give a sketch of the argumentation.

Let $\varepsilon > 0$. We define the the semiconcave approximation of u

$$u_\varepsilon(x) : \mathbb{R}^N \to \mathbb{R}, \qquad u_\varepsilon(x) := \inf_{y \in \overline{\Omega}} \left(u(y) + \frac{|y-x|^2}{\varepsilon} \right).$$

Let $x \in \Omega$ and $\phi \in C^2(\mathbb{R}^N)$ such that x is a local minimum of $u - \phi$ in Ω. Then $(D\phi(x), D^2\phi(x)) \in J^{2,-}_\Omega u(x)$.

It can be shown that there exists a minimum point $x_\varepsilon \in \mathbb{R}^N$ of $u_\varepsilon - \phi$ such that $x_\varepsilon \to x$ (see [IL90]).

For now, fix ε. Since u_ε is semiconcave, Jensen's Lemma A.3 and Aleksandrov's Theorem A.2 tell us that we can find a sequence $x^m_\varepsilon \to x_\varepsilon$ and $p^m_\varepsilon \to 0$ such that x^m_ε is the minimum point of $y \mapsto u_\varepsilon(y) - \phi(y) - \langle p^m_\varepsilon, y \rangle$, and u_ε is twice differentiable at x^m_ε. In particular, we have:

$$Du_\varepsilon(x^m_\varepsilon) = D\phi(x^m_\varepsilon) + p^m_\varepsilon, \tag{2.9}$$

as well as

$$D^2 u_\varepsilon(x^m_\varepsilon) \geq D^2\phi(x^m_\varepsilon). \tag{2.10}$$

Sending $m \to \infty$ we obtain from (2.9),

$$Du_\varepsilon(x^m_\varepsilon) \to D\phi(x_\varepsilon).$$

Let $A_\varepsilon := \lim_{m\to\infty} D^2 u_\varepsilon(x_\varepsilon^m)$. Now we have because of (2.10) and because ϕ is C^2

$$A_\varepsilon \geq D^2\phi(x_\varepsilon). \tag{2.11}$$

Also, u_ε is convex (see [CIL92]), so it holds:

$$A_\varepsilon \geq 0 \tag{2.12}$$

We have that $(D\phi(x_\varepsilon), A_\varepsilon) \in \overline{J}^{2,-} u_\varepsilon(x_\varepsilon)$.

Let $y_\varepsilon \in \Omega$ be such that

$$y_\varepsilon \to x. \tag{2.13}$$

Since u is continuous in Ω, we have:

$$u(y_\varepsilon) \to u(x). \tag{2.14}$$

It can be shown that $\overline{J}^{2,-} u_\varepsilon(x_\varepsilon) \subset \overline{J}^{2,-} u(y_\varepsilon)$ (see [CIL92], Appendix, Lemma A.5). So we have

$$(D\phi(x_\varepsilon), A_\varepsilon) \in \overline{J}^{2,-} u(y_\varepsilon). \tag{2.15}$$

Finally,

$$\begin{aligned} 0 &\leq F(y_\varepsilon, u(y_\varepsilon), D\phi(x_\varepsilon), A_\varepsilon) \\ &\leq F(y_\varepsilon, u(y_\varepsilon), D\phi(x_\varepsilon), D^2\phi(x_\varepsilon)) \\ &\quad \to F(x, u(x), D\phi(x), D^2\phi(x)). \end{aligned}$$

Here, we first used (2.8) and (2.15), as well as continuity of F, next we used (2.11) and elliptic degeneracy of F, and finally continuity of F, (2.13), (2.14), and the fact that $\phi \in C^2(\mathbb{R}^N)$.

Considering the inequality (2.12), this proves the assertion. ∎

2.1.2. Concavity and Subjets

Let us now also see how concavity relates to subjets. We will need these results in the following chapter, regarding the partial convexity of solutions of second order boundary value problems.

A smooth function u is concave if and only if $D^2u(x) \leq 0$ for all $x \in \Omega$. In the viscosity sense, this translates as a property of the subjet $J_\Omega^{2,-}u(x)$.

Lemma 2.1.12
Let Ω be a convex domain. Let $u \in LSC(\overline{\Omega})$ satisfy

$$\forall\ x \in \Omega \quad \forall\ (p, A) \in J_\Omega^{2,-} u(x) \quad A \leq 0.$$

Then u is concave in Ω.

Proof Assume, for contradiction, that there exist $y \neq z \in \Omega$ and $\lambda \in (0,1)$ such that

$$u(\lambda y + (1-\lambda)z) < \lambda u(y) + (1-\lambda)u(z).$$

Translating, rotating and dilating the coordinate system, we assume, w.l.o.g.

$$y = 0, \quad z = e_1.$$

In the following, we represent a point $x \in \Omega$ as $(x_1, \hat{x})$, with $x_1 = \langle x, e_1 \rangle$ and $\hat{x} = (x_2, \ldots, x_N)$.

Adding a linear function to u, we can assume, w.l.o.g.

$$\begin{aligned} u(0,0) = u(1,0) &> 0, \\ u(1-\lambda, 0) &< 0. \end{aligned}$$

Since u is lower semicontinuous, we can choose $\varepsilon > 0$ such that

$$u(0,\hat{x}),\ u(1,\hat{x}) \geq 0 \quad \text{for } |\hat{x}| \leq \varepsilon,$$

and

$$\hat{\Omega} := \{(x_1, \hat{x}) \mid x_1 \in (0,1),\ |\hat{x}| < \varepsilon\} \subset \Omega.$$

We fix $C > 0$ so large that

$$-C\varepsilon^2 \leq \inf_{\hat{\Omega}} u,$$

and $\delta > 0$ so small that

$$u(1-\lambda, 0) < -\delta\lambda(1-\lambda).$$

Define $\phi : \hat{\Omega} \to \mathbb{R}$ as follows:

$$\phi(x_1, \hat{x}) := -\delta x_1(1-x_1) - C\,|\hat{x}|^2.$$

Let us look at ϕ on $\partial\hat{\Omega}$:

- Let $x_1 \in \{0,1\}$ and $|\hat{x}| \leq \varepsilon$. Then

$$\begin{aligned} \phi(x_1, \hat{x}) &= -C\,|\hat{x}|^2 \\ &\leq 0 \\ &\leq u(x_1, \hat{x}). \end{aligned}$$

- Let $x_1 \in (0,1)$ and $|\hat{x}| = \varepsilon$. Then

$$\begin{aligned} \phi(x_1, \hat{x}) &= -\delta x_1(1-x_1) - C\,|\varepsilon|^2 \\ &\leq \underbrace{-\delta x_1(1-x_1)}_{<0} + \inf_{\hat{\Omega}} u \\ &\leq \inf_{\hat{\Omega}} u \\ &\leq u(x_1, \hat{x}). \end{aligned}$$

So, we conclude $\phi \le u$ on $\partial\hat{\Omega}$.

On the other hand,

$$\phi(1-\lambda,0) = -\delta(1-\lambda)\lambda > u(1-\lambda,0).$$

Therefore $u-\phi$ achieves an interior minimum point in $\hat{\Omega}$. But in $\hat{\Omega}$ it holds:

$$\frac{\partial^2\phi}{\partial x_1^2} = 2\delta,$$

or, written differently

$$\left\langle D^2\phi(x)e_1, e_1\right\rangle = 2\delta > 0.$$

This contradicts our assumptions on $J_\Omega^{2,-}u(x)$. ■

Remark 2.1.13
The converse is also true, i.e. if u is concave, $x \in \Omega$ and $(p,A) \in J_\Omega^{2,-}u(x)$, then $A \le 0$.

Lemma 2.1.14
Let Ω be a convex bounded domain and $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ be degenerate elliptic. Assume u is a concave function in Ω. Then u is a viscosity supersolution of $F=0$ in Ω if and only if

$$F(x,u(x),Du(x),D^2u(x)) \ge 0 \quad \text{at every point } x \in \Omega \text{ of twice differentiability.} \tag{2.16}$$

Proof Let $x \in \Omega$ be a point of twice differentiability of u. Since $(Du(x), D^2u(x)) \in J_\Omega^{2,-}u(x)$,

$$F(x,u(x),Du(x),D^2u(x)) \ge 0.$$

Conversely, let $x \in \Omega$, and let $\phi \in C^2(\Omega)$ be such that x is a strict minimum point of $u-\phi$. We have to show that, if $F(y,u(y),Du(y),D^2u(y)) \ge 0$ at every point $y \in \Omega$ of twice differentiability, then $F(x,u(x),D\phi(x),D^2\phi(x)) \ge 0$. We apply Jensen's Lemma A.3 and Aleksandrov's Theorem A.2 to the semiconcave function $u-\phi$:

we can find sequences $x_m \to x$ and $p_m \to 0$ such that x_m is a minimum point of the mapping $y \mapsto u(y)-\phi(y)-\langle p_m, y\rangle$ and u is twice differentiable at x_m. In particular,

$$Du(x_m) = D\phi(x_m) + p_m, \qquad D^2u(x_m) \ge D^2\phi(x_m). \tag{2.17}$$

Now, we have:

$$\begin{aligned} 0 &\le F(x_m,u(x_m),Du(x_m),D^2u(x_m)) \\ &\le F(x_m,u(x_m),D\phi(x_m)+p_m,D^2\phi(x_m)) \\ &\to F(x,u(x),D\phi(x),D^2\phi(x)). \end{aligned}$$

Here, we used (2.16), ellipticity of F along with (2.17), and finally the continuity of F. ■

Remark 2.1.15
Note that it is also sufficient that u is semiconcave.

Also, if Ω is a nonconvex domain, it is enough that if $K \subset \Omega$ is closed and convex, then $u\Big|_{\mathrm{Int}K}$ is (semi-)concave.

For more details on viscosity sub- and superjets, see [CIL92].

2.2. Convex Envelope

Now, we want to examine the properties of the convex envelope u_* of u. Let us remember how u_* was defined:

$$u_*(x) := \inf\left\{\sum_{i=1}^{k}\lambda_i u(x_i) \mid x = \sum_{i=1}^{k}\lambda_i x_i, \text{ with } x_i \in \overline{\Omega},\ \sum_{i=1}^{k}\lambda_i = 1,\ \lambda_i > 0,\ k \leq N+1\right\}$$

Lemma 2.2.1
Let Ω be bounded, and $u \in LSC(\overline{\Omega})$. Then, $u_* \in LSC(\overline{\Omega})$. Additionally, if there exists a convex ϕ such that $u = \phi$ on $\partial\Omega$, then $u = u_*$ on $\partial\Omega$.

Proof Obviously, because Ω is bounded, for every $x \in \Omega$ there are $x_1, \ldots, x_k \in \overline{\Omega}$ and $\lambda_1, \ldots, \lambda_k > 0$ $(k < N+1)$ such that:

$$u_*(x) = \sum_{i=1}^{k}\lambda_i u(x_i), \quad x = \sum_{i=1}^{k}\lambda_i x_i \quad \text{and} \quad \sum_{i=1}^{k}\lambda_i = 1. \tag{2.18}$$

Now, let's consider a sequence $\overline{\Omega} \ni x^m \to x$. Because of (2.18), for each $m \in \mathbb{N}$, we can find $x_1^m, \ldots, x_{N+1}^m \in \overline{\Omega}$, $\lambda_1^m, \ldots, \lambda_{N+1}^m \geq 0$ such that

$$u_*(x^m) = \sum_{i=1}^{N+1}\lambda_i^m u(x_i^m), \quad x^m = \sum_{i=1}^{N+1}\lambda_i^m x_i^m \quad \text{and} \quad \sum_{i=1}^{N+1}\lambda_i^m = 1. \tag{2.19}$$

Since Ω is bounded, and of course all (λ_i^m) are bounded, the sequence $(x_1^m, \ldots, x_{N+1}^m, \lambda_1^m, \ldots, \lambda_{N+1}^m)$ has a converging subsequence $(x_1^{m'}, \ldots, x_{N+1}^{m'}, \lambda_1^{m'}, \ldots, \lambda_{N+1}^{m'})$. Obviously, x is the convex combination of the corresponding limit point $(x_1, \ldots, x_{N+1}, \lambda_1, \ldots, \lambda_{N+1})$.

Now we have

$$\begin{aligned} \liminf_{m'\to\infty} u_*(x^{m'}) &= \liminf_{m'\to\infty} \sum_{i=1}^{N+1}\lambda_i^{m'} u(x_i^{m'}) \\ &\geq \sum_{i=1}^{N+1}\lambda_i u(x_i) \\ &\geq u_*(x). \end{aligned}$$

Here, we used first (2.19), then lower semicontinuity of u, and finally the definition of u_* (with the fact stated above, namely that x is the convex combination of $(x_1, \ldots, x_{N+1}, \lambda_1, \ldots, \lambda_{N+1})$). We thus proved the lower semicontinuity of u_*.

Ω is convex, so a boundary point x can be a convex combination of only boundary points. Let $x_i \in \partial\Omega$, $i = 1, \ldots, k$ be as in (2.18). Then we have:

$$u_*(x) = \sum_{k=1}^{N}\lambda_i u(x_i) = \sum_{k=1}^{N}\lambda_i \phi(x_i) \underset{\phi \text{ convex}}{\geq} \phi\left(\sum_{k=1}^{N}\lambda_i x_i\right) = \phi(x) = u(x). \quad \blacksquare$$

Corollary 2.2.2
If $u \in C(\overline{\Omega})$ and $u_* = u$ on $\partial\Omega$, then $u_* \in C(\overline{\Omega})$.

Proof Clearly, u_* is continuous at every interior point.

Let $(x_m) \subset \overline{\Omega}$ be a sequence with a limit in $x \in \partial\Omega$. Then we have

$$u_*(x) \leq \liminf u_*(x_m) \leq \limsup u_*(x_m) \leq \limsup u(x_m) = u(x) = u_*(x),$$

i.e. $u_* \in C(\overline{\Omega})$. ■

Suppose now that no x_i is in $\partial\Omega$ in the representation (2.18) of u_*, i.e.

$$\forall x \in \Omega \quad \exists x_1, \dots, x_k \in \Omega \text{ and } \exists \lambda_1, \dots, \lambda_k > 0 \text{ such that:}$$
$$u_*(x) = \sum_{i=1}^{k} \lambda_i u(x_i), \quad x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1. \tag{2.20}$$

Lemma 2.2.3
Let Ω be a convex bounded domain. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and let u_* be its convex envelope. Suppose that (2.20) holds, and let $x_1, \dots, x_k$ be like in (2.20). Then, for each x, if $(p, A) \in J^{2,-}_{\Omega} u_*(x)$, the following holds:

$$p = Du(x_i) \quad i = 1 \dots k \tag{2.21}$$

and

$$A \leq (\lambda_1 D^2 u(x_1)^{-1} + \dots + \lambda_k D^2 u(x_k)^{-1})^{-1}. \tag{2.22}$$

Proof Let $x \in \Omega$ and $(p, A) \in J^{2,-}_{\Omega} u_*(x)$ (such (p, A) exists, see Remark 2.1.8). Fix $h \in \mathbb{R}^N$ so small that $x + h, x - h \in \overline{\Omega}$ and $\overline{B}\left(x_1, \frac{h}{\lambda_1}\right) \subset \overline{\Omega}$. ($\overline{B}\left(x_1, \frac{h}{\lambda_1}\right)$ is the closed ball around x_1 with radius $\frac{h}{\lambda_1}$.) Now we have:

$$\begin{aligned} u_*(x) + \langle p, h\rangle &\leq u_*(x+h) \leq \lambda_1 u\left(x_1 + \frac{h}{\lambda_1}\right) + \lambda_2 u(x_2) + \dots + \lambda_k u(x_k) \\ &= u_*(x) + \langle Du(x_1), h\rangle + \frac{1}{2\lambda_1}\left\langle D^2 u(x_1)h, h\right\rangle + o(|h|^2). \end{aligned} \tag{2.23}$$

Here we used first Lemma 2.1.9, then the definition of the convex hull, and finally Taylor's expansion and (2.18). So, we obtained:

$$\langle p, h\rangle - \langle Du(x_1), h\rangle \leq \frac{1}{2\lambda_1}\left\langle D^2 u(x_1)h, h\right\rangle + o(|h|^2)$$

Analogously, (for $-h$ instead of h), we get:

$$\langle p, h\rangle - \langle Du(x_1), h\rangle \geq \frac{-1}{2\lambda_1}\left\langle D^2 u(x_1)h, h\right\rangle + o(|h|^2)$$

Sending $|h| \to 0$, we conclude $\langle p, h\rangle - \langle Du(x_1), h\rangle = 0$, i.e. $p = Du(x_1)$.

Now, if μ is an eigenvalue of $D^2 u(x_1)$ it holds:

$$\frac{\mu}{2\lambda_1}|h|^2 + o(|h|^2) \geq 0.$$

Sending $|h| \to 0$, we conclude $\mu \geq 0$. This means that $D^2 u(x_1) \geq 0$.

These assertions remain true for every i by permutation. So we proved (2.21), and in addition we have that $D^2u(x_i)$ is invertible for every $i = 1, \dots, k$. (We ignore for now the possibility $D^2u(x_i) = 0$.)

Now we estimate A. Let $h \in \mathbb{R}^N$, and let $h_1, \dots, h_k \in \mathbb{R}^N$ be such that $h = \lambda_1 h_1 + \dots + \lambda_k h_k$. Fix $r > 0$ so small that:

- $x + rh \in \overline{\Omega}$

- $x_i + rh_i \in \overline{\Omega}$ for every $i = 1, \dots, k$

- $u_*(x) + r\langle p, h\rangle + \frac{r^2}{2}\langle Ah, h\rangle + o(r^2) \leq u_*(x + rh)$ (definition of subjet)

So, we have:

$$
\begin{aligned}
u_*(x) \quad + \quad & r\langle p, h\rangle + \frac{r^2}{2}\langle Ah, h\rangle + o(r^2) \\
& \leq u_*(x + rh) \\
& \leq \lambda_1 u(x_1 + rh_1) + \dots + \lambda_k u(x_k + rh_k) \\
& \leq u_*(x) + r\langle p, h\rangle + \frac{\lambda_1 r^2}{2}\left\langle D^2u(x_1)h_1, h_1\right\rangle + \dots + \frac{\lambda_k r^2}{2}\left\langle D^2u(x_k)h_k, h_k\right\rangle + o(r^2)
\end{aligned}
$$

After simplification, sending $r \to 0$ yields:

$$
\langle Ah, h\rangle \leq \lambda_1\left\langle D^2u(x_1)h_1, h_1\right\rangle + \dots + \lambda_k\left\langle D^2u(x_k)h_k, h_k\right\rangle
$$

If we choose h_i in the following way:

$$
h_i = D^2u(x_i)^{-1}\left(\sum_{j=1}^{k}\lambda_j D^2u(x_j)^{-1}\right)^{-1} h,
$$

and plug it into the above equation, we obtain the estimate (2.22):

$$
\begin{aligned}
\langle Ah, h\rangle \;\le\; & \lambda_1 \left\langle D^2u(x_1)D^2u(x_1)^{-1}\left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h,\right.\\
& \left. D^2u(x_1)^{-1}\left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h\right\rangle + \cdots +\\
& \lambda_k \left\langle D^2u(x_k)D^2u(x_k)^{-1}\left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h,\right.\\
& \left. D^2u(x_k)^{-1}\left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h\right\rangle\\
= & \left\langle \left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h, \lambda_1 D^2u(x_1)^{-1}\left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h\right\rangle\\
& + \cdots + \left\langle \left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h, \lambda_k D^2u(x_k)^{-1}\left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h\right\rangle\\
= & \left\langle \left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h, \sum_{i=1}^k \left[\lambda_i D^2u(x_i)^{-1}\left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h\right]\right\rangle\\
= & \left\langle \left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h, \left(\sum_{i=1}^k \lambda_i D^2u(x_i)^{-1}\right)\left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h\right\rangle\\
= & \left\langle \left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1} h, h\right\rangle
\end{aligned}
$$

i.e.

$$
A \le \left(\sum_{j=1}^k \lambda_j D^2u(x_j)^{-1}\right)^{-1}. \quad \blacksquare
$$

Remark 2.2.4
It is possible that $D^2u(x_i) = 0$. In this case we replace it with $D^2u(x_i) + \frac{1}{n}I$ and send $n \to \infty$. Now

$$
\left(\sum_{i=1}^k \lambda_i \left(D^2u(x_i) + \frac{1}{n}I\right)^{-1}\right)^{-1}
$$

is a decreasing sequence, so the right hand side in

$$
A \le \lim_{n\to\infty} \left(\sum_{i=1}^k \lambda_i \left(D^2u(x_i) + \frac{1}{n}I\right)^{-1}\right)^{-1}
$$

is well defined and makes sense. We keep this remark in mind every time we use (2.22).

Now let us see what are the sufficient conditions for u_* to be a viscosity supersolution of $F = 0$. The following Theorem was proved by Alvarez-Lasry-Lions in [ALL97]:

Theorem 2.2.5
Let Ω be a convex bounded domain. Let $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ be elliptic degenerate in the sense of (2.4), and such that the mapping

$$H : \mathbb{R}^N \times \mathbb{R} \times S^N_{++} \to \mathbb{R}, \qquad H(x, r, A) := F(x, r, p, A^{-1})$$

is concave. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical solution of $F = 0$ in Ω. Then, if (2.20) holds, its convex envelope u_* is a viscosity supersolution of $F = 0$ in Ω.

Proof Fix $x \in \Omega$, and let $(p, A) \in J^{2,-}_\Omega u_*(x)$ (such (p, A) exists; see Remark 2.1.8). W.l.o.g. let $A \in S^N_+$ (see Lemma 2.1.11). Then

$$\begin{aligned}
F(x, u_*(x), p, A) &\underset{(2.20)}{=} F\left(\sum_{i=1}^k \lambda_i x_i, \sum_{i=1}^k \lambda_i u(x_i), p, A\right) \\
&\underset{(2.22),(2.4)}{\geq} F\left(\sum_{i=1}^k \lambda_i x_i, \sum_{i=1}^k \lambda_i u(x_i), p, \left(\sum_{i=1}^k \lambda_i D^2u(x_i)^{-1}\right)^{-1}\right) \\
&\underset{H \text{ concave}}{\geq} \sum_{i=1}^k \lambda_i F\left(x_i, u(x_i), p, D^2u(x_i)\right) \\
&\underset{(2.21)}{=} \sum_{i=1}^k \lambda_i F\left(x_i, u(x_i), Du(x_i), D^2u(x_i)\right) \\
&= 0.
\end{aligned}$$

The last equality comes of course from u being a classical solution of $F = 0$. So we see that $F(x, u_*(x), p, A) \geq 0$ for $x \in \Omega$, i.e. u_* is a viscosity supersolution of $F = 0$ on Ω. ■

Remark 2.2.6
We will essentially follow the same idea in chapter 4.

The requirement that H is concave can be weakened. Let $\varepsilon > 0$. Now define M_ε in the following way:

$$M_\varepsilon := H^{-1}\left([-\varepsilon, \infty)\right).$$

Corollary 2.2.7
Let Ω be a convex bounded domain. Let $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ be elliptic degenerate in the sense of (2.4), and let M_ε as defined above be convex for every $\varepsilon > 0$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution of $F = 0$ in Ω. Then, if (2.20) holds, its convex envelope u_* is a viscosity supersolution of $F = 0$ in Ω.

Proof Fix $x \in \Omega$, and let $(p, A) \in J^{2,-}_\Omega u_*(x)$ (such (p, A) exists; see Remark 2.1.8). W.l.o.g. let $A \in S^N_+$ (see Lemma 2.1.11).

Let $\varepsilon > 0$. Since u is a solution of $F = 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$:

$$-\varepsilon \leq F\left(x_i, u(x_i), Du(x_i), D^2u(x_i) + \frac{1}{n}\right) \qquad \forall i = 1, \ldots, k$$

So $\left(x_i, u(x_i), \left(D^2u(x_i) + \frac{1}{n}\right)^{-1}\right) \in M$ for every $i = 1, \ldots, k$, and since M is convex

$$\left(\underbrace{\sum_{i=1}^{k} \lambda_i x_i}_{=x}, \underbrace{\sum_{i=1}^{k} \lambda_i u(x_i)}_{=u_*(x)}, \sum_{i=1}^{k} \lambda_i \left(D^2u(x_i) + \frac{1}{n} \right)^{-1} \right) \in M.$$

Lemma 2.2.3 says that $Du(x_i) = p$ for every $i = 1, \ldots, k$, so next we have

$$F\left(x, u_*(x), p, \left(\sum_{i=1}^{k} \lambda_i \left(D^2u(x_i) + \frac{1}{n} \right)^{-1} \right)^{-1} \right) \geq -\varepsilon.$$

F is continuous, so

$$F\left(x, u_*(x), p, \lim_{n\to\infty} \left(\sum_{i=1}^{k} \lambda_i \left(D^2u(x_i) + \frac{1}{n} \right)^{-1} \right)^{-1} \right) \geq -\varepsilon.$$

Finally, because F is elliptic degenerate and (again Lemma 2.2.3)

$$A \leq \lim_{n\to\infty} \left(\sum_{i=1}^{k} \lambda_i \left(D^2u(x_i) + \frac{1}{n} I \right)^{-1} \right)^{-1},$$

we have:

$$F(x, u_*(x), p, A) \geq F\left(x, u_*(x), p, \lim_{n\to\infty} \left(\sum_{i=1}^{k} \lambda_i \left(D^2u(x_i) + \frac{1}{n} \right)^{-1} \right)^{-1} \right) \geq -\varepsilon.$$

The above holds for every $\varepsilon > 0$, so sending $\varepsilon \to 0$, we obtain

$$F(x, u_*(x), p, A) \geq 0,$$

i.e. u_* is a viscosity supersolution of $F = 0$. ∎

3. Partial Convexity of Solutions of Second Order Boundary Value Problems

As mentioned in chapter 1, we will be interested in the question of *partial convexity*:

Let $N',\ N'' \in \mathbb{N}$ such that $N' + N'' = N$ and $\Omega \subset \mathbb{R}^N$ is a convex bounded domain. If $x \in \Omega$, we write $x = (x', x'')$, where $x' \in \mathbb{R}^{N'}$ and $x'' \in \mathbb{R}^{N''}$. Suppose we have a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the boundary value problem:

$$\begin{cases} F(x, u, Du, D^2u) & = 0 \qquad \text{in } \Omega \\ \qquad\qquad\qquad u & = \phi \qquad \text{on } \partial\Omega. \end{cases}$$

We want to prove that u is convex with respect to the first variable, i.e. that if we fix x'', the mapping

$$x' \longmapsto u(x', x'')$$

is convex.

We define the *partial convex envelope* $u'_* : \overline{\Omega} \to \mathbb{R}$ of u:

$$u'_*(x', x'') := \inf \left\{ \sum_{i=1}^{k} \lambda_i u(x'_i, x'') \mid x' = \sum_{i=1}^{k} \lambda_i x'_i, \text{ with } (x'_i, x'') \in \overline{\Omega}, \right.$$
$$\left. \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i > 0,\ k \leq N' + 1 \right\},$$

and we follow the same scheme as in chapter 2:

Obviously, u_* is convex with respect to the first variable and

$$u'_* \leq u \quad \text{on } \overline{\Omega} \tag{3.1}$$

If we can prove that:

i) $u'_* = u$ on $\partial\Omega$,

ii) u'_* is a viscosity supersolution of $F = 0$ in Ω,

iii) a suitable comparison principle holds,

then $u'_* \geq u$, that is, $u'_* = u$. Hence, u is convex with respect to the first variable.

As previously, one can see that for every $(x', x'') \in \Omega$ there are $(x'_1, x''), \ldots, (x'_k, x'') \in \overline{\Omega}$ and $\lambda_1, \ldots, \lambda_k > 0$ $(k < N' + 1)$ such that:

$$u'_*(x', x'') = \sum_{i=1}^{k} \lambda_i u(x'_i, x''), \quad x' = \sum_{i=1}^{k} \lambda_i x'_i \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1, \tag{3.2}$$

and that $u'_* \in$LSC$(\overline{\Omega})$.

Analogously to Lemma 2.1.9, we have:

If $((p', p''), A) \in J^{2,-}_{\Omega} u'_*(x', x'')$, we have for every $(y', x'') \in \overline{\Omega}$:

$$u'_*(x', x'') + \langle p', y' - x' \rangle \quad \leq \quad u'_*(y', x'') \tag{3.3}$$

As before, we have $J^{2,-}_{\Omega} u'_*(x) \neq \emptyset$ for every $x \in \Omega$.

Also, we will assume:

$$\text{in representation (3.2), none of the } (x'_i, x''),\ i = 1, \ldots k \text{ are from } \partial\Omega \tag{3.4}$$

Before we go on to see under which conditions is u'_* a viscosity supersolution of $F = 0$, we introduce additional notations:

- we shall write $p \in \mathbb{R}^N$ in the form $(p', p'') \in \mathbb{R}^{N'} \times \mathbb{R}^{N''}$;
- we shall write $A \in S^N$ in the form $\begin{pmatrix} a & b \\ b^T & c \end{pmatrix}$ where $a \in S^{N'},\ b \in \mathbb{R}^{N' \times N''},\ c \in S^{N''}$
- we set

$$G(x', x''; r; p', p''; a, b, c) := F\left((x', x''), r, (p', p''), \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}\right). \tag{3.5}$$

What is a sufficient condition for u'_* to be a viscosity supersolution of $F = 0$?

Theorem 3.0.8
Let Ω be a convex bounded domain. Let $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ be elliptic degenerate in the sense of (2.4) *and let it satisfy:*

$$(x', r, p'', a, b, c) \longmapsto G(x', x''; r; p', p''; a^{-1}, a^{-1}b, c + b^T a^{-1} b) \quad \text{is concave} \tag{3.6}$$

for every $(x', r, p'', a, b, c) \in \mathbb{R}^{N'} \times \mathbb{R} \times \mathbb{R}^{N''} \times S^{N'} \times \mathbb{R}^{N' \times N''} \times S^{N''}$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical solution of $F = 0$ in Ω. Then, if (3.4) *holds, its partial convex envelope u'_* is a viscosity supersolution of $F = 0$ in Ω.*

Proof Let $(x', x'') \in \Omega$ and $(p, A) \in J^{2,-}_{\Omega} u'_*(x', x'')$. Let $x_i,\ i = 1, \ldots, k$ be as in (3.4). Let us introduce the following notations:

- $x_i = (x'_i, x''), \quad x = (x', x'')$
- $(p'_i, p''_i) = Du(x_i)$
- $\begin{pmatrix} a_i & b_i \\ b_i^T & c_i \end{pmatrix} = D^2u(x_i)$

Now, for every $h'_i \in \mathbb{R}^{N'}$, $h' = \sum_{i=1}^{k} \lambda_i h'_i$, $h'' \in \mathbb{R}^{N''}$ small enough it holds (if $h = (h', h'')$ and $h_i = (h'_i, h''))$:

$$\begin{aligned} u'_*(x', x'') \quad + \quad & \langle p', h' \rangle + \langle p'', h'' \rangle + \frac{1}{2} \langle Ah, h \rangle + o(|h|^2) \\ & \leq u'_*(x' + h', x'' + h'') \\ & \leq \sum_{i=1}^{k} \lambda_i u(x'_i + h'_i, x'' + h'') \\ & \leq u'_*(x', x'') + \sum_{i=1}^{k} \lambda_i \langle p'_i, h'_i \rangle + \sum_{i=1}^{k} \lambda_i \langle p''_i, h'' \rangle + \sum_{i=1}^{k} \frac{\lambda_i}{2} \langle D^2u(x_i)h_i, h_i \rangle \\ & \quad + o(\sum_{i=1}^{k} |h_i|^2). \end{aligned}$$

Here, we used here the definition of subjet, then the definition of partial convex envelope, and finally Taylor expansion.

a) First, let i be arbitrary, and set

$$h'_j = \begin{cases} h'/\lambda_i & j = i, \\ 0 & j \neq i \end{cases}$$

and

$$h'' = 0.$$

By partial convexity (see (3.3)), we have:

$$\begin{aligned} u'_*(x', x'') + \langle p', h' \rangle \quad & \leq \quad u'_*(x' + h', x'') \\ & \leq \quad \sum_{j=1}^{k} \lambda_j u(x'_j + h'_j, x'') \\ & \leq \quad u'_*(x', x'') + \langle p'_i, h' \rangle + \frac{1}{2\lambda_i} \langle D^2u(x_i)h_i, h_i \rangle + o(|h_i|^2). \end{aligned}$$

´ As in proof of Lemma 2.2.2, we conclude:

$$p' = p'_i \qquad a_i \geq 0. \tag{3.7}$$

b) Next, let h'' be arbitrary and set

$$h_j = 0 \qquad \forall j$$

We conclude, by sending $|h''| \to 0$:

$$p'' = \sum_{i=1}^{k} \lambda_i p''_i. \tag{3.8}$$

c) Finally, let $h \in \mathbb{R}^N$ be arbitrary. We are left with the following inequality:

$$\begin{aligned}
\langle Ah, h\rangle &\leq \sum_{i=1}^{k} \lambda_i \left\langle D^2 u(x_i) h_i, h_i \right\rangle \\
&= \sum_{i=1}^{k} \lambda_i \left\langle \begin{pmatrix} a_i h_i' + b_i h'' \\ b_i^T h_i' + c_i h'' \end{pmatrix}, \begin{pmatrix} h_i' \\ h'' \end{pmatrix} \right\rangle \\
&= \sum_{i=1}^{k} \lambda_i \left[\langle a_i h_i', h_i' \rangle + 2 \langle b_i h'', h_i' \rangle + \langle c_i h'', h'' \rangle \right] \qquad (3.9)
\end{aligned}$$

We choose h_i' in the following way:

$$h_i' = a_i^{-1} \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} \right)^{-1} \left[h' + \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} b_j \right) h'' \right] - a_i^{-1} b_i h''.$$

Plugging h_i' into (3.9), we conclude that:

$$A \leq \begin{pmatrix} a' & b' \\ b'^T & c' \end{pmatrix} \qquad (3.10)$$

where

$$\begin{aligned}
a' &= \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} \right)^{-1} \\
b' &= \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} \right)^{-1} \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} b_i \right) \qquad (3.11) \\
c' &= \sum_{i=1}^{k} \lambda_i \left(c_i - b_i^T a_i^{-1} b_i \right) + \left(\sum_{i=1}^{k} \lambda_i b_i^T a_i^{-1} \right) \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} \right)^{-1} \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} b_i \right).
\end{aligned}$$

(See appendix, section B for this calculation, since it is lengthy.)

u is a solution of $F = 0$, i.e:

$$G(x_i', x''; u(x_i', x''); p_i', p_i''; a_i, b_i, c_i) = 0.$$

Set

$$\begin{aligned}
\alpha_i &:= a_i^{-1} \\
\beta_i &:= a_i^{-1} b_i \qquad (3.12) \\
\gamma_i &:= c_i - b_i^T a_i^{-1} b_i = c_i - \beta_i^T \alpha_i^{-1} \beta_i.
\end{aligned}$$

Then, we have

$$\begin{aligned}
a_i &= \alpha_i^{-1} \\
b_i &= a_i \beta_i = \alpha_i^{-1} \beta_i \\
c_i &= \gamma_i + \beta_i^T \alpha_i^{-1} \beta_i.
\end{aligned}$$

Now, with above defined α_i, β_i and γ_i, and with (3.7), the fact that u is a solution reads:

$$G(x_i', x''; u(x_i', x''); p', p_i''; \alpha_i^{-1}, \alpha_i^{-1}\beta_i, \gamma_i + \beta_i^T \alpha_i^{-1} \beta_i) = 0. \qquad (3.13)$$

Finally, we have:

$$\begin{aligned}
0 \underset{(3.13)}{=} & \sum_{i=1}^{k} \lambda_i G(x_i', x''; u(x_i', x''); p', p_i''; \alpha_i^{-1}, \alpha_i^{-1}\beta_i, \gamma_i + \beta_i^T \alpha_i^{-1}\beta_i) \\
\underset{(3.6)}{\leq} & G\left(\sum_{i=1}^{k} \lambda_i x_i', x''; \sum_{i=1}^{k} \lambda_i u(x_i', x''); p', \sum_{i=1}^{k} \lambda_i p_i''; \left(\sum_{i=1}^{k} \lambda_i \alpha_i\right)^{-1}, \right. \\
& \left(\sum_{i=1}^{k} \lambda_i \alpha_i\right)^{-1} \left(\sum_{i=1}^{k} \lambda_i \beta_i\right), \left(\sum_{i=1}^{k} \lambda_i \gamma_i\right) + \\
& \left. \left(\sum_{i=1}^{k} \lambda_i \beta_i^T\right) \left(\sum_{i=1}^{k} \lambda_i \alpha_i\right)^{-1} \left(\sum_{i=1}^{k} \lambda_i \beta_i\right)\right) \\
\underset{(3.8),(3.2),(3.12)}{=} & G\left(x', x''; u_*'(x', x''); p', p''; \left(\sum_{i=1}^{k} \lambda_i a_i^{-1}\right)^{-1}, \right. \\
& \left(\sum_{i=1}^{k} \lambda_i a_i^{-1}\right)^{-1} \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} b_i\right), \sum_{i=1}^{k} \lambda_i \left(c_i - b_i^T a_i^{-1} b_i\right) + \\
& \left. \left(\sum_{i=1}^{k} \lambda_i b_i^T a_i^{-1}\right) \left(\sum_{i=1}^{k} \lambda_i a_i^{-1}\right)^{-1} \left(\sum_{i=1}^{k} \lambda_i a_i b_i\right)\right) \\
\underset{(3.11)}{=} & G(x', x''; u_*'(x', x''); p', p''; a', b', c') \\
\underset{(2.4),(3.10)}{\leq} & F(x, u_*'(x), p, A).
\end{aligned}$$

So, we proved that u_*' is a supersolution of $F = 0$. ■

Remark 3.0.9
Note that the proof does not require Ω to be convex. It is enough that the set

$$\left\{x' \in \mathbb{R}^{N'} \mid (x', x'') \in \Omega\right\}$$

is convex for every $x'' \in \mathbb{R}^{N''}$.

Corollary 3.0.10
Let Ω be a convex bounded domain, and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical solution of $F = 0$ in Ω. Then, if (3.4) holds, its partial convex envelope u_*' is semiconcave in every compact subset of Ω.

Proof Let K be a compact subset of Ω. Then the set of all (x_i', x'') as in (3.4) for every $(x', x'') \in K$ is clearly compact. Since u is $C^2(\Omega)$ there exists a constant $C > 0$ such that $D^2u(x_i', x'') \leq CI$ for such (x_i', x'').

Also, we have $J^{2,-}_{\text{Int}K} u_*'(x', x'') = J^{2,-}_{\Omega} u_*'(x', x'')$. Let $(p, A) \in J^{2,-}_{\text{Int}K} u_*'(x', x'')$.

We set $h'_i = h'$ in (3.9), which now reads

$$\begin{aligned}\langle Ah, h\rangle &\leq \sum_{i=1}^{k} \lambda_i \left[\langle a_i h', h'\rangle + 2\langle b_i h'', h'\rangle + \langle c_i h'', h''\rangle\right] \\ &= \sum_{i=1}^{k} \lambda_i \langle D^2 u(x_i) h, h\rangle \\ &\leq \langle CIh, h\rangle\end{aligned}$$

So, we obtain the following inequality

$$\langle (A - CI)h, h\rangle \leq 0.$$

Since h is arbitrary, this means $A - CI \leq 0$. Now, if K is convex, Lemma 2.1.12 says that $u'_* - C\,\|(x', x'')\|^2$ is concave in Int$K$, i.e. that u'_* is semiconcave in Int$K$. ∎

Remark 3.0.11
If Ω is as in Remark 3.0.9,then the above Corollary also holds.

Later, we will be interested in proving partial convexity in only one direction, i.e. when $N' = 1$. In this case the condition (3.6) can be greatly simplified.

Theorem 3.0.12
Let $N' = 1$, and let Ω be a convex bounded domain. Let $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ be elliptic degenerate (i.e. let (2.4) hold) and satisfy:

$$(x', r, p'', c) \longmapsto G(x', x''; r; p', p''; 0, b, c) \quad \text{is concave} \tag{3.14}$$

for every $(x', r, p'', c) \in \mathbb{R}^{N'} \times \mathbb{R} \times \mathbb{R}^{N''} \times S^{N''}$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a classical solution of $F = 0$ in Ω. Then, if (3.4) holds, its partial convex envelope u'_* is a viscosity supersolution of $F = 0$ in Ω.

Proof We keep the notations of the proof of Theorem 3.0.8.

Corollary 3.0.10 tells us that u'_* is semiconcave, so we can apply Lemma 2.1.14: to prove that u'_* is a supersolution, we only need to prove that

$$F(x, u'_*(x), Du'_*(x), D^2u'_*(x)) \geq 0 \quad \text{at every point } x \text{ of twice differentiability of } u'_*.$$

So, we set $A = D^2u'_*(x)$, and because of partial convexity we have

$$a \geq 0. \tag{3.15}$$

We claim that (3.15) implies $a = 0$, unless $u'_*(x) = u(x)$.

Because of partial convexity of u'_*, we have:

$$u'_*(x', x'') + p'h' \leq u'_*(x' + h', x'').$$

Choose $h' = x'_i - x'$:

$$\begin{aligned}u'_*(x', x'') + p'(x'_i - x') &\leq u'_*(x'_i, x'') \\ &\leq u(x'_i, x''),\end{aligned}$$

i.e.

$$u(x_i', x'') - u_*'(x', x'') - p'(x_i' - x') \geq 0.$$

But

$$\begin{aligned}
\sum_{i=1}^{k} \lambda_i \underbrace{\left(u(x_i', x'') - u_*'(x', x'') - p'(x_i' - x'\right)}_{\geq 0} &= \underbrace{\sum_{i=1}^{k} \lambda_i u(x_i', x'')}_{=u_*'(x',x'')} - u_*'(x', x'') \cdot \underbrace{\sum_{i=1}^{k} \lambda_i}_{=1} \\
&\quad -p'\Big[\underbrace{\sum_{i=1}^{k} \lambda_i x_i'}_{=x'} - x' \cdot \underbrace{\sum_{i=1}^{k} \lambda_i}_{=1}\Big] \\
&= 0.
\end{aligned}$$

So, we conclude:

$$u(x_i', x'') - u_*'(x', x'') - p'(x_i' - x') = 0. \tag{3.16}$$

By definition of subjet, we have ($x = (x', x'')$ as before):

$$\begin{aligned}
u_*'(x + r(x_i - x)) &\geq u_*'(x) + rp(x_i - x) + \frac{r^2}{2} \langle A(x_i - x)(x_i - x)) \rangle \\
&= u_*'(x) + rp'(x_i' - x') + \frac{r^2}{2} a(x_i' - x')^2.
\end{aligned}$$

On the other hand (if $r \in (0, 1)$):

$$\begin{aligned}
u_*'(x + r(x_i - x)) &\leq (1 - r)u_*'(x) + ru_*'(x_i) \\
&\leq (1 - r)u_*'(x) + ru(x_i) \\
&= u_*'(x) + rp'(x_i' - x').
\end{aligned}$$

Here we used first the partial convexity of u_*', then its definition, and finally (3.16). Combining the above two strings of inequalities and sending $r \to 0$, we obtain:

$$a(x_i' - x')^2 \leq 0. \tag{3.17}$$

But (3.15) holds ($a \geq 0$), so necessarily

$$a = 0. \tag{3.18}$$

We assumed that $u_*'(x) \neq u(x)$; otherwise (3.17) would read $a \cdot 0 = 0$, and we could not reach the conclusion $a = 0$. But since u is a supersolution of $F = 0$, in points where $u_*'(x) = u(x)$ there is nothing to prove.

From the proof of Theorem 3.0.8 we have the inequality (3.9):

$$\langle Ah, h \rangle \leq \sum_{i=1}^{k} \lambda_i \left[a_i(h_i')^2 + 2h_i' \cdot b_i h'' + \langle c_i h'', h'' \rangle \right].$$

Plugging in $a = 0$, we get:

$$h' \cdot 2bh'' + \langle ch'', h'' \rangle \leq \sum_{i=1}^{k} \lambda_i \left[a_i(h_i')^2 + 2h_i' \cdot b_i h'' + \langle c_i h'', h'' \rangle \right].$$

Choosing $h_i' = -\frac{(b_i-b)^T(b_i-b)}{a_i}$ (assuming as before w.l.o.g. $a_i > 0$), we obtain

$$c \leq \sum_{i=1}^{k} \lambda_i \left(c_i - \frac{(b_i - b)^T(b_i - b)}{a_i} \right). \tag{3.19}$$

On the other hand, since

$$\begin{pmatrix} \alpha & \beta \\ \beta^T & \frac{\beta^T\beta}{\alpha} \end{pmatrix} \geq 0 \quad \forall \alpha > 0 \quad \forall \beta \in \mathbb{R}^{1\times N''},$$

it holds:

$$\begin{pmatrix} a_i & b_i \\ b_i^T & c_i \end{pmatrix} \geq \begin{pmatrix} 0 & b \\ b^T & c_i - \frac{(b_i-b)^T(b_i-b)}{a_i} \end{pmatrix}. \tag{3.20}$$

Finally, we have:

$$\begin{aligned} 0 &= \sum_{i=1}^{k} \lambda_i G(x_i', x''; u(x_i, x''); p_i', p_i''; a_i, b_i, c_i) \\ &\underset{(3.7)}{=} \sum_{i=1}^{k} \lambda_i G(x_i', x''; u(x_i, x''); p', p_i''; a_i, b_i, c_i) \\ &\underset{(2.4),(3.20)}{\leq} \sum_{i=1}^{k} \lambda_i G\left(x_i', x''; u(x_i, x''); p', p_i''; 0, b, c_i - \frac{(b_i - b)^T(b_i - b)}{a_i}\right) \\ &\underset{(3.14),(3.8),(3.19)}{\leq} G(x', x''; u_*(x, x''); p', p''; 0, b, c) \\ &\underset{(3.18),(3.5)}{=} F(x, u_*'(x), p, A). \quad \blacksquare \end{aligned}$$

Remark 3.0.13
In light of remarks 2.1.15, 3.0.9 and 3.0.11, we have that the above Theorem holds even if Ω is not convex, but such that the set

$$\left\{x' \in \mathbb{R} \mid (x', x'') \in \Omega\right\}$$

is convex for every $x'' \in \mathbb{R}^{N''}$.

Remark 3.0.14
Analogously to Corollary 2.2.7, conditions (3.6) and (3.14) can be weakened. However, later in our specific equation in chapter 6, it will not make a difference, so we write the conditions as presented by Alvarez-Lasry-Lions.

4. Concavity of the Solution of the Isotropic BVP

In this chapter, we turn to the following specific problem:

$$\begin{cases} \Delta u + c(x)u^{-\gamma} = 0 & \text{in} \quad \Omega \\ u(x) = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.1}$$

where Ω is a convex, bounded domain.

4.1. Existence and Uniqueness Results

The following Theorem was proved by Lazer-McKenna in [LM91]:

Theorem 4.1.1
Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded domain with smooth boundary $\partial\Omega$ (of class $C^{2+\alpha}, 0 < \alpha < 1$). If $c \in C^\alpha(\overline{\Omega}), c(x) > 0$ for all $x \in \overline{\Omega}$ and $\gamma > 0$, then there exists a unique function $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ such that $u(x) > 0$ for all $x \in \Omega$ and u is a solution of (4.1).

In [LM91] a lower and upper bound for the solution u is also given:

Theorem 4.1.2
Let $\gamma > 1$, and let Φ_1 denote an eigenfunction corresponding to the smallest eigenvalue λ_1 of the problem

$$\begin{cases} \Delta\Phi + \lambda\Phi = 0 & \textit{in} \quad \Omega \\ \Phi = 0 & \textit{on} \quad \partial\Omega \end{cases}$$

such that $\Phi_1(x) > 0$ on Ω.
Then there exist positive constants b_1 and b_2 such that

$$b_1\Phi_1(x)^{\frac{2}{1+\gamma}} \leq u(x) \leq b_2\Phi_1(x)^{\frac{2}{1+\gamma}} \quad \textit{on} \quad \overline{\Omega}. \tag{4.2}$$

This result will be useful for us later on.

4.2. Concavity for $\gamma \geq 2$ and a Concave c, and for $\gamma > 1$ and a Constant c

We examine under which conditions is the unique solution of the problem (4.1) concave, or equivalently, when is $v := -u$ convex. If we rewrite (4.1) for v, it becomes:

$$\begin{cases} -\Delta v + c(x)(-v)^{-\gamma} = 0 & \text{in} \quad \Omega \\ \quad v(x) = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.3}$$

We follow the method presented in chapter 2. Therefore, we assume now that Ω is convex.

Let v be the unique solution of (4.3), and let v_* be the its convex envelope:

$$v_*(x) := \inf\left\{\sum_{i=1}^{k} \lambda_i v(x_i) \mid x = \sum_{i=1}^{k} \lambda_i x_i, \text{ with } x_i \in \overline{\Omega}, \sum_{i=1}^{k} \lambda_i = 1, \; \lambda_i > 0, \; k \leq N+1\right\}$$

Let us repeat conditions from chapter 2 which we need to prove that v is convex:

i) $v_* = v = 0$ on $\partial\Omega$,

ii) v_* is a viscosity supersolution of (4.1) in Ω,

iii) a suitable comparison principle holds, i.e. if w is a viscosity supersolution of (4.1) and v is a classical solution of (4.1), then $w \geq v$.

Let us see if and when these conditions hold:

i): Clearly, because Ω is convex, and $\phi \equiv 0$ is convex, by Lemma 2.2.1:

$$v_*(x) = 0 \quad \text{on } \partial\Omega \tag{4.4}$$

ii): How would we define F as in chapter 2? An obvious way would be (since $v_* < 0$)

$$F : \Omega \times (-\infty, 0) \times S^N \to \mathbb{R} \qquad F(x, r, A) := -\text{trace}A + c(x)(-r)^{-\gamma}.$$

F defined in that way is clearly continuous and elliptic degenerate in the sense of (2.4). Thus, according to Lemma 2.1.11, v_* is a viscosity supersolution of $F = 0$ in Ω if and only for every $x \in \Omega$

$$F(x, v_*(x), A) \geq 0 \qquad \forall (p, A) \in J_{\Omega}^{2,-} v_*(x) \text{ such that } A \geq 0.$$

However, the mapping

$$H : \Omega \times (-\infty, 0) \times S^N \to \mathbb{R} \qquad H(x, r, A) := -\text{trace}A^{-1} + c(x)(-r)^{-\gamma}$$

is not concave. Therefore, we cannot use theorem 2.2.5 directly. Since the above condition is not necessary for proving that v_* is a viscosity supersolution of $F = 0$, we try to go around it.

So, we have that v_* will be a viscosity supersolution of (4.1) in Ω if and only if for every $x \in \Omega$:

$$-\text{trace}(A) + c(x)(-v_*)^{-\gamma} \geq 0 \qquad \forall (p, A) \in J_{\Omega}^{2,-} v_*(x) \text{ such that } A \geq 0.$$

So let $x \in \Omega$ and $(p, A) \in J_{\Omega}^{2,-} v_*(x)$ be such that $A \geq 0$.

If trace$(A) = 0$, the above obviously holds, since $c > 0$ and $v_* < 0$. So now suppose trace$(A) > 0$. Then v_* will be a supersolution if and only if:

$$\frac{1}{\operatorname{trace}(A)} \geq \frac{(-v_*(x))^\gamma}{c(x)}. \tag{4.5}$$

Next, let $x_1, \ldots, x_k \in \overline{\Omega}$ and $\lambda_1, \ldots, \lambda_k > 0$ be such that:

$$v_*(x) = \sum_{i=1}^k \lambda_i v(x_i), \quad x = \sum_{i=1}^k \lambda_i x_i \quad \text{and} \quad \sum_{i=1}^k \lambda_i = 1.$$

We will show later that none of the x_i are on $\partial\Omega$. We assume for the time being that this is so. In this case Lemma 2.2.3 gives us the estimate

$$A \leq (\lambda_1 D^2 v(x_1)^{-1} + \cdots + \lambda_k D^2 v(x_k)^{-1})^{-1}.$$

for $(p, A) \in J_\Omega^{2,-} v_*(x)$.

The trace operator is monotone in the cone of nonnegative matrices, so we have:

$$\operatorname{trace}(A) \leq \operatorname{trace}((\lambda_1 D^2 v(x_1)^{-1} + \cdots + \lambda_k D^2 v(x_k)^{-1})^{-1}) \tag{4.6}$$

Also, $Q \to (\operatorname{trace}(Q^{-1}))^{-1}$ is concave in the cone of positive matrices (see [Tru]), so:

$$\begin{aligned} \left[\operatorname{trace}\left(\sum_{i=1}^k \lambda_i D^2 v(x_i)^{-1}\right)^{-1}\right]^{-1} &\geq \sum_{i=1}^k \lambda_i \left[\operatorname{trace}(D^2 v(x_i))\right]^{-1} \\ &= \sum_{i=1}^k \lambda_i \left[\Delta v(x_i)\right]^{-1} \end{aligned} \tag{4.7}$$

(4.6) and (4.7) now give:

$$\begin{aligned} \frac{1}{\operatorname{trace}(A)} &\geq \sum_{i=1}^k \lambda_i \left[\Delta v(x_i)\right]^{-1} \\ &= \sum_{i=1}^k \lambda_i \frac{(-v(x_i))^\gamma}{c(x_i)}. \end{aligned} \tag{4.8}$$

If the mapping $(x, s) \longmapsto \frac{(-s)^\gamma}{c(x)}$ is convex, then (4.8) and (2.20) give us:

$$\begin{aligned} \frac{1}{\operatorname{trace}(A)} &\geq \sum_{i=1}^k \lambda_i \frac{(-v(x_i))^\gamma}{c(x_i)} \\ &\geq \frac{\left(-\sum_{i=1}^k \lambda_i v(x_i)\right)^\gamma}{c\left(\sum_{i=1}^k \lambda_i x_i\right)} \\ &= \frac{(-v_*(x))^\gamma}{c(x)}, \end{aligned}$$

and this is exactly (4.5).

Lemma 4.2.1
The mapping $(x,s) \longmapsto \frac{(-s)^\gamma}{c(x)}$ *where* $(x,s) \in \mathbb{R}^N \times (-\infty, 0)$ *is convex, if*

- $\gamma \geq 2$ *and* $c \in C^2(\Omega)$ *is a concave function such that* $c > 0$ *on* $\overline{\Omega}$*;*

- $\gamma \geq 1$ *and* $c \equiv m$*, where* m *is a positive constant.*

Proof Let $g(x,s) := \frac{(-s)^\gamma}{c(x)}$, and let us write the Hessian matrix of g.

$$\begin{aligned}
\frac{\partial^2 g}{\partial x_j \partial x_i} &= \frac{2(-s)^\gamma}{[c(x)]^3} \cdot \frac{\partial c}{\partial x_j} \cdot \frac{\partial c}{\partial x_i} - \frac{(-s)^\gamma}{[c(x)]^2} \cdot \frac{\partial^2 c}{\partial x_j \partial x_i} \\
\frac{\partial^2 g}{\partial x_j \partial s} &= \frac{\gamma(-s)^{\gamma-1}}{[c(x)]^2} \cdot \frac{\partial c}{\partial x_j} \\
\frac{\partial^2 g}{\partial s^2} &= \frac{\gamma(\gamma-1)(-s)^{\gamma-2}}{c(x)}
\end{aligned}$$

Now, we can write the Hessian in the following way:

$$D^2 g(x,s) = q^T q + \mathbf{M} \tag{4.9}$$

where

$$q := \sqrt{\frac{2(-s)^\gamma}{c^3(x)}} \begin{pmatrix} \frac{\partial c}{\partial x_1} \\ \vdots \\ \frac{\partial c}{\partial x_k} \\ \frac{\gamma c(x)}{2(-s)} \end{pmatrix}$$

and

$$\mathbf{M} := \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \frac{\gamma(\gamma-2)(-s)^{\gamma-2}}{2c(x)} \end{pmatrix}, \quad (\mathbf{B})_{i,j} = -\frac{(-s)^\gamma}{c^2(x)} \cdot \frac{\partial^2 c}{\partial x_j \partial x_i}(x)$$

$q^T q$ is clearly positive semidefinite. Namely, if λ is its eigenvalue, then:

$$q^T q x = \lambda x \Rightarrow x^T q^T q x = \lambda x^T x, \text{ so } \lambda = \frac{x^T q^T q x}{x^T x} = \frac{\|qx\|}{\|x\|} \geq 0.$$

On the other hand, if c is concave, then $\mathbf{B} \geq 0$. If, in addition, $\gamma \geq 2$, then $\mathbf{M} \geq 0$. From (4.9) it is now clear that $D^2 g(x,s) \geq 0$, i.e. that $g(x,s)$ is convex.

If c is a positive constant, then the Hessian of g is the following matrix

$$D^2 g(x,s) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\gamma(\gamma-1)(-s)^{\gamma-2}}{m} \end{pmatrix}.$$

Now the second assertion is clear. ∎

To obtain that v_* is a viscosity supersolution (for $\gamma \geq 2$, $c \in C^2$ concave, or $\gamma \geq 1$, c a constant), it still has to be shown that none of the x_i as above are on $\partial\Omega$. We will show that holds for $\gamma > 1$. We first need an additional result.

Lemma 4.2.2
Let $d_{\partial\Omega} : \mathbb{R}^N \to \mathbb{R}$ *be the distance from* $\partial\Omega$*, i.e.*

$$d_{\partial\Omega}(x) := \inf_{y \in \partial\Omega} |x - y|.$$

Then, there exist constants $C_1, C_2 > 0$ *such that*

$$C_1 \; d_{\partial\Omega}(x) \leq \Phi_1(x) \leq C_2 \; d_\partial\Omega(x) \qquad \forall x \in \overline{\Omega}. \tag{4.10}$$

Proof Suppose, to the contrary, that for every $C_1 > 0$ there exists $x_0 \in \overline{\Omega}$ such that

$$C_1 d_{\partial\Omega}(x_0) > \Phi_1(x_0),$$

i.e. that there exists $(x_k)_{k\in\mathbb{N}} \subset \Omega$ such that $x_k \to x_0$ and

$$\frac{\Phi_1(x_k)}{d_{\partial\Omega}(x_k)} \underset{x_k \to x_0}{\to} 0. \tag{4.11}$$

If $x_0 \in \Omega$, this is obviously not true since $d_{\partial\Omega}(x_0) > 0$ and $\Phi_1(x_0) > 0$.

Hence, let $x_0 \in \partial\Omega$.

It is known that Φ_1 is superharmonic, so we have by Hopf's Lemma (see Theorem A.4)

$$\frac{\partial \Phi_1}{\partial \nu}(x_0) < 0.$$

On the other hand, it is also clear that

$$\left| \frac{\partial d_{\partial\Omega}}{\partial \nu}(x_0) \right| = 1,$$

And, because $\Phi_1 = d_{\partial\Omega} = 0$ on $\partial\Omega$, we have (τ is of course a tangential vector at x_0)

$$\begin{aligned} \frac{\partial \Phi_1}{\partial \tau}(x_0) &= 0 \\ \frac{\partial d_{\partial\Omega}}{\partial \tau}(x_0) &= 0. \end{aligned}$$

So, we obtained:

$$|\nabla \Phi_1(x_0)| \cdot \frac{1}{|\nabla d_{\partial\Omega}(x_0)|} > 0. \tag{4.12}$$

Now we have

$$\begin{aligned} \frac{\Phi_1(x_k)}{d_{\partial\Omega}(x_k)} &= \frac{\Phi_1(x_k) - \Phi_1(x_0)}{d_{\partial\Omega}(x_k) - d_{\partial\Omega}(x_0)} \\ &= \frac{\Phi_1(x_k) - \Phi_1(x_0)}{|x_k - x_0|} \cdot \frac{|x_k - x_0|}{d_{\partial\Omega}(x_k) - d_{\partial\Omega}(x_0)} \\ &\underset{x_k \to x_0}{\to} |\nabla \Phi_1(x_0)| \cdot \frac{1}{|\nabla d_{\partial\Omega}(x_0)|} \underset{(4.12)}{>} 0. \end{aligned}$$

Thus, we obtained a contradiction to (4.11), i.e. there is such $C_1 > 0$ which fulfills $C_1 \ d_{\partial\Omega}(x_0) \leq \Phi_1(x_0)$, where $x_0 \in \overline{\Omega}$ is arbitrary.

Analogously, one proves that there is a $C_2 > 0$ which fulfills $C_2 \ d_{\partial\Omega}(x) \geq \Phi_1(x)$, for every $x \in \overline{\Omega}$. ∎

Theorem 4.2.3
Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a convex bounded domain with smooth boundary $\partial\Omega$ (of class $C^{2+\alpha}, 0 < \alpha < 1$). Let $c \in C^\alpha(\overline{\Omega}), c(x) > 0$ for all $x \in \Omega$ and $\gamma > 1$. Let v be the unique solution of (4.3), and let v_* be its convex envelope. For $x \in \Omega$ let $x_1, \ldots, x_k \in \overline{\Omega}$ and $\lambda_1, \ldots, \lambda_k > 0$ be such that

$$v_*(x) = \sum_{i=1}^{k} \lambda_i v(x_i), \quad x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1.$$

Then, $x_i \notin \partial\Omega$ for every $i = 1, \ldots, k$.

Proof Let $(p, A) \in J_{\Omega}^{2,-} u_*(x)$ (such (p, A) exists, see Remark 2.1.8). Suppose, for contradiction, and w.l.o.g. that $x_1 \in \partial\Omega$. Let ν be the inner normal unit vector at x_1, and choose $t > 0$ so small that $x_1 + \frac{t\nu}{\lambda_1} \in \overline{\Omega}$. Now we have:

$$\begin{aligned} v_*(x) + \langle p, t\nu\rangle &\leq v_*(x + t\nu) \\ &\leq \lambda_1 v\left(x_1 + \frac{t\nu}{\lambda_1}\right) + \lambda_2 v(x_2) + \cdots + \lambda_k v(x_k) + \underbrace{\lambda_1 v(x_1)}_{=0} \\ &= v_*(x) + \lambda_1 v\left(x_1 + \frac{t\nu}{\lambda_1}\right) \end{aligned}$$

Here, we used Lemma 2.1.9, definition of convex envelope, and (2.18), respectively. Next, we have (since $\gamma > 1$):

$$\begin{aligned} \langle p, t\nu\rangle &\leq \lambda_1 v\left(x_1 + \frac{t\nu}{\lambda_1}\right) \\ &\underset{(4.2)}{\leq} -\lambda_1 b_1 \Phi_1^{\frac{2}{1+\gamma}}\left(x_1 + \frac{t\nu}{\lambda_1}\right) \\ &\underset{(4.10)}{\leq} -K\left[d_{\partial\Omega}\left(x_1 + \frac{t\nu}{\lambda_1}\right)\right]^{\frac{2}{1+\gamma}}, \quad K > 0 \text{ const.} \end{aligned}$$

So, this means that (for t small enough):

$$\overbrace{-\langle p, t\nu\rangle}^{t|p| \geq \langle p, -t\nu\rangle =} \geq K\left[d_{\partial\Omega}\left(x_1 + \frac{t\nu}{\lambda_1}\right)\right]^{\frac{2}{1+\gamma}} = \tilde{K} t^{\frac{2}{1+\gamma}}, \quad \tilde{K} > 0 \text{ const.}$$

Finally, we have:

$$t \geq \tilde{\tilde{K}} t^{\frac{2}{1+\gamma}}, \quad \tilde{\tilde{K}} > 0 \text{ const.}$$

Sending $t \to 0$ we obtain a contradiction. ■

We proved that v_* is a viscosity supersolution of (4.1) if $\gamma \geq 2$ and c is a concave C^2 function, or if $\gamma > 1$ and c is constant.

iii): The suitable comparison principle will be for a viscosity supersolution and a classical solution of (4.1).

Lemma 4.2.4
Let w be a viscosity supersolution of (4.1)*, and v its unique classical solution. Then $w \geq v$.*

Proof Suppose, for contradiction, that $w - v$ achieves a negative minimum in $\hat{x} \in \Omega$. Let $\varepsilon > 0$. Theorem (A.1) now says that there exists $A \in S^N$ such that $(Dv(\hat{x}), A) \in \overline{J}_{\Omega}^{2,-} w(\hat{x})$, and the following inequality holds:

$$A \geq D^2 v(\hat{x}) - \varepsilon \left(D^2 v(\hat{x})\right)^2. \tag{4.13}$$

Trace operator is linear and monotone, so this means:

$$\text{trace}(A) \geq \underbrace{\text{trace}\left(D^2 v(\hat{x})\right)}_{=\Delta v(\hat{x})} - \varepsilon \cdot \text{trace}\left(\left(D^2 v(\hat{x})\right)^2\right). \tag{4.14}$$

Since w is a supersolution of (4.1), and the mapping

$$(x, r, p, A) \mapsto -\text{trace}A + c(\hat{x})\,(-w(\hat{x}))^{-\gamma}$$

is continuous, it holds:

$$0 \leq -\text{trace}A + c(\hat{x})\,(-w(\hat{x}))^{-\gamma}\,. \tag{4.15}$$

On the other hand, since $w(\hat{x}) < v(\hat{x})$, we have

$$-c(\hat{x})\,(-w(\hat{x}))^{-\gamma} \geq -c(\hat{x})\,(-v(\hat{x}))^{-\gamma} + \mu \tag{4.16}$$

where $\mu > 0$. Now we have:

$$\begin{aligned} 0 \underset{(4.15)}{\geq} & \ \text{trace}(A) - c(\hat{x})\,(-w(\hat{x}))^{-\gamma} \\ \underset{(4.16)}{\geq} & \ \text{trace}(A) - c(\hat{x})\,(-v(\hat{x}))^{-\gamma} + \mu \\ \underset{(4.14)}{\geq} & \ \Delta v(\hat{x}) - \varepsilon \cdot \text{trace}\left(\left(D^2 v(\hat{x})\right)^2\right) - c(\hat{x})\,(-v(\hat{x}))^{-\gamma} + \mu. \end{aligned}$$

Sending $\varepsilon \to 0$, we have, since v is the solution of (4.1):

$$0 \geq \Delta v(\hat{x}) - c(\hat{x})\,(-v(\hat{x}))^{-\gamma} + \mu = \mu.$$

However, $\mu > 0$, so we obtained a contradiction. ■

So, we proved that if in addition to the conditions of Theorem 4.1.1 we have that

- Ω is convex,
- $\gamma \geq 2$,
- c is concave and in $C^2(\Omega)$,

then v is convex. This means of course that $u = -v$ is concave. Thus, we have the following Theorem:

Theorem 4.2.5
Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded convex domain with smooth boundary $\partial\Omega$ (of class $C^{2+\alpha}, 0 < \alpha < 1$). Let $c \in C^{\alpha}(\overline{\Omega}) \cap C^2(\Omega)$ be a concave function such that $c(x) > 0$ for all $x \in \Omega$. Let $\gamma \geq 2$. Then the unique solution of

$$\begin{cases} \Delta u + c(x) u^{-\gamma} = 0 & \textit{in} \quad \Omega \\ u(x) = 0 & \textit{on} \quad \partial\Omega \end{cases}$$

is concave.

Also, we proved that if in addition to the conditions of Theorem 4.1.1 we have that

- Ω is convex,
- $\gamma > 1$,
- c is a constant,

then v is convex. This means of course that $u = -v$ is concave. Thus, we have the following Theorem:

Theorem 4.2.6
Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a bounded convex domain with smooth boundary $\partial\Omega$ (of class $C^{2+\alpha}, 0 < \alpha < 1$). Let $m > 0$ and $\gamma > 1$. Then the unique solution of

$$\begin{cases} \Delta u + m u^{-\gamma} = 0 & \text{in} \quad \Omega \\ u(x) = 0 & \text{on} \quad \partial\Omega \end{cases}$$

is concave.

5. Lower Bound for a Solution of the Anisotropic BVP

We consider the following problem:

$$\begin{cases} u^\alpha u_{xx} + u^\beta u_{yy} + q(x,y) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^2$ is a convex, bounded domain, $\alpha > \beta \geq 0$. In this chapter, we wish to find a lower bound for a solution u in case it exists.

5.1. The Existence Result and the Comparison Lemma

The following Theorem was proved by Choi-Lazer-McKenna in [CLM95]:

Theorem 5.1.1
Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary $\partial\Omega$ (of class $C^{2+\gamma}, 0 < \gamma < 1$). If $q \in C^\gamma(\overline{\Omega}), q(x) > 0$ for all $x \in \overline{\Omega}$ and $\alpha > \beta \geq 0$, then there exists a function $u \in C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$ such that $u(x) > 0$ for all $x \in \Omega$ and u is a solution of (5.1).

Choi-Lazer-McKenna also proved the following comparison principle:

Lemma 5.1.2
Let q be continuous, and suppose there exist positive constants m and M such that $m \leq q \leq M$ on $\overline{\Omega}$. If $u \in C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$ is a positive solution of (5.1), and $w \in C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$ is a positive subsolution of (5.1) such that $w_{xx} \leq 0$, then $w \leq u$ on $\overline{\Omega}$. Analogously, if $v \in C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$ is a positive supersolution of (5.1) such that $v_{xx} \leq 0$, then $v \geq u$ on $\overline{\Omega}$.

As of yet, for the anisotropic problem, we do not have an analogous result to Theorem 4.1.2. Here, we obtain that result partly:

We prove that there exists a $K > 0$ such that

$$u(x) \geq K \left[d_{\partial\Omega}(x,y)\right]^{\frac{2}{1+\beta}} \qquad \forall (x,y) \in \overline{\Omega} \tag{5.2}$$

if in addition to the conditions of the above Theorem, $\beta > 1$.

5.2. The Lower Bound on Small Domains

Here, we prove that in domains Ω which are "small", the solution of the isotropic boundary value problem

$$\begin{cases} \Delta u + mu^{-\beta} = 0 & \text{in} \quad \Omega \\ u(x) = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{5.3}$$

is a subsolution for the problem (5.1) when $\beta > 1$, and $q \geq m > 0$. Because of the comparison principle 5.1.2, Theorem 4.1.2, and Lemma 4.2.2, this would give us the wanted lower bound (5.2) on such domains.

To prove this, we first need the following Lemma:

Lemma 5.2.1
Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain, $t \in \mathbb{R}$ and $\Omega_t := t\Omega$. Let ω_t be the solution of (5.3) on Ω_t. Then there exists a $t_0 > 0$ such that for every t such that $t_0 > t > 0$

$$\|\omega_t\|_\infty \leq 1. \tag{5.4}$$

Proof If ω is the solution on Ω, then

$$\omega_t(x, y) := t^{\frac{2}{1+\beta}} \omega\left(\frac{x}{t}, \frac{y}{t}\right)$$

is the solution on Ω_t:

$$\begin{aligned} \Delta\omega_t + m\omega_t^{-\beta} &= \Delta\left[t^{\frac{2}{1+\beta}}\omega\left(\frac{x}{t}, \frac{y}{t}\right)\right] + m\left[t^{\frac{2}{1+\beta}}\omega\left(\frac{x}{t}, \frac{y}{t}\right)\right]^{-\beta} \\ &= t^{\frac{-2\beta}{1+\beta}}\left[\Delta\omega\left(\frac{x}{t}, \frac{y}{t}\right) + m\omega\left(\frac{x}{t}, \frac{y}{t}\right)^{-\beta}\right] \\ &= 0. \end{aligned}$$

So, we can conclude:

$$t \to 0 \Rightarrow \|\omega_t\|_\infty \to 0,$$

i.e. we can choose t so small so that:

$$\|\omega_t\|_\infty \leq 1. \quad \blacksquare$$

Definition 5.2.2
Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary $\partial\Omega$. Let $\beta \geq 0$. We will call Ω ***small*** *if the unique solution ω of*

$$\begin{cases} \Delta\omega + m\omega^{-\beta} = 0 & \textit{in} \quad \Omega \\ \omega(x) = 0 & \textit{on} \quad \partial\Omega \end{cases}$$

is such that $\|\omega\|_\infty \leq 1$ on Ω. Otherwise, we will call Ω ***large****.*

Remark 5.2.3
Obviously, if Ω is small, then if $\tilde{\Omega} \subset \Omega$ is bounded convex domain with smooth boundary, then $\tilde{\Omega}$ is also small. To prove this assertion, assume ω is the unique solution of (5.3) on Ω, and $\tilde{\omega}$ on $\tilde{\Omega}$. We know that ω is positive inside Ω, i.e. also on $\partial\tilde{\Omega}$. This means, since of course ω solves the partial differential equation inside $\tilde{\Omega}$, that ω is a supersolution of

(5.3) on $\tilde{\Omega}$. We also know that ω is concave (Theorem 4.2.6), i.e. that $\omega_{xx} \le 0$. Sine $\tilde{\omega}$ is positive, we can use now Lemma 5.1.2. We conclude that

$$\omega \ge \tilde{\omega} \quad \text{on } \tilde{\Omega}.$$

Since Ω is small (in the sense of the definition above), $\|\omega\|_\infty \le 1$ on Ω, and of course on $\tilde{\Omega}$. Now it is clear that $\|\tilde{\omega}\|_\infty \le 1$ on $\tilde{\Omega}$, i.e. that $\tilde{\Omega}$ is small.

Now we can easily see that on small domains Ω, if $\alpha > \beta > 1$ and $q \ge m$, the solution ω of (5.3) is a subsolution of the anisotropic boundary value problem (5.1):

Since $\beta > 1$, Theorem 4.2.6 says that ω is concave, which implies:

$$\omega_{xx} \le 0. \tag{5.5}$$

Now we have:

$$\begin{aligned} \omega^\alpha \omega_{xx} + \omega^\beta \omega_{yy} + q(x,y) &= \omega^\beta \left[\omega^{\alpha-\beta}\omega_{xx} + \omega_{yy}\right] + q(x,y) \\ &\underset{(5.5),(5.4)}{\ge} \omega^\beta \left[\omega_{xx} + \omega_{yy}\right] + m \\ &= 0, \end{aligned}$$

i.e. ω is a subsolution of the problem (5.1) on Ω.

Suppose that u is a positive solution of the problem (5.1) on Ω. If q is continuous, then we have by the comparison Lemma 5.1.2:

$$u \ge \omega \qquad \text{on} \quad \Omega \tag{5.6}$$

Now, (4.2), (4.10) and (5.6) give us the lower bound for u on Ω:

$$u(x,y) \ge \; K\left[d_{\partial\Omega_t}(x,y)\right]^{\frac{2}{1+\beta}} \qquad \forall (x,y) \in \overline{\Omega}_t \tag{5.7}$$

where K is a positive constant.
We have proved:

Lemma 5.2.4
Let $\Omega \subset \mathbb{R}^2$ be a small bounded convex domain with smooth boundary $\partial\Omega$ (of class $C^{2+\gamma}, 0 < \gamma < 1$). Let $\alpha > \beta > 1$ and let $q \in C^\gamma(\overline{\Omega}) \cap C(\overline{\Omega})$, $q(x) \ge m > 0$. Let u solve (5.1) on Ω. Then, there exists a $K > 0$ such that

$$u(x,y) \ge \; K\left[d_{\partial\Omega}(x,y)\right]^{\frac{2}{1+\beta}} \qquad \forall (x,y) \in \overline{\Omega}.$$

5.3. The Lower Bound on Large Domains

Clearly, the requirement that the domain be small is undesirable, so we want to remove this restriction in this chapter. For this, we need the following Lemma.

Lemma 5.3.1
For $i = 1, \dots, m$ let Ω, Ω_i be convex bounded domains with smooth boundaries $\partial\Omega$, $\partial\Omega_i$ such that it holds: $\Omega = \bigcup_{i=1,..,k} \Omega_i$. Let u be a solution of (5.1) on Ω, and u_i on Ω_i. Also suppose that for every Ω_i there exists a $K_i > 0$ such that it holds:

$$u_i(x,y) \ge \; K_i\left[d_{\partial\Omega_i(x,y)}\right]^{\frac{2}{1+\beta}} \qquad \forall (x,y) \in \overline{\Omega}_i.$$

Then there exists $K > 0$ such that it holds:

$$u(x,y) \ge \; K\left[d_{\partial\Omega(x,y)}\right]^{\frac{2}{1+\beta}} \qquad \forall (x,y) \in \overline{\Omega}.$$

Proof Let $u_1, \ldots, u_m$ solve (5.1) on $\Omega_1, \ldots, \Omega_m$, respectively. We have, because of classical maximum principle

$$u(x,y) \geq u_1(x,y) \qquad \forall (x,y) \in \overline{\Omega}_1,$$
$$\vdots$$
$$u(x,y) \geq u_m(x,y) \qquad \forall (x,y) \in \overline{\Omega}_m.$$

We claim that for every $j \in \{1, \ldots, m\}$ there exists a $K'_j > 0$ such that

$$u(x,y) \geq \ K'_j \left[d_{\partial\Omega}(x,y)\right]^{\frac{2}{1+\beta}} \qquad \forall (x,y) \in \overline{\Omega}_j.$$

Suppose to the contrary that there is $j \in \{1, \ldots, m\}$ such that for every $K > 0$ there exists $(x_0, y_0) \in \overline{\Omega}_j$ such that

$$u(x_0,y_0) \leq \ K \left[d_{\partial\Omega}(x_0,y_0)\right]^{\frac{2}{1+\beta}},$$

i.e. that there exists $((x_k, y_k))_{k\in\mathbb{N}} \subset \overline{\Omega}_j$ such that $(x_k, y_k) \to (x_0, y_0) \in \overline{\Omega}_j$ and

$$\frac{u(x_k,y_k)}{\left[d_{\partial\Omega}(x_0,y_0)\right]^{\frac{2}{1+\beta}}} \underset{(x_k,y_k)\to(x_0,y_0)}{\longrightarrow} 0.$$

Of course, if $(x_0, y_0) \in \overline{\Omega}_j \setminus \partial\Omega$, (i.e. $(x_0, y_0) \in \Omega$), this cannot be true, since $u(x_0, y_0) > 0$, and $d_{\partial\Omega}(x_0, y_0) > 0$.

On the other hand, if $(x_0, y_0) \in \partial\Omega$, (i.e. since $\Omega_j \subset \Omega$ this means $(x_0, y_0) \in \partial\Omega_j$), we have, by above inequalities:

$$u(x_0,y_0) \geq u_j(x_0,y_0) \geq \ K_j \left[d_{\partial\Omega_j}(x_0,y_0)\right]^{\frac{2}{1+\beta}} = K_j \left[d_{\partial\Omega}(x_0,y_0)\right]^{\frac{2}{1+\beta}}.$$

So, we have a contradiction, and taking

$$K := \min\left\{K'_1, K'_2, \ldots, K'_m\right\}$$

we get

$$u(x,y) \geq \ K \left[d_{\partial\Omega}(x,y)\right]^{\frac{2}{1+\beta}} \qquad \forall (x,y) \in \overline{\Omega}. \qquad \blacksquare$$

Let Ω be a large domain, and suppose u solves (5.1) on Ω. Let $\sigma > 0$.

$$\Omega_1 := \{(x,y) \in \Omega \ : \ d_{\partial\Omega}(x,y) \geq \sigma\}, \quad \Omega_2 := \Omega \setminus \Omega_1.$$

It is clear that there exists a $K_1 > 0$ such that

$$u(x,y) \geq \ K_1 \left[d_{\partial\Omega}(x,y)\right]^{\frac{2}{1+\beta}} \qquad \forall (x,y) \in \overline{\Omega}_1. \tag{5.8}$$

The idea is (because of lemmas 5.2.4 and 5.3.1) to cover Ω_2 with finitely many small domains, $\Omega_2^{(1)}, \ldots, \Omega_2^{(m)}$. Then if $\alpha > \beta > 1$ and $q \geq m > 0$ is continuous, we obtain the lower bound on Ω.

We have to construct $\Omega_2^{(i)}$, and choose a suitable σ.

5.3.1. The Construction of the Cover

Let $\varepsilon_1 > 0$ be defined as follows:

$$\varepsilon_1 = \sup\{r \mid B_r(0,0) \text{ is small in the sense of definition 5.2.2}\}.$$

($B_r(x_0, y_0)$ is of course the open ball around (x_0, y_0) of radius r.)

Let $(x_0, y_0) \in \partial\Omega$. Now, let $\varepsilon_2 : \partial\Omega \to \mathbb{R}$ be such that:

$$\begin{aligned}\varepsilon_2(x_0, y_0) = \sup\{r \mid \partial\Omega \cap B_r(x_0, y_0) &\text{ is a graph of a } C^{2+\gamma} \text{ function} \\ &\text{(after a suitable coordinate system movement)}\}.\end{aligned}$$

For every $(x_0, y_0) \in \partial\Omega$ choose $\varepsilon > 0$ in the following way:

$$\varepsilon =: \varepsilon(x_0, y_0) = \min\{\varepsilon_1, \varepsilon_2(x_0, y_0)\}.$$

Let $(x_0, y_0) \in \partial\Omega$ be arbitrary. Rotate the coordinate system so that $\partial\Omega \cap B_\varepsilon(x_0, y_0)$ is a graph of a convex function $\varphi \in C^{2+\gamma}$. (Choice of ε allows us to do that.) Also, translate the origin of the coordinate system in a point $(\tilde{x}, \tilde{y}) \in \Omega$ such that

$$d_{\partial\Omega}(\tilde{x}, \tilde{y}) = \|(x_0, y_0) - (\tilde{x}, \tilde{y})\| = \frac{\varepsilon}{3}. \tag{5.9}$$

After this movement of the coordinate system, (x_0, y_0) becomes $(\tilde{x}_0, \tilde{y}_0)$, and φ becomes $\tilde{\varphi}$. Since $\tilde{\varphi}$ is convex and smooth, and because of the choice of the origin, $\tilde{\varphi}(\tilde{x}_0) < 0$. (It can't be 0 because that is a contradiction to smoothness of $\tilde{\varphi}$.) It is clear from the choice of the origin, that

$$\begin{aligned}|\tilde{x}_0| &\leq \frac{\varepsilon}{3} \\ |\tilde{\varphi}(\tilde{x}_0)| &\leq \frac{\varepsilon}{3}.\end{aligned}$$

Since $\tilde{\varphi}$ is continuous, there exist $a \in [-\frac{\sqrt{2}\varepsilon}{3}, \tilde{x}_0) \setminus \{0\}$ and $b \in (\tilde{x}_0, \frac{\sqrt{2}\varepsilon}{3}] \setminus \{0\}$ such that

$$\tilde{\varphi}(x) < 0 \quad \text{and } \tilde{\varphi}(x) \geq -\frac{\sqrt{2}\varepsilon}{3} \qquad \forall x \in [a, b]. \tag{5.10}$$

Let $a_0 := a + \frac{|\tilde{x}_0 - a|}{2}$, and $b_0 := b - \frac{|\tilde{x}_0 - b|}{2}$, and define $\psi : [a, b] \to \mathbb{R}$ in the following way:

$$\psi(x) := \begin{cases} -\tilde{\varphi}(a)\left[1 - \frac{1}{a_0 - a}\sqrt{-(x-a)^2 + 2(x-a)(a_0 - a)}\right]^4 & a \leq x \leq a_0; \\ 0 & a_0 < x < b_0; \\ -\tilde{\varphi}(b)\left[1 - \frac{1}{b - b_0}\sqrt{-(x-b)^2 + 2(b-x)(b-b_0)}\right]^4 & b_0 \leq x \leq b. \end{cases}$$

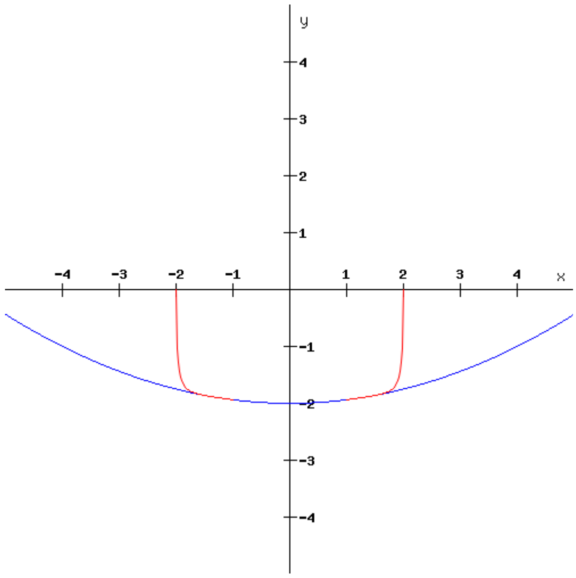


Figure 5.1.: Here, $\varepsilon = 6, a = -2, a_0 = -1, b_0 = 1$ and $b = 2$. The blue graph is of the function $\tilde{\varphi}$, and the red of $\tilde{\varphi} + \psi$.

Now $\tilde{\varphi} + \psi : [a, b]$ is a $C^{2+\gamma}$ function. This is clear since $\tilde{\varphi}, \psi \in C^{2+\gamma}$ and:

$$\begin{aligned} \psi(a_0) &= (b_0) = 0; \\ \psi'(a_0) &= \psi'(b_0) = 0; \\ \psi''(a_0) &= \psi''(b_0) = 0; \\ \psi^{(3)}(a_0) &= \psi^{(3)}(b_0) = 0. \end{aligned} \tag{5.11}$$

Furthermore, $\tilde{\varphi} + \psi$ is convex, since both $\tilde{\varphi}$ and ψ are convex.

Also, clearly

$$(\tilde{\varphi} + \psi)(a) = (\tilde{\varphi} + \psi)(b) = 0.$$

Suppose $(x_k^1)_{k \in \mathbb{N}}$ is such that $x_k^1 \underset{k \to \infty}{\to} a$, and $(x_k^2)_{k \in \mathbb{N}}$ is such that $x_k^2 \underset{k \to \infty}{\to} b$. Then it holds:

$$\begin{aligned} &\psi'(x_k^1) \underset{k \to \infty}{\to} -\infty, \\ &\psi'(x_k^2) \underset{k \to \infty}{\to} -\infty. \end{aligned} \tag{5.12}$$

The idea now is to mirror $\tilde{\varphi} + \psi$ around the x-axis, i.e. our domain U will be the area bounded by graphs of functions $\pm(\tilde{\varphi} + \psi) : [a, b] \to \mathbb{R}$.

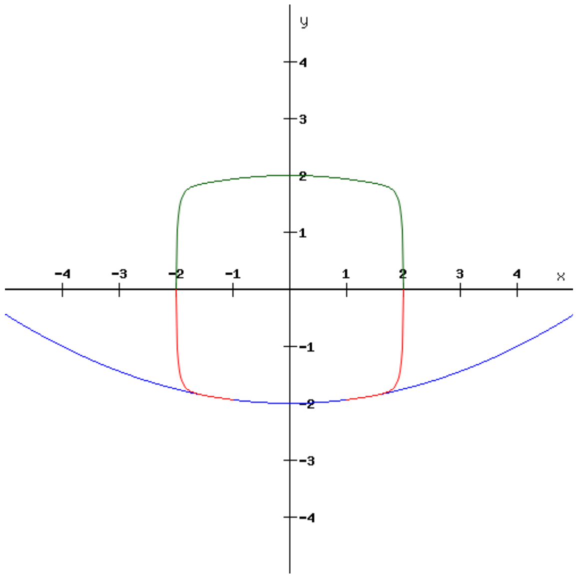


Figure 5.2.: Mirroring around the x-axis.

Remark 5.3.2
We still need to prove the claims stated above, namely convexity of ψ, (5.11), and (5.12). We check this in appendix, section C. That still is not enough though to show that U is indeed $C^{2+\gamma}$. Namely, we may have problems in points $(a,0)$ and $(b,0)$. We need in addition the following (if $f:[a,a_0]\to\mathbb{R}$ is such that $f(x):=\psi|_{[a,a_0]}$ and $g:[b_0,b]\to\mathbb{R}$ is such that $g(x):=\psi|_{[b_0,b]}$):

$$\begin{aligned} \frac{f''}{f'}(a) &= 0, & \frac{f^{(3)}}{(f')^4}(a) &= 0,\\ \frac{g''}{g'}(b) &= 0, & \frac{g^{(3)}}{(g')^4}(b) &= 0. \end{aligned}$$

That we need above conditions and that they hold is also checked in appendix, section C.

Note that because of (5.10) the area U is a subset of the rectangle $[-\frac{\sqrt{2}\varepsilon}{3},\frac{\sqrt{2}\varepsilon}{3}]\times[-\frac{\sqrt{2}\varepsilon}{3},\frac{\sqrt{2}\varepsilon}{3}]$ $\subset B_{\frac{2\varepsilon}{3}}(\tilde{x},\tilde{y})$, and it holds: $\|(x_0,y_0)-(\tilde{x},\tilde{y})\|=\frac{\varepsilon}{3}$. ((5.9)). Thus, for every $(x,y)\in U$ it holds:

$$\begin{aligned} \|(\tilde{x}_0,\tilde{y}_0)-(x,y)\| &= \|(\tilde{x}_0,\tilde{y}_0)-(\tilde{x},\tilde{y})+(\tilde{x},\tilde{y})-(x,y)\|\\ &\le \|(\tilde{x}_0,\tilde{y}_0)-(\tilde{x},\tilde{y})\|+\|(\tilde{x},\tilde{y})-(x,y)\|\\ &\le \frac{\varepsilon}{3}+\frac{2\varepsilon}{3}=\varepsilon, \end{aligned}$$

i.e. $(x,y)\in B_\varepsilon(\tilde{x}_0,\tilde{y}_0)$. So, $U\subset B_\varepsilon(\tilde{x}_0,\tilde{y}_0)$. From the choice of ε, we see that U is small.

Let $A\in\mathbb{R}^2$ be an open set such that $A\cap\Omega=\emptyset$ and $\emptyset\neq(\partial A\cap\partial\Omega)\subset(\partial U\cap\partial\Omega)$. Now we set $U_{(x_0,y_0)}:=U\cup A$.

We can construct U $(U_{(x_0,y_0)})$ as above for every $(x_0,y_0)\in\partial\Omega$.

$\{U_{(x_0,y_0)}\mid(x_0,y_0)\in\partial\Omega\}$ covers $\partial\Omega$, and since $\partial\Omega$ is compact, there exists a finite subcover

$$M := \left\{U_{(x_i,y_i)}\mid(x_i,y_i)\in\partial\Omega, i=1,\dots m\right\}.$$

The set $\partial\Omega \cap \partial U_{(x_i,y_i)}$ contains exactly two points for every i: $(x_1^i, y_1^i),\ (x_2^i, y_2^i)$.

Let ν_j^i be the inner unit normal vector at (x_j^i, y_j^i) and let

$$t_j^i := \sup\left\{t \mid (x_j^i, y_j^i) + t\nu_j^i \in U_{(x_i,y_i)}\right\},$$

and finally we choose σ:

$$\sigma := \min_{i,j} t_j^i,$$

and $\Omega_2^{(i)}$:

$$\Omega_2^{(i)} := \Omega \cap U_{(x_i,y_i)}.$$

Let us remember how we defined Ω_1 and Ω_2:

$$\Omega_1 := \{(x,y) \in \Omega \ : \ d_{\partial\Omega}(x,y) > \sigma\}, \quad \Omega_2 = \Omega \setminus \Omega_1.$$

Now it is clear from the choice of σ and convexity of $\Omega_2^{(i)}$ that $\left\{\Omega_2^{(i)} \mid i = 1, \dots .m\right\}$ covers Ω_2.

Because $\Omega_2^{(i)}$ is small (for $i = 1, \dots, m$) with a smooth boundary and because of (5.8), we can now use Lemma 5.3.1. So, we obtained the desired result also on large domains. In conclusion we have the following Theorem:

Theorem 5.3.3
Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary $\partial\Omega$, let $\alpha > \beta > 1$ and let $q \in C^\gamma(\overline{\Omega}) \cap C(\overline{\Omega})$, $q(x) \geq m > 0$. Suppose u is a solution of

$$\begin{cases} u^\alpha u_{xx} + u^\beta u_{yy} + q(x,y) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Then there exists $K > 0$ such that it holds:

$$u(x,y) \geq \ K\left[d_{\partial\Omega}(x,y)\right]^{\frac{2}{1+\beta}} \qquad \forall (x,y) \in \overline{\Omega}.$$

Remark 5.3.4
Note that we only need the continuity of q on Ω_2. We can choose σ very small, and thus lessen the restriction on q.

6. Partial Concavity of a Solution of the Anisotropic BVP. Consequent Uniqueness.

In this chapter we consider the concavity properties of the anisotropic problem:

$$\begin{cases} u^\alpha u_{xx} + u^\beta u_{yy} + q(x,y) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6.1}$$

($\Omega \subset \mathbb{R}^2$ is a convex, bounded domain, and $\alpha > \beta \geq 0$.)

Choi-Lazer-McKenna gave the following uniqueness result in [CLM95]:

Lemma 6.0.5
If there exist $m, M > 0$ such that $M \geq q(x) \geq m > 0$ for all $x \in \overline{\Omega}$, then there is at most one positive solution u of (6.1) with $u_{xx} \leq 0$.

Thus, we wish to see when a positive solution u is concave in x-direction, as that would also give us the uniqueness of the solution.

6.1. Partial Concavity

Equivalently, we wish to see when $v := -u$ is convex in x-direction. We rewrite (6.1) for v:

$$\begin{cases} -(-v)^\alpha v_{xx} - (-v)^\beta v_{yy} + q(x,y) = 0 & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6.2}$$

The above is equivalent to the following:

$$\begin{cases} -(-v)^{\alpha-\beta} v_{xx} - v_{yy} + (-v)^{-\beta} q(x,y) = 0 & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6.3}$$

Let v be a negative solution of (6.3), and let v'_* be its partial convex envelope (as in section 3):

$$v'_*(x,y) := \inf \left\{ \sum_{i=1}^{k} \lambda_i v(x_i, y) \mid x = \sum_{i=1}^{k} \lambda_i x_i, \text{ with } (x_i, y) \in \overline{\Omega}, \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i > 0,\ k \le 2 \right\}.$$

So, in this case,

$$F : \Omega \times (-\infty, 0) \times \mathbb{R}^N \times S^N \to R,$$
$$F\left((x,y), r, (p', p''), \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}\right) := -(-r)^{\alpha-\beta} a - c + (-r)^{-\beta} q(x,y)$$

is our F from (2.1), and $\phi \equiv 0$. Of course:

$$G(x,y;r;p',p'';a,b,c) = F\left((x,y), r, (p', p''), \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}\right).$$

Let us repeat at this point the conditions we need in order for v to be convex with respect to the x-variable:

i) $v'_* = 0$ on $\partial\Omega$,

ii) v'_* is a viscosity supersolution of $F = 0$ in Ω,

iii) a suitable comparison principle holds, i.e. if v is a classical solution of $F = 0$, and its partial convex envelope v'_* is a viscosity supersolution of $F = 0$, then $v'_* \ge v$.

Let's see if and when these conditions hold:

i): $v'_* = 0$ clearly holds on $\partial\Omega$.

ii): Let's repeat the sufficient conditions that v'_* is a viscosity supersolution of $F = 0$ in Ω:

1) For every $((x,y), r, p) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$,
$F((x,y), r, p, A) \le F((x,y), r, p, B)$ if $B \le A$ (elliptic degeneracy),

2) $F \in C(\Omega \times (-\infty, 0) \times \mathbb{R}^N \times S^N)$,

3) For $x \in \Omega$ let $x_1, x_2 \in \overline{\Omega}$ and $\lambda > 0$ be such that:

$$v'_*(x) = \lambda v(x_1) + (1-\lambda) v(x_2), \quad x = \lambda x_1 + (1-\lambda) x_2.$$

Then neither x_1 nor x_2 are in $\partial\Omega$.

4) The mapping $(x, r, p'', c) \longmapsto G(x,y;r;p',p'';0,b,c)$ is concave.

Let us see if and when these conditions hold:

1): F is clearly elliptic degenerate.

2): Since $r < 0$, F is continuous.

3): For $\alpha > \beta > 1$ and a continuous q, because of Theorem 5.3.3, we can obtain this result exactly as before in Theorem 4.2.3.

4): When is the mapping $(x', r, p'', c) \mapsto G(x', x''; r; p', p''; 0, b, c)$ concave?

We have:

$$G(x, y; r; p', p''; 0, b, c) = -c + (-r)^{-\beta} q(x, y),$$

so (3.6) comes down to concavity of the following mapping:

$$(x, r, c) \mapsto -c + (-r)^{-\beta} q(x, y).$$

The Hessian of this mapping is then:

$$H(x, r, c) = \begin{pmatrix} (-r)^{-\beta} \frac{\partial^2 q}{\partial x^2} & \beta(-r)^{-\beta-1} \frac{\partial q}{\partial x} & 0 \\ \beta(-r)^{-\beta-1} \frac{\partial q}{\partial x} & \beta(\beta+1)(-r)^{-\beta-2} q & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, it comes down to the following matrix:

$$\tilde{H}(x, r) := \begin{pmatrix} (-r)^{-\beta} \frac{\partial^2 q}{\partial x^2} & \beta(-r)^{-\beta-1} \frac{\partial q}{\partial x} \\ \beta(-r)^{-\beta-1} \frac{\partial q}{\partial x} & \beta(\beta+1)(-r)^{-\beta-2} q \end{pmatrix}.$$

If $\beta = 0$, $\det\tilde{H} = 0$, and $\operatorname{trace}\tilde{H} = \frac{\partial^2 q}{\partial x^2}$.

Thus, if q is concave in x-direction, $\operatorname{trace}\tilde{H} \le 0$. This means $\tilde{H}$ is negative semidefinite, meaning, in turn, that the mapping $(x, r, c) \mapsto -c + (-r)^{-\beta} q(x, y)$ is concave.

Sadly, we were not able to get concavity of $(x', r, p'', c) \mapsto G(x', x''; r; p', p''; 0, b, c)$ for $\beta > 1$, for which 3) holds.

The only hope for proving that v'_* is a viscosity supersolution of $F = 0$ is for $\beta = 0$, q concave in x-direction. We prove in the next section 6.2 that in case $\beta = 0$ 3) holds on some special Ω.

iii): We compare a classical solution v with its partial convex envelope v'_* which is also a supersolution.

Lemma 6.1.1
Let Ω be a convex bounded domain. Let v be a classical solution of (6.2)*, and v'_* its partial convex envelope (such that 3) holds) and a viscosity supersolution of* (6.2)*. Then $v'_* \ge v$.*

Proof Assume, for contradiction, that $v'_* - v$ has a strict negative minimum, i.e. there exists $(\hat{x}, \hat{y}) \in \Omega$ such that

$$(v'_* - v)(\hat{x}, \hat{y}) < 0. \tag{6.4}$$

Then, for $\varepsilon > 0$ small enough, it also holds

$$(v'_* + \varepsilon h - v)(\hat{x}, \hat{y}) < 0,$$

where $h(x, y) := e^{(x-\hat{x})^2 + (y-\hat{y})^2}$. The function $v'_* + \varepsilon h - v$ is semiconcave (Corollary 3.0.10 tells us v'_* is semiconcave, h and v are twice continuously differentiable in Ω), so we can use Jensen's Lemma A.3 and Aleksandrov's Theorem: A.2:

There exist sequences $(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) \to (\hat{x}, \hat{y})$ and $p^\varepsilon_m \to 0$ such that v'_* is twice differentiable at $(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m)$ and $(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m)$ is the minimum point of the mapping

$$(x, y) \mapsto v'_*(x, y) + \varepsilon h(x, y) - v(x, y) - \langle p^\varepsilon_m, (x, y)\rangle \, .$$

In particular,

$$D^2 v'_*(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) \geq D^2 v(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) - \varepsilon D^2 h(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m).$$

Since the Hessian is symmetric, it holds:

$$v'_{*,xx}(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) \geq v_{xx}(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) - \varepsilon h_{xx}(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m), \tag{6.5}$$

$$v'_{*,yy}(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) \geq v_{yy}(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) - \varepsilon h_{yy}(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m). \tag{6.6}$$

Since v'_* is convex in x-direction, it holds:

$$v'_{*,xx}(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) \geq 0 \quad \forall m. \tag{6.7}$$

Let $M \in \mathbb{N}$ be such that $(v'_* - v)(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) < 0$ for $m \geq M$. Now we have:

$$\begin{aligned} 0 \;&\geq\; \left((-v'_*)^{\alpha-\beta} v'_{*,xx} + v'_{*,yy} - q(-v'_*)^{-\beta}\right)(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) \\ &\geq\; \left((-v)^{\alpha-\beta} v'_{*,xx} + v'_{*,yy} - q(-v)^{-\beta}\right)(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) + \mu \quad (\text{for some } \mu > 0) \\ &\geq\; \left((-v)^{\alpha-\beta}(v_{xx} - \varepsilon h_{xx}) + (v_{yy} - \varepsilon h_{yy}) - q(-v)^{-\beta}\right)(\hat{x}^\varepsilon_m, \hat{y}^\varepsilon_m) + \mu. \end{aligned}$$

Here we used first the fact that v'_* is a supersolution of (6.2), then (6.7) and (6.4), and finally (6.5) and (6.6).

Sending $m \to \infty$ yields:

$$0 \geq \left((-v)^{\alpha-\beta}(v_{xx} - \varepsilon h_{xx}) + (v_{yy} - \varepsilon h_{yy}) - q(-v)^{-\beta}\right)(\hat{x}, \hat{y}) + \mu.$$

Finally, sending $\varepsilon \to 0$, we obtain:

$$\begin{aligned} 0 \;&\geq\; \underbrace{\left((-v)^{\alpha-\beta} v_{xx} + v_{yy} - q(-v)^{-\beta}\right)(\hat{x}, \hat{y})}_{=0} + \mu \\ &=\; \mu, \end{aligned}$$

which is a contradiction because $\mu > 0$. ■

Remark 6.1.2
Actually, the proof only uses the following facts regarding v'_*:

- v'_* is semiconcave,
- v'_* is convex in x-direction.

Therefore, the above Lemma can be generalized:

Corollary 6.1.3
Let Ω be a convex bounded domain. Let v be a classical solution of $F = 0$, and w a viscosity supersolution of $F = 0$ such that w is semiconcave and convex in x-direction. Then, $w \geq v$.

Remark 6.1.4
If Ω is not convex, but such that

$$\{x \in \mathbb{R} \mid (x, y) \in \Omega\}$$

is convex for every $y \in \mathbb{R}$, then, in view of the Remark 3.0.11, Theorem 6.1.1 still holds.

The condition "w is semiconcave" is then accordingly replaced with "if $K \subset \Omega$ is a closed convex subset of Ω, then $w\Big|_{\mathrm{Int}K}$ is semiconcave" in Corollary 6.1.3.

6.2. Partial Concavity In Case $\beta = 0$ On Certain Types of Domains

We look at the anisotropic boundary value problem when $\alpha > \beta = 0$:

$$\begin{cases} u^\alpha u_{xx} + u_{yy} + q(x, y) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6.8}$$

where $\Omega \in R^2$ is a bounded domain, $\alpha > 0$, and q is a C^2 concave function such that $q \geq m > 0$.

6.2.1. Partial Concavity In Case $\beta = 0$ On Rectangles

Let $\Omega = (a, b) \times (c, d)$ where $a, b, c, d \in \mathbb{R}$ are such that $a < b$, and $c < d$.

Suppose there exists a positive solution u of (6.8) on Ω.

Let $v := -u$, and let v'_* be its partial convex envelope. Fix $y \in (c, d)$. Let $x \in (a, b)$, and let $x_1 \leq x_2 \in [a, b]$ and $\lambda > 0$ be such that

$$v'_*(x, y) = \lambda v(x_1, y) + (1 - \lambda)v(x_2, y).$$

We prove that x_1 and x_2 are in (a, b).

This is the missing piece in proving that u is concave in x-direction.

Proof Suppose, for contradiction, and w.l.o.g. that $x_1 = a$. Let $\varepsilon > 0$ be so small that

- $a + 2\varepsilon < b$, $y - 2\varepsilon > c$, and $y + 2\varepsilon < d$.

  Then $\underbrace{(a, a + 2\varepsilon) \times (y - \varepsilon, y + \varepsilon)}_{=:R} \subset \Omega$.

  Also, for $(x, y) \in R$ such that $x < x + \varepsilon$:

$$d_{\partial\Omega}(x, y) = d_{\partial R}(x, y) = x - a.$$

- if w_1 is the solution of

$$\begin{cases} w_1'' + \frac{m}{2} w_1^{-\alpha} = 0 & \text{in} \quad (a, a+2\varepsilon) \\ w_1(-\varepsilon) = w_1(\varepsilon) = 0, \end{cases}$$

then $w_1 \leq 1$. (w_1 is nonnegative.) (Such ε exists, see Lemma 5.2.1.)

- if w_2 is the solution of

$$\begin{cases} w_2'' + \frac{m}{2} = 0 & \text{in} \quad (y-\varepsilon, y+\varepsilon) \\ w_2(y-\varepsilon) = w_2(y+\varepsilon) = 0, \end{cases}$$

then $w_2 \leq 1$. (w_2 is nonnegative.)

Now, if u is the solution of (6.8) on Ω, then, by maximum principle, it is a supersolution of (6.8) on R.

Next we prove that $w(x,y) := w_1(x)w_2(y)$ (w_1, w_2 as above) is a subsolution of (6.8) on R:

$$\begin{aligned} w(x,y)^\alpha w_{xx}(x,y) + w_{yy}(x,y) + q(x,y) &= w_1^\alpha(x) w_2^{\alpha+1}(y) w_1'' + w_1(x) w_2''(y) + q(x,y) \\ &= -\frac{m}{2} \underbrace{w_2^{\alpha+1}(y)}_{\leq 1} - \frac{m}{2} \underbrace{w_1(x)}_{\leq 1} + q(x,y) \\ &\geq -\frac{m}{2} - \frac{m}{2} + m \\ &= 0. \end{aligned}$$

By comparison Lemma 6.1.1 we have

$$u \geq w \qquad \text{on } R. \tag{6.9}$$

Now, let t be such that $t < 2\varepsilon$ and $\frac{t}{\lambda} < \varepsilon$. Then

$$\begin{aligned} &(a+t, y) \in R \\ &d_{\partial\Omega}\left(a + \frac{t}{\lambda}, y\right) = d_{\partial R}\left(a + \frac{t}{\lambda}, y\right) = \frac{t}{\lambda}, \end{aligned}$$

so by (6.9), Theorem 4.1.2 and Lemma 4.2.2 we have for some $\tilde{K} > 0$:

$$\begin{aligned} u\left(a + \frac{t}{\lambda}, y\right) &\geq w\left(a + \frac{t}{\lambda}, y\right) \\ &= w_1\left(a + \frac{t}{\lambda}\right) w_2(y) \\ &\geq \tilde{K} \left(\frac{t}{\lambda}\right)^{\frac{1}{2+\alpha}} \underbrace{w_2(y)}_{>0} \\ &= \tilde{\tilde{K}} t^{\frac{1}{2+\alpha}}, \end{aligned}$$

i.e.

$$v\left(a + \frac{t}{\lambda}, y\right) \leq -\tilde{\tilde{K}} t^{\frac{1}{2+\alpha}}.$$

On the other hand we have, if $((p', p''), A) \in J^{2,-}_{\Omega} v'_*(x)$ (such $((p', p''), A)$ exists, see Remark 2.1.8).:

$$\begin{aligned} v'_*(x,y) + p't &\leq v'_*(x+t,y) \\ &\leq \lambda v\left(a + \frac{t}{\lambda}, y\right) + (1-\lambda)v(x_2,y) + \underbrace{\lambda v(a,y)}_{=0} \\ &= v'_*(x,y) + \lambda v\left(a + \frac{t}{\lambda}, y\right). \end{aligned}$$

Here, we used (3.3), definition of partial convex envelope, and (3.2), respectively. Next, we have:

$$\begin{aligned} p't &\leq \lambda v\left(a + \frac{t}{\lambda}, y\right) \\ &\leq -\tilde{\tilde{K}} t^{\frac{1}{2+\alpha}}. \end{aligned}$$

Finally, we have

$$t \geq K t^{\frac{1}{2+\alpha}}$$

where $K > 0$. Sending $t \to \infty$, we obtain a contradiction. ∎

So, we proved the following Theorem:

Theorem 6.2.1
Let $\Omega = (a,b) \times (c,d)$, where $a, b, c, d \in \mathbb{R}$ are such that $a < b$ and $c < d$. Suppose there exists a positive solution u of

$$\begin{cases} u^{\alpha} u_{xx} + u_{yy} + q(x,y) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $q \in C^2(\Omega)$ is concave in x- direction and such that $q \geq m > 0$, then u is concave in x-direction.

Lemma 6.0.5 now gives us the following uniqueness result:

Corollary 6.2.2
Let $\Omega = (a,b) \times (c,d)$, where $a, b, c, d \in \mathbb{R}$ are such that $a < b$ and $c < d$. If $q \in C^2(\Omega)$ is concave in x-direction and such that $q \geq m > 0$, then there is at most one positive solution of

$$\begin{cases} u^{\alpha} u_{xx} + u_{yy} + q(x,y) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

6.2.2. Extension of the Result On Certain Nonconvex Domains

Motivated by remarks 3.0.13 and 6.1.4, we wish to extend this result to some nonconvex domains.

Let $\Omega \cap (\mathbb{R} \times \{y\}) = I_y$ where I_y is either an open interval or empty and let $\Omega = \bigcup_{i=1,\ldots m} \Omega_i$, where $\Omega_i = (a_i, b_i) \times (c_i, d_i)$. We assume $a_i < b_i$, $c_i < d_i$, $c_i = d_{i-1}$ for $i = 2 \ldots m$, and

$c_i < c_{i+1}$ for $i = 1 \dots m-1$.

Suppose that there exists a positive solution u of (6.8) on Ω. Let $v := -u$, and let v'_* be its partial convex envelope.

Fix $y \in (c_1, d_m)$. Let $x \in \mathbb{R}$ be such that $(x, y) \in \Omega$, and let x_1, x_2 and $\lambda > 0$ be such that $(x_1, y), (x_2, y) \in \overline{\Omega}$ and

$$v'_*(x, y) = \lambda v(x_1, y) + (1 - \lambda) v(x_2, y).$$

We can prove exactly as in previous section that neither (x_1, y) nor (x_2, y) are on $\partial\Omega$.

This is the missing piece in proving that u is concave in x-direction. So, we have the following corollaries:

Theorem 6.2.3
Let $\Omega \cap (\mathbb{R} \times \{y\}) = I_y$ where I_y is either an open interval or empty and let $\Omega = \bigcup_{i=1,\dots m} \Omega_i$, where $\Omega_i = (a_i, b_i) \times (c_i, d_i)$ ($a_i < b_i$, $c_i < d_i$). Suppose there exists a positive solution u of

$$\begin{cases} u^\alpha u_{xx} + u_{yy} + q(x,y) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $q \in C^2(\Omega)$ is concave in x- direction and such that $q \geq m > 0$, then u is concave in x-direction.

Corollary 6.2.4
Let $\Omega \cap (\mathbb{R} \times \{y\}) = I_y$ where I_y is either an open interval or empty and let $\Omega = \bigcup_{i=1,\dots m} \Omega_i$, where $\Omega_i = (a_i, b_i) \times (c_i, d_i)$ ($a_i < b_i$, $c_i < d_i$). If $q \in C^2(\Omega)$ is concave in x-direction and such that $q \geq m > 0$, then there is at most one positive solution of

$$\begin{cases} u^\alpha u_{xx} + u_{yy} + q(x,y) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

6.3. Evaluation of the Uniqueness Result

At this point, we want to compare our uniqueness result with a previously existing one, given in [Rei97], which holds on domains with the *interior rectangle condition.*

Definition 6.3.1
A bounded domain $\Omega \in \mathbb{R}^2$ with $0 \in \Omega$ satisfies the interior rectangle condition (IRC) if for each $(x, y) \in \partial\Omega$ the rectangle $\{(\tau_1 x, \tau_2 y) \mid \tau_i \in [0, 1)\}$ is a subset of Ω.

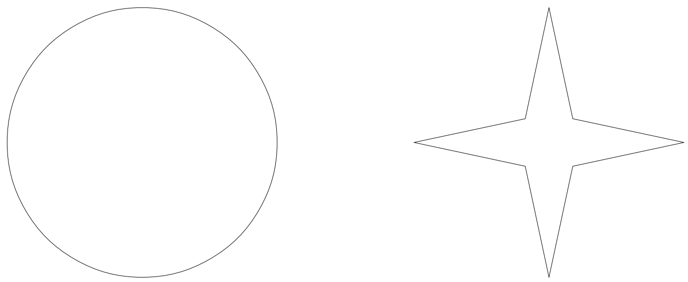
Theorem 6.3.2
Let $\Omega \subset \mathbb{R}^2$, be a bounded domain which satisfies the interior rectangle condition and $q : \Omega \to \mathbb{R}$ satisfies

$$q(\tau_1 x, \tau_2 y) \geq q(x, y) \quad \text{for } (x, y) \in \Omega,\ \tau_i \in [0, 1]. \tag{6.10}$$

Then there is at most one positive solution of (6.1).

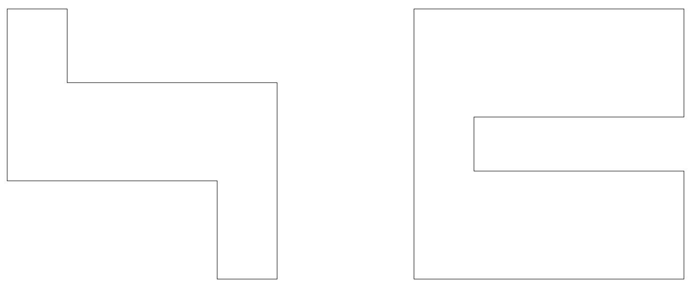
Let us break down conditions of Theorems 6.2.3 and 6.3.2, and compare them:

Ω1) Clearly, there are a lot of domains Ω which satisfy (IRC), but are not as in Theorem 6.2.3. The ball is an obvious example. There are also nonconvex domains which satisfy (IRC)



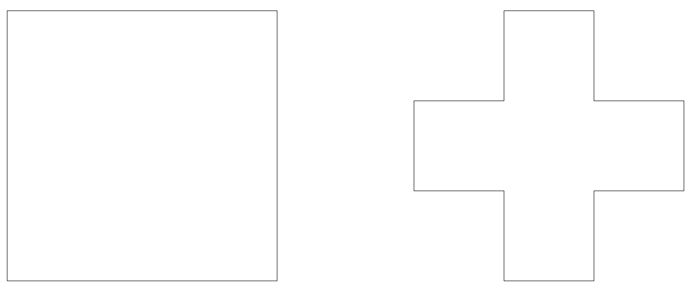
On such domains, we have a uniqueness result (if q satisfies (6.10)).

Ω2) Also, there are such domains that satisy conditions of Theorem 6.2.3, but not (IRC).



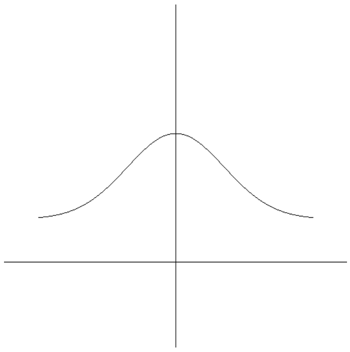
On such domains we obtained a new uniqueness result.

Ω3) Clearly, the rectangle satisfies both (IRC) and the conditions of 6.2.3. There are of course more domains with that property.

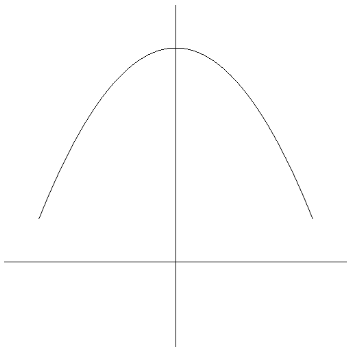


q1) On such domains we also obtained a new uniqueness result, since there are such q that satisfy the conditions of 6.2.3, but not (6.10). For example, on the rectangle $(0,1) \times (0,1)$ the function $q(x,y) = x + M$, where $M > 0$.

q2) Of course, there are functions that satisfy (6.10), but not the conditions of Theorem 6.2.3. For example, on the figure below we have a graph of the function $\phi(x) : (-1,1) \to \mathbb{R}$. If we define $\tilde{\phi}(x,y) : (-1,1) \times (-1,1) \to \mathbb{R}$, $\tilde{\phi}(x,y) := \phi(x)$, then $\tilde{\phi}$ is one of such functions (on a rectangle that contains 0).



q3) There are such functions that satisfy both (6.10) and conditions of Theorem 6.2.3. For example, on the figure below we have a graph of the function $\phi(x) : (-1,1) \to \mathbb{R}$. If we define $\tilde{\phi}(x,y) : (-1,1) \times (-1,1) := \phi(x)$, then $\tilde{\phi}$ is one of such functions (on a rectangle that contains 0).



In this case, the uniqueness result in [Rei97] implies our new one.

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Appendix

A. Some Useful Results

Theorem A.1
Let Ω be a bounded domain. Let $u \in$USC(Ω) and φ be twice continuously differentiable in Ω. Suppose $u - \varphi$ achieves a local minimum in $\hat{x} \in \Omega$. Then, for every $\varepsilon > 0$ there exists $A \in S^N$ such that $(D\varphi(\hat{x}), A) \in \overline{J}^{2,-}_{\Omega} u(\hat{x})$, and the following inequality holds:

$$A \geq D^2\varphi(\hat{x}) - \varepsilon \left(D^2\varphi(\hat{x})\right)^2 . \tag{6.11}$$

Theorem A.2 (Aleksandrov's Theorem)
Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be semiconvex. Then φ is twice differentiable almost everywhere on $\mathbb{R}^N$.

Theorem A.3 (Jensen's Lemma)
Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be semiconvex and $\hat{x}$ be a strict local maximum point of φ. For $p \in \mathbb{R}^N$, set $\varphi_p(x) = \varphi(x) + \langle p, x\rangle$. Then, for $r, \delta > 0$,

$$K = \{x \in B_r(\hat{x}) : \; \exists \, p \in B_\delta \text{ such that } \varphi_p \text{ has a local maximum at } x\}$$

has positive measure.

Theorem A.4 (Hopf's Boundary Point Lemma)
Suppose u is a superharmonic function in a domain Ω, $u > 0$ in Ω, and $u(x_0) = 0$ for some $x_0 \in \partial\Omega$. Assume Ω satisfies an interior sphere condition at x_0, i.e. there exists an open ball $B \subset \Omega$ such that $\overline{B} \cap \partial\Omega = x_0$. Then the exterior normal derivative $\frac{\partial u}{\partial \nu}(x_0)$ is strictly negative if it exists.

Remark A.5
Let $\Omega \in \mathbb{R}^N$ be a domain. If $\partial\Omega$ is smooth (of class C^2), Ω satisfies the interior sphere condition.

B. Calculation, Chapter 3

Let us here prove that, if we plug

$$h_i' = a_i^{-1} \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} \right)^{-1} \left[h' + \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} b_j \right) h'' \right] - a_i^{-1} b_i h''.$$

into (3.9):

$$\langle Ah, h\rangle \leq \sum_{i=1}^{k} \lambda_i \left[\langle a_i h_i', h_i' \rangle + 2 \langle b_i h'', h_i' \rangle + \langle c_i h'', h'' \rangle \right] ,$$

(where $h = (h', h'')$, $\sum_{i=1}^{k} \lambda_i h'_i = h'$) we obtain

$$A \leq \begin{pmatrix} a' & b' \\ b'^T & c' \end{pmatrix}$$

where

$$\begin{aligned}
a' &= \left(\sum_{i=1}^{k} \lambda_i a_i^{-1}\right)^{-1} \\
b' &= \left(\sum_{i=1}^{k} \lambda_i a_i^{-1}\right)^{-1} \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} b_i\right) \\
c' &= \sum_{i=1}^{k} \lambda_i \left(c_i - b_i^T a_i^{-1} b_i\right) + \left(\sum_{i=1}^{k} \lambda_i b_i^T a_i^{-1}\right) \left(\sum_{i=1}^{k} \lambda_i a_i^{-1}\right)^{-1} \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} b_i\right).
\end{aligned}$$

Let us first see that indeed $\sum_{i=1}^{k} \lambda_i h'_i = h'$:

$$\begin{aligned}
\sum_{i=1}^{k} \lambda_i h'_i &= \sum_{i=1}^{k} \lambda_i \left[a_i^{-1} \left(\sum_{j=1}^{k} \lambda_j a_j^{-1}\right)^{-1} \left[h' + \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} b_j\right) h''\right] - a_i^{-1} b_i h''\right] \\
&= \left(\sum_{i=1}^{k} \lambda_i a_i^{-1}\right) \left(\sum_{j=1}^{k} \lambda_j a_j^{-1}\right)^{-1} \left[h' + \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} b_j\right) h''\right] - \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} b_i\right) h'' \\
&= h' + \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} b_j\right) h'' - \left(\sum_{i=1}^{k} \lambda_i a_i^{-1} b_i\right) h'' \\
&= h'.
\end{aligned}$$

Let us now calculate $\sum_{i=1}^{k} \lambda_i \langle a_i h'_i, h'_i \rangle$ and $\sum_{i=1}^{k} \lambda_i \langle b_i h'', h'_i \rangle$:

$$\begin{aligned}
\sum_{i=1}^{k} \lambda_i \langle a_i h'_i, h'_i \rangle &= \sum_{i=1}^{k} \lambda_i \left\langle a_i \left[a_i^{-1} \left(\sum_{j=1}^{k} \lambda_j a_j^{-1}\right)^{-1} \left[h' + \left(\sum_{j_1}^{k} \lambda_j a_j^{-1} b_j\right) h''\right] - a_i^{-1} b_i h''\right], \right. \\
&\qquad \left. a_i^{-1} \left(\sum_{j=1}^{k} \lambda_j a_j^{-1}\right)^{-1} \left[h' + \left(\sum_{j_1}^{k} \lambda_j a_j^{-1} b_j\right) h''\right] - a_i^{-1} b_i h'' \right\rangle \\
&= \sum_{i=1}^{k} \lambda_i \left\langle \left(\sum_{j=1}^{k} \lambda_j a_j^{-1}\right)^{-1} \left[h' + \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} b_j\right) h''\right], \right. \\
&\qquad \left. a_i^{-1} \left(\sum_{j=1}^{k} \lambda_j a_j^{-1}\right)^{-1} \left[h' + \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} b_j\right) h''\right] - a_i^{-1} b_i h'' \right\rangle \\
&\quad - \sum_{i=1}^{k} \lambda_i \left\langle b_i h'', a_i^{-1} \left(\sum_{j=1}^{k} \lambda_j a_j^{-1}\right)^{-1} \left[h' + \left(\sum_{j=1}^{k} \lambda_j a_j^{-1} b_j\right) h''\right] - a_i^{-1} b_i h'' \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle \left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left[h' + \left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right)h''\right],\right.\\
&\qquad \left.\left(\sum_{i=1}^{k}\lambda_i a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left[h' + \left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right)h''\right] - \left(\sum_{i=1}^{k}\lambda_i a_i^{-1} b_i\right)h''\right\rangle\\
&\quad -\sum_{i=1}^{k}\lambda_i\left\langle h'', b_i^T a_i^{-1}\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left[h' + \left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right)h''\right] - b_i^T a_i^{-1} b_i h''\right\rangle\\
&= \left\langle \left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left[h' + \left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right)h''\right], h'\right\rangle\\
&\quad -\left\langle h'', \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left[h' + \left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right)h''\right] - \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1} b_i\right)h''\right\rangle\\
&= \left\langle \left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1} h', h'\right\rangle\\
&\quad + \underbrace{\left\langle \left[\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right)\right]h'', h'\right\rangle - \left\langle h'', \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1} h'\right\rangle}_{=0}\\
&\quad -\left\langle h'', \left[\left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right) - \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1} b_i\right)\right]h''\right\rangle\\
&= \left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\langle h', h'\rangle\\
&\quad -\left[\left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right) - \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1} b_i\right)\right]\langle h'', h''\rangle
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{k}\lambda_i\langle b_i h'', h_i'\rangle &= \sum_{i=1}^{k}\lambda_i\langle h'', b_i^T h_i'\rangle\\
&= \sum_{i=1}^{k}\lambda_i\left\langle h'', b_i^T a_i^{-1}\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left[h' + \left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right)h''\right] - b_i^T a_i^{-1} b_i h''\right\rangle\\
&= \left\langle h'', \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left[h' + \left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right)h''\right]\right.\\
&\qquad \left. - \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1} b_i\right)h''\right\rangle
\end{aligned}
$$

$$\begin{aligned}
&= \left\langle h'', \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1} h'\right\rangle \\
&\quad + \left\langle h'', \left[\left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right) - \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1} b_i\right)\right] h''\right\rangle \\
&= \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left\langle h'', h'\right\rangle \\
&\quad + \left[\left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right) - \left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1} b_i\right)\right]\left\langle h'', h''\right\rangle
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{i=1}^{k}\lambda_i\left[\left\langle a_i h_i', h_i'\right\rangle + 2\left\langle b_i h'', h_i'\right\rangle + \left\langle c_i h'', h''\right\rangle\right] &= \left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left\langle h', h'\right\rangle \\
&\quad +2\left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left\langle h'', h'\right\rangle \\
&\quad + \left[\left(\sum_{i=1}^{k}\lambda_i b_i^T a_i^{-1}\right)\left(\sum_{j=1}^{k}\lambda_j a_j^{-1}\right)^{-1}\left(\sum_{j=1}^{k}\lambda_j a_j^{-1} b_j\right)\right. \\
&\quad \left. + \left(\sum_{i=1}^{k}\lambda_i\left(c_i - b_i^T a_i^{-1} b_i\right)\right)\right]\left\langle h'', h''\right\rangle
\end{aligned}$$

This gives the assertion.

C. The Function ψ from Section 5.3.1

Let $f : [a, a_0] \to \mathbb{R}$ be $\psi\Big|_{[a,a_0]}$, i.e.

$$f(x) = -\tilde{\varphi}(a)\left[1 - \frac{1}{a_0 - a}\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}\right]^4.$$

Clearly:

$$f(a_0) = 0. \tag{6.12}$$

Now we calculate the first derivative.

$$\begin{aligned}
f'(x) &= -4\tilde{\varphi}(a)\left[1 - \frac{1}{a_0 - a}\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}\right]^3 \frac{1}{a_0 - a}\cdot \\
&\quad \frac{x - a_0}{\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}}.
\end{aligned}$$

We see that

$$f'(a_0) = 0. \tag{6.13}$$

Also

$$f'(x) = -4\underbrace{\tilde{\varphi}(a)}_{\leq 0}\left[1 - \underbrace{\frac{1}{a_0 - a}\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}}_{\leq 1}\right]^3 \underbrace{\frac{1}{a_0 - a}}_{\geq 0} \cdot \underbrace{\frac{x - a_0}{\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}}}_{\leq 0},$$

so

$$f'(x) \underset{x \to a}{\longrightarrow} -\infty. \tag{6.14}$$

Note that it then also holds

$$\frac{1}{f'}(x) \underset{x \to a}{\longrightarrow} 0. \tag{6.15}$$

Now to calculate the second derivative:

$$\begin{aligned} f''(x) &= -4\tilde{\varphi}(a)\left[1 - \frac{1}{a_0 - a}\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}\right]^2 \cdot \\ &\left[\frac{3}{(a_0-a)^2} \cdot \frac{(x-a_0)^2}{-(x-a)^2 + 2(x-a)(a_0-a)} + \right. \\ &\left[1 - \frac{1}{a_0 - a}\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}\right]\frac{1}{a_0-a} \cdot \\ &\left.\left(\frac{1}{\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}} + \frac{(x-a_0)^2}{(-(x-a)^2 + 2(x-a)(a_0-a))^{\frac{3}{2}}}\right)\right]. \end{aligned}$$

We see that

$$f''(a_0) = 0. \tag{6.16}$$

Also, it is clear that $f''(x) \geq 0$ for all x, i.e. that f is convex.

Now, let's rewrite f'' a little bit:

$$\begin{aligned} f''(x) &= -4\tilde{\varphi}(a)\left[1 - \frac{1}{a_0 - a}\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}\right]^2 \cdot \\ &\frac{1}{\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}} \cdot F(x), \end{aligned}$$

where

$$\begin{aligned} F(x) &:= \frac{2}{(a_0-a)^2} \cdot \frac{(x-a_0)^2}{\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}} + \frac{1}{a_0-a} \\ &+ \frac{1}{a_0-a} \cdot \frac{(x-a_0)^2}{-(x-a)^2 + 2(x-a)(a_0-a)} \\ &- \frac{1}{(a_0-a)^2}\frac{1}{\sqrt{-(x-a)^2 + 2(x-a)(a_0-a)}}. \end{aligned}$$

Now let's calculate $\lim_{x\to a} F(x)$ (we use of course L'Hôspitals rule):

$$\begin{aligned}
\lim_{x\to a} \frac{(x-a_0)^2}{\sqrt{-(x-a)^2+2(x-a)(a_0-a)}} &= \lim_{x\to a} \frac{2(x-a_0)}{\frac{a_0-x}{\sqrt{-(x-a)^2+2(x-a)(a_0-a)}}} \\
&= \lim_{x\to a} \frac{2(x-a_0)\sqrt{-(x-a)^2+2(x-a)(a_0-a)}}{a_0-x} \\
&= 0; \\
\lim_{x\to a} \frac{1}{\sqrt{-(x-a)^2+2(x-a)(a_0-a)}} &= \lim_{x\to a} \frac{\sqrt{-(x-a)^2+2(x-a)(a_0-a)}}{a_0-x} \\
&= 0; \\
\lim_{x\to a} \frac{(x-a_0)^2}{-(x-a)^2+2(x-a)(a_0-a)} &= \lim_{x\to a} \frac{2(x-a_0)}{2(a_0-x)} \\
&= -1.
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{x\to a} F(x) &= \frac{2}{(a_0-a)^2}\cdot 0 + \frac{1}{a_0-a} + \frac{1}{a_0-a}\cdot(-1) - \frac{1}{(a_0-a)^2}\cdot 0 \\
&= 0.
\end{aligned}$$

Now we can calculate $\lim_{x\to a} \frac{f''}{f'}(x)$:

$$\begin{aligned}
\lim_{x\to a} \frac{f''}{f'}(x) &= \lim_{x\to a} \frac{F(x)}{\left[1-\frac{1}{a_0-a}\sqrt{-(x-a)^2+2(x-a)(a_0-a)}\right]\frac{1}{a_0-a}\cdot(x-a_0)} \\
&= \frac{0}{-1} = 0.
\end{aligned}$$

So, it holds:

$$\frac{f''}{f'}(x) \underset{x\to a}{\longrightarrow} 0. \tag{6.17}$$

Let's calculate now the third derivative:

$$\begin{aligned}
f^{(3)}(x) = \; & -8\tilde{\varphi}(a)\left[1-\frac{1}{a_0-a}\sqrt{-(x-a)^2+2(x-a)(a_0-a)}\right]\cdot \\
& \frac{1}{a_0-a}\cdot\frac{x-a_0}{-(x-a)^2+2(x-a)(a_0-a)}\cdot F(x) \\
& -4\tilde{\varphi}(a)\left[1-\frac{1}{a_0-a}\sqrt{-(x-a)^2+2(x-a)(a_0-a)}\right]^2\cdot \\
& \frac{x-a_0}{(-(x-a)^2+2(x-a)(a_0-a))^{\frac{3}{2}}}\cdot F(x) \\
& -4\tilde{\varphi}(a)\left[1-\frac{1}{a_0-a}\sqrt{-(x-a)^2+2(x-a)(a_0-a)}\right]^2\cdot \\
& \frac{1}{\sqrt{-(x-a)^2+2(x-a)(a_0-a)}}\cdot F'(x),
\end{aligned}$$

while

$$\begin{aligned}F'(x) &= \frac{2}{(a_0-a)^2}\cdot\left(\frac{2(x-a_0)}{\sqrt{-(x-a)^2+2(x-a)(a_0-a)}}+\frac{(x-a_0)^3}{(-(x-a)^2+2(x-a)(a_0-a))^{\frac{3}{2}}}\right)\\ &\quad+\frac{1}{a_0-a}\cdot\left(\frac{2(x-a_0)}{(-(x-a)^2+2(x-a)(a_0-a))^{\frac{3}{2}}}+\frac{2(x-a_0)^3}{(-(x-a)^2+2(x-a)(a_0-a))^2}\right)\\ &\quad-\frac{1}{(a_0-a)^2}\cdot\frac{x-a_0}{(-(x-a)^2+2(x-a)(a_0-a))^{\frac{3}{2}}}.\end{aligned}$$

It is easily seen that

$$f^{(3)}(a_0)=0. \tag{6.18}$$

Let

$$\begin{aligned}G(x) &:= F'(x)\cdot(-(x-a)^2+2(x-a)(a_0-a))^{\frac{3}{2}}\\ &= \frac{2}{(a_0-a)^2}\cdot\left(2(x-a_0)\left(-(x-a)^2+2(x-a)(a_0-a)\right)+(x-a_0)^3\right)\\ &\quad+\frac{1}{a_0-a}\cdot\left(2(x-a_0)+\frac{2(x-a_0)^3}{\sqrt{-(x-a)^2+2(x-a)(a_0-a)}}\right)\\ &\quad-\frac{1}{(a_0-a)^2}\cdot(x-a_0).\end{aligned}$$

and let's calculate $\lim_{x\to a} G(x)$ (again we use L'Hôspitals rule):

$$\begin{aligned}\lim_{x\to a}\frac{2(x-a_0)^3}{\sqrt{-(x-a)^2+2(x-a)(a_0-a)}} &= \lim_{x\to a}\frac{6(x-a_0)^2}{\frac{a_0-x}{\sqrt{-(x-a)^2+2(x-a)(a_0-a)}}}\\ &= \lim_{x\to a}\frac{6(x-a_0)^2\sqrt{-(x-a)^2+2(x-a)(a_0-a)}}{a_0-x}\\ &= 0.\end{aligned}$$

So,

$$\lim_{x\to a} G(x)=0.$$

Now, to calculate $\lim_{x\to a}\frac{f^{(3)}}{(f')^4}(x)$:

$$\begin{aligned}\lim_{x\to a}\frac{f^{(3)}}{(f')^4}(x) &= \lim_{x\to a}\frac{(x-a_0)\cdot F(x)-(x-a)^2+2(x-a)(a_0-a)}{32\tilde{\varphi}(a)^3\left[1-\frac{1}{a_0-a}\sqrt{-(x-a)^2+2(x-a)(a_0-a)}\right]^{11}\frac{(x-a_0)^4}{(a_0-a)^3}}\\ &\quad+\lim_{x\to a}\frac{F(x)\cdot\sqrt{-(x-a)^2+2(x-a)(a_0-a)}}{64\tilde{\varphi}(a)^3\left[1-\frac{1}{a_0-a}\sqrt{-(x-a)^2+2(x-a)(a_0-a)}\right]^{10}\frac{(x-a_0)^3}{(a_0-a)^4}}\\ &\quad+\lim_{x\to a}\frac{G(x)}{64\tilde{\varphi}(a)^3\left[1-\frac{1}{a_0-a}\sqrt{-(x-a)^2+2(x-a)(a_0-a)}\right]^{10}\frac{(x-a_0)^4}{(a_0-a)^4}}\\ &= 0.\end{aligned}$$

So, it holds:

$$\frac{f^{(3)}}{(f')^4}(x)\underset{x\to a}{\longrightarrow}0. \tag{6.19}$$

As we mentioned earlier in section 5.3.1, the conditions (6.12), (6.13), (6.16) and (6.18) give us that $\tilde{\varphi} + \psi$ is $C^{2+\gamma}$ on $(a, \tilde{x}_0]$. However, we also need $(\tilde{\varphi} + \psi)^{-1}$ to be $C^{2+\gamma}$ in 0 ($(\tilde{\varphi} + \psi)(a) = 0$). It is enough to show that ψ^{-1} is C^3 in 0.

Let $h(x) := \psi^{-1}(x)$. Since $\psi \in C^3$, by the inverse function theorem we have

$$\begin{aligned} h'(x) &= \frac{1}{\psi'(h(x))}; \\ h''(x) &= \frac{-\psi''(h(x))}{(\psi')^3(h(x))}; \\ h^{(3)}(x) &= \frac{-\psi^{(3)}(h(x))}{(\psi')^4(h(x))} + \frac{3(\psi'')^2(h(x))}{(\psi')^5(h(x))}. \end{aligned}$$

Now since (6.15), (6.17) and (6.19) hold, we see that h is indeed C^3 in 0.