

Characterization of Inclusions in Impedance Tomography

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Abstract

The inverse problem of electrical impedance tomography (EIT) is to recover the conductivity inside an investigated object from boundary measurements of current and voltage. There is a variety of methods to localize inclusions, i.e. domains in which the conductivity is different from the background conductivity such as e.g. the Factorization method. However, these qualitative methods don't provide any information about the conductivity inside the inclusions.

In this work we present a method to compute the conductivity inside inclusions after they have been localized. This method is based on a new factorization with three operators that are different from those in the original Factorization method for EIT. In particular, we investigate the spectrum of the operator that appears in the middle of this new factorization and show that it is closely related to the conductivity of the inclusions.

1 Introduction

In electrical impedance tomography (EIT) a current pattern f is injected at the surface of the investigated object B and the resulting electrical potential is measured at the surface. The electrical potential u solves the boundary value problem

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{in } B, \quad \gamma \partial_\nu u = f \quad \text{on } \partial B.$$

From data sets of this type one tries to extract information about the conductivity γ inside the subject. This work is based on the assumption that all possible measurement data sets $\{f, u|_{\partial B}\}$ are known, that is we assume to know the Neumann-to-Dirichlet map Λ that maps every current pattern f to its corresponding boundary potential $u|_{\partial B}$. This idealized model is called the continuum model.

There exists a variety of reconstruction methods that are used to detect inclusions or anomalies inside B from the knowledge of the Neumann-to-Dirichlet map Λ ; that is, they localize domains in which the conductivity is different from the background conductivity that is assumed to be known a priori. One of these methods is the Factorization method that has first been suggested by Kirsch in [17] for scattering problems. The Factorization method has been carried over

to EIT in [2], [4], [3] (see also [20]). It has been extended and improved continuously since then (see e.g. [18], [15] [9], [10]). There are other qualitative reconstruction methods for EIT such as methods based on the concept of the source support (see e.g. [12], [13] and [14]). For a more extensive overview on various reconstruction methods for EIT we refer to the review articles [8],[1],[24] and the references therein.

However, one possible drawback of these qualitative methods is that the value of the conductivity inside the inclusion remains unknown. The aim of this paper is to derive a method for approximating the conductivity inside the inclusion after it has been identified by a qualitative method, such as, e.g., the Factorization method. This plan is comparable to the considerations in, e.g., [5], [7] and [6], where the authors present methods of approximating the surface impedance or the index of refraction of a scatterer whose shape has been determined before.

We assume that the conductivity is of the type $\gamma = 1 + \chi_\Omega q$, where χ_Ω is the characteristic function of the inclusion Ω . The main idea of our method is to make use of a factorization of $\Lambda - \Lambda_0$ that appears in a new version of the Factorization method which has been derived in [19]. Our method of determining the conductivity is restricted to the case of piecewise constant conductivities. Furthermore, we emphasize that this problem is still an ill-posed problem.

The outline of this work is as follows: In Section 2 we present the new factorization and show substantial properties of the appearing operators. In Section 3 we focus on the middle operator \mathcal{T} of the factorization and especially its spectrum that is closely related to the conductivity inside the inclusions. In Subsection 3.1 we first consider the case of only one inclusion having a constant conductivity contrast, while in Subsection 3.2 we generalize our results to the case of several different conductivity contrasts. Afterwards, in Section 4, we show how this spectrum and the conductivity can be computed numerically and in Section 5 we present some numerical results with this new method.

2 Factorization of the relative NtD map

We start by stating our assumptions on the problem setting. Let the object $B \subset \mathbb{R}^d$ ($d = 2, 3$) be a simply connected C^2 -domain, let the inclusion Ω be a (possibly disconnected) open set with C^2 -boundary such that $\bar{\Omega} \subset B$ and $B \setminus \bar{\Omega}$ is connected.

For the conductivity distribution we restrict ourselves to piecewise constant and real-valued conductivities of the type $\gamma(x) = 1 + q(x)\chi_\Omega(x)$ ($x \in B$), and q is assumed to be real-valued and constant on each component of Ω .

Throughout this work the following function spaces will play an important role:

$$\begin{aligned} L_\diamond^2(\partial B) &= \left\{ g \in L^2(\partial B) : \int_{\partial B} g \, ds = 0 \right\}, \\ L^2(\Omega, \mathbb{R}^d) &= \{ h : B \rightarrow \mathbb{R}^d : h_j \in L^2(B), j = 1, \dots, d \}, \\ H_\diamond^1(B) &= \left\{ u \in H^1(B) : \int_{\partial B} u \, ds = 0 \right\}. \end{aligned}$$

The direct problem of EIT for a given current pattern $f \in L^2_\diamond(\partial B)$ is to find a weak solution $u \in H^1_\diamond(B)$ to the boundary value problem

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{in } B, \quad \partial_\nu u = f \quad \text{on } \partial B, \quad (1)$$

that is

$$\iint_B \gamma \nabla u \cdot \nabla \psi \, dx = \int_{\partial B} f \psi \, ds \quad \text{for all } \psi \in H^1_\diamond(B).$$

Since this problem is well-posed we obtain, using also the trace theorem, that the Neumann-to-Dirichlet operator $\Lambda : L^2_\diamond(\partial B) \rightarrow L^2_\diamond(\partial B)$ that maps the current pattern f to the trace $u|_{\partial B}$ is a well-defined and bounded linear operator.

Furthermore, we also need to consider the background direct problem that is to find a weak solution $u_0 \in H^1_\diamond(B)$ to

$$\Delta u_0 = 0 \quad \text{in } B, \quad \partial_\nu u_0 = f \quad \text{on } \partial B. \quad (2)$$

This problem leads to the background Neumann-to-Dirichlet operator $\Lambda_0 : L^2_\diamond(\partial B) \rightarrow L^2_\diamond(\partial B)$ by $f \mapsto u_0|_{\partial B}$.

We proceed by defining the operators that will appear in our new factorization and start with the operator $\mathcal{A} : L^2_\diamond(\partial B) \rightarrow L^2(\Omega, \mathbb{R}^d)$, defined by $f \mapsto \nabla u_0|_\Omega$, where $u_0 \in H^1_\diamond(B)$ solves (2) in the weak sense. Hence \mathcal{A} is defined via the background direct problem.

The corresponding adjoint $\mathcal{A}^* : L^2(\Omega, \mathbb{R}^d) \rightarrow L^2_\diamond(\partial B)$ is given by $h \mapsto w|_{\partial B}$ where $w \in H^1_\diamond(B)$ solves

$$\iint_B \nabla w \cdot \nabla \psi \, dx = \iint_\Omega h \cdot \nabla \psi \, dx \quad \text{for all } \psi \in H^1_\diamond(B). \quad (3)$$

This is the weak formulation for the problem to find $w \in H^1_\diamond B$ that solves

$$\begin{aligned} \Delta w &= 0 \quad \text{in } B \setminus \bar{\Omega}, & \partial_\nu w &= 0 \quad \text{on } \partial B, \\ \Delta w &= h \quad \text{in } \Omega, & \partial_\nu w|_+ - \partial_\nu w|_- &= \nu \cdot h \quad \text{on } \partial\Omega, \end{aligned}$$

where we denote the trace from the exterior and from the interior by $|_\pm$, respectively. For the corresponding proof we refer to [23].

Using the unique continuation principle for harmonic functions we note that \mathcal{A} is injective which also implies that \mathcal{A}^* has dense range in $L^2_\diamond(\partial B)$. In the following lemma we show representations for the closure of the range of \mathcal{A} and the nullspace of its adjoint \mathcal{A}^* .

Lemma 2.1. *The operators $\mathcal{A}, \mathcal{A}^*$ defined above have the following properties:*

- (a) *The nullspace $\mathcal{N}(\mathcal{A}^*)$ consists of all $h \in L^2(\Omega, \mathbb{R}^d)$ such that there is a solution $w \in H^1(\Omega)$ to*

$$\iint_\Omega \nabla w \cdot \nabla \psi \, dx = \iint_\Omega h \cdot \nabla \psi \, dx \quad \text{for all } \psi \in H^1_\diamond(\Omega)$$

satisfying $w = 0$ on $\partial\Omega$.

(b)

$$\overline{\mathcal{R}(\mathcal{A})} = \{ \nabla u : u \in H^1(\Omega), \Delta u = 0 \text{ in } \Omega \}.$$

For the corresponding proof we refer again to [23]. For $h \in \overline{\mathcal{R}(\mathcal{A})}$ we find the classical formulation of (3) using the fact that $h = \nabla v$ for some $v \in H^1(B)$ and $\Delta v = 0$. As a result we obtain the following transmission problem: $\mathcal{A}^*h = w|_{\partial B}$, where $w \in H_{\diamond}^1(B)$ satisfies

$$\begin{aligned} \Delta w &= 0 \text{ in } B \setminus \partial\Omega, \\ \partial_{\nu} w &= 0 \text{ on } \partial B, \\ \partial_{\nu} w|_{-} - \partial_{\nu} w|_{+} &= \nu \cdot h = \partial_{\nu} v \text{ on } \partial\Omega. \end{aligned} \quad (4)$$

In addition, the operator $\mathcal{T} : L^2(\Omega, \mathbb{R}^d) \rightarrow L^2(\Omega, \mathbb{R}^d)$ is defined by $h \mapsto q(h - \nabla w)$, where $w \in H_{\diamond}^1(B)$ solves

$$\iint_B (1 + q\chi_{\Omega}) \nabla w \cdot \nabla \psi \, dx = \iint_{\Omega} q h \cdot \nabla \psi \, dx \quad \text{for all } \psi \in H_{\diamond}^1(B). \quad (5)$$

Again, for $h = \nabla u \in \overline{\mathcal{R}(\mathcal{A})}$ we can formulate this as a transmission problem: $\mathcal{T}h = q \nabla(u - w)|_{\Omega}$, where $w \in H_{\diamond}^1(B)$ satisfies

$$\begin{aligned} \Delta w &= 0 \text{ in } B \setminus \partial\Omega, \\ \partial_{\nu} w &= 0 \text{ on } \partial B, \\ (1 + q) \partial_{\nu} w|_{-} - \partial_{\nu} w|_{+} &= q(\nu \cdot h) = q \partial_{\nu} u \text{ on } \partial\Omega. \end{aligned} \quad (6)$$

Throughout this work we restrict \mathcal{T} to $\overline{\mathcal{R}(\mathcal{A})}$, and we observe that $\mathcal{T} : \overline{\mathcal{R}(\mathcal{A})} \rightarrow \overline{\mathcal{R}(\mathcal{A})}$ since q is constant on each component of Ω and the potential w that solves the transmission problem (6) satisfies $\nabla w|_{\Omega} \in \overline{\mathcal{R}(\mathcal{A})}$. In addition, it is easy to show that \mathcal{T} is self-adjoint.

By construction of the operators the following factorization can be shown as it is done in [19] and [23].

Lemma 2.2. *Let \mathcal{A}, \mathcal{T} be defined as above. Then:*

$$\Lambda_0 - \Lambda = \mathcal{A}^* \mathcal{T} \mathcal{A}. \quad (7)$$

The main idea of our method to determine the conductivity inside Ω is to make use of the factorization (7): the relative Neumann-to-Dirichlet operator $\Lambda_0 - \Lambda$ is known from our measurement data. Moreover, the operators $\mathcal{A}, \mathcal{A}^*$ only contain information about the inclusion's boundary, hence they are also known after the inclusion has been located. The information about the conductivity inside Ω is contained in the middle operator \mathcal{T} .

Therefore, our goal is to recover information about the operator \mathcal{T} from the knowledge of $\Lambda_0 - \Lambda$ and $\mathcal{A}, \mathcal{A}^*$. In the following section we take a closer look at \mathcal{T} and especially its spectrum, while afterwards we show how the knowledge about this spectrum can be used to compute the conductivity contrast q .

3 The Spectrum of the Middle Operator

In this section we show some important properties of the operator \mathcal{T} and in particular of its spectrum $\sigma(\mathcal{T})$. We start our considerations by giving bounds for $\sigma(\mathcal{T})$ and note that the first result even holds for arbitrary (space-dependent) q

Lemma 3.1. *The spectrum of \mathcal{T} satisfies the following bounds: $\frac{q_{\min}}{1+q_{\min}} \leq \lambda < q_{\max}$ for every $\lambda \in \sigma(\mathcal{T})$, where $q_{\min} = \min \{q(x) : x \in \Omega\}$ and $q_{\max} = \max \{q(x) : x \in \Omega\}$.*

Proof. Since \mathcal{T} is a bounded self-adjoint operator, $\sigma(\mathcal{T})$ is a compact set in \mathbb{R} . Furthermore, it is a well-known fact that

$$\sigma(\mathcal{T}) \subset \left\{ \langle \mathcal{T}h, h \rangle : h \in L^2(\Omega, \mathbb{R}^d), \|h\|_{L^2(\Omega, \mathbb{R}^d)} = 1 \right\}.$$

For the upper bound we use the weak formulation (5) and obtain for $h \in L^2(\Omega, \mathbb{R}^d)$ with $\|h\|_{L^2(\Omega, \mathbb{R}^d)} = 1$

$$\begin{aligned} \langle \mathcal{T}h, h \rangle &= \iint_{\Omega} q (h - \nabla w) \cdot h \, dx \\ &= \iint_{\Omega} q |h|^2 \, dx - \iint_{\Omega} q (\nabla w \cdot h) \, dx \\ &\leq q_{\max} - \iint_B (1 + q\chi_{\Omega}) |\nabla w|^2 \, dx < q_{\max}. \end{aligned}$$

Equality would imply that $\nabla w \equiv 0$ in B from which we can easily conclude that $h = 0$, in contradiction to $\|h\|_{L^2(\Omega, \mathbb{R}^d)} = 1$. Now it remains to show the lower bound where we again assume $\|h\|_{L^2(\Omega, \mathbb{R}^d)} = 1$. Using (5) we obtain

$$\begin{aligned} \langle \mathcal{T}h, h \rangle &= \iint_{\Omega} q (h - \nabla w) \cdot h \, dx \\ &= \iint_{\Omega} q |h - \nabla w|^2 \, dx + \iint_{\Omega} q (h - \nabla w) \cdot \nabla w \, dx \\ &= \iint_{\Omega} q |h - \nabla w|^2 \, dx + \iint_B |\nabla w|^2 \, dx \\ &\geq \iint_{\Omega} \left(q |h|^2 - 2q (h \cdot \nabla w) + (1 + q) |\nabla w|^2 \right) \, dx \\ &= \iint_{\Omega} \left(\left| \sqrt{1+q} \nabla w - \frac{q}{\sqrt{1+q}} h \right|^2 + \frac{q}{1+q} |h|^2 \right) \, dx \\ &\geq \min \left\{ \frac{q(x)}{1+q(x)} : x \in \Omega \right\}. \end{aligned}$$

It remains to show that $\frac{q}{1+q} \geq \frac{q_{\min}}{1+q_{\min}}$ on Ω , which can be obtained from the equality $\frac{q}{1+q} = 1 - \frac{1}{1+q}$. \square

In the following considerations we first restrict ourselves to the case of a constant conductivity contrast q , while later we extend our results to the case of several inclusions with different conductivity contrasts.

3.1 Constant Conductivity contrast

We now assume that the conductivity contrast q is a constant. Our first step is to show a connection between the operator \mathcal{T} and a boundary integral operator on the inclusion boundary $\partial\Omega$. Therefore, we have to define this integral operator and the single layer potential using the Neumann function.

The Neumann function N for the domain B and a fixed point $y \in B$ is defined to be the distributional solution to

$$\begin{aligned}\Delta_x N(x, y) &= \delta(x - y), \quad x \in B, \\ \partial_{\nu(x)} N(x, y) &= |\partial B|^{-1}, \quad x \in \partial B, \\ \int_{\partial B} N(x, y) ds(x) &= 0.\end{aligned}$$

Let $SL : H_\diamond^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^1(B)$ be the single layer potential, that is for smooth φ ,

$$(SL\varphi)(x) = \int_{\partial\Omega} N(x, y)\varphi(y)ds(y), \quad x \in B \setminus \partial\Omega.$$

In [23] it is shown that the range of the single layer potential is a closed subspace of $H_\diamond^1(B)$ that can be represented as follows:

$$\mathcal{R}(SL) = \{w \in H_\diamond^1(B) : \Delta w = 0 \text{ in } B \setminus \partial\Omega, \partial_\nu w = 0 \text{ on } \partial B\}. \quad (8)$$

In addition, the open mapping theorem implies that SL is boundedly invertible on $\mathcal{R}(SL)$.

Let us consider the integral operator $D^* : H_\diamond^{-\frac{1}{2}}(\partial\Omega) \rightarrow H_\diamond^{-\frac{1}{2}}(\partial\Omega)$ which is for smooth φ given by

$$(D^*\varphi)(x) = \int_{\partial\Omega} \partial_{\nu_x} N(x, y)\varphi(y)ds(y), \quad x \in \partial\Omega.$$

We proceed by deriving a connection between \mathcal{T} and D^* in order to characterize the eigenvalues of \mathcal{T} . In Lemma 3.1 we have already shown that $\sigma(\mathcal{T}) \subset \left[\frac{q}{1+q}, q\right)$ which allows us to restrict our considerations to this interval.

Theorem 3.2. *λ is an eigenvalue of \mathcal{T} if and only if $\mu = -\frac{2+q}{2q} + \frac{1}{\lambda}$ is an eigenvalue of D^* and $\mu \neq -\frac{1}{2}$.*

Proof. “ \implies ”: Let $\lambda \in \left[\frac{q}{1+q}, q\right)$ be an eigenvalue of \mathcal{T} , which implies that there exists $h \in \overline{\mathcal{R}(\mathcal{A})}$, $h \neq 0$; such that $\mathcal{T}h = \lambda h$ and $h = \nabla u$ for some $u \in H^1(\Omega)$ satisfying $\Delta u = 0$ in Ω . From the definition of \mathcal{T} we obtain

$$q(\nabla u - \nabla w) = \lambda \nabla u \quad \text{in } \Omega, \quad (9)$$

where w is the weak solution of the transmission boundary value problem (6). From (9) we obtain $(q-\lambda)\nabla u = q\nabla w$ and thus $u = \frac{q}{q-\lambda}w + c$ with some constant c ($\lambda < q$ by assumption). The transmission condition in (6) can now be written in terms of w :

$$\underbrace{\left(1 + q - \frac{q^2}{q-\lambda}\right)}_{:=\rho} \partial_\nu w|_- - \partial_\nu w|_+ = 0 \quad \text{on } \partial\Omega. \quad (10)$$

Since $w \in \mathcal{R}(SL)$, it can be represented by a single layer potential: $w = SL\varphi$ with a density $\varphi \in H_\diamond^{-\frac{1}{2}}(\partial\Omega)$. Using the jump relations for the normal derivative of the single layer potential (see e.g. [21]) and (10) leads to the integral equation

$$\rho \left(\frac{1}{2}\varphi + D^*\varphi \right) - \left(-\frac{1}{2}\varphi + D^*\varphi \right) = 0$$

on $\partial\Omega$ which can be simplified to

$$\frac{1}{2} \frac{\rho+1}{\rho-1} \varphi + D^*\varphi = 0. \quad (11)$$

We note that $\rho \neq 1$, since $\rho = 1$ would imply $\lambda = 0$, a contradiction to $\sigma(\mathcal{T}) \subset \left[\frac{q}{1+q}, q \right)$. Equation (11) has a nontrivial solution φ if and only if $\mu := -\frac{1}{2} \frac{\rho+1}{\rho-1}$ is an eigenvalue of D^* . It remains to show that $-\frac{1}{2} \frac{\rho+1}{\rho-1} = -\frac{2+q}{2q} + \frac{1}{\lambda}$ which is a simple computation. In addition, $\mu \neq -\frac{1}{2}$ since otherwise we have $\lambda = q$, again a contradiction to $\sigma(\mathcal{T}) \subset \left[\frac{q}{1+q}, q \right)$.

“ \Leftarrow ”: Now let $\mu \neq -\frac{1}{2}$ be an eigenvalue of D^* . Then the integral equation

$$-\mu\varphi + D^*\varphi = 0$$

has a nontrivial solution $\varphi \in H_\diamond^{-\frac{1}{2}}(\partial\Omega)$, and the single layer ansatz $w = SL\varphi$ provides $w \in H_\diamond^1(B)$ such that $\Delta w = 0$ in $B \setminus \partial\Omega$, and $\partial_\nu w = 0$ on ∂B . Defining $\rho := \frac{2\mu-1}{2\mu+1}$, we realize that (10) holds. Define $\lambda := \frac{q(1-\rho)}{(1-\rho)+q}$ and $u := \frac{q}{q-\lambda}w$, then w solves (6), and we obtain $q(\nabla u - \nabla w) = \lambda\nabla u$ in Ω . Since $\mu \neq -\frac{1}{2}$ we know from the jump relations that $\partial_\nu w|_- \neq 0$ and thus w, u are non-constant in Ω . \square

Since we assumed the inclusion boundary $\partial\Omega$ to be of class C^2 , the integral operator D^* is a compact operator, hence its spectrum consists at most of zero and a sequence of eigenvalues converging towards zero. From the correspondence of eigenvalues in Theorem 3.2 we deduce that the only possible accumulation point of eigenvalues in the spectrum of \mathcal{T} is $\frac{2q}{2+q}$.

This observation leads to the supposition that the difference operator $\mathcal{T} - \frac{2q}{2+q}I$ is compact, which is subject to the following theorem.

Theorem 3.3. *For $\lambda^* = \frac{2q}{2+q}$ the operator $K : \overline{\mathcal{R}(\mathcal{A})} \rightarrow \overline{\mathcal{R}(\mathcal{A})}$, defined by $K := \mathcal{T} - \lambda^*I$, is compact.*

Proof. Let $\nabla u \in \overline{\mathcal{R}(\mathcal{A})}$, then $K\nabla u = q(\nabla u - \nabla w) - \frac{2q}{2+q}\nabla u = \frac{q^2}{2+q}\nabla u - q\nabla w$, and $w \in H_\diamond^1(B)$ is the solution to (5) for $h = \nabla u$. We define $v \in H^1(\Omega)$ by $v := \frac{q^2}{2+q}u - qw$. Then, since q is constant, we obtain $\nabla v \in \overline{\mathcal{R}(\mathcal{A})}$

Our aim is to decompose the map $K : \nabla u \mapsto \nabla v$ into several bounded operators out of which one is compact and start with (5):

$$\iint_B (1 + q\chi_\Omega) \nabla w \cdot \nabla \psi dx = \iint_\Omega q \nabla u \cdot \nabla \psi dx \quad \text{for all } \psi \in H_\diamond^1(B),$$

and obtain, using $q\nabla u = \frac{2+q}{q}\nabla v + (2+q)\nabla w$, that

$$\begin{aligned} & \iint_{B \setminus \overline{\Omega}} \nabla w \cdot \nabla \psi dx + \iint_\Omega \nabla w \cdot \nabla \psi dx + \iint_\Omega q \nabla w \cdot \nabla \psi dx \\ &= \iint_\Omega \frac{2+q}{q} \nabla v \cdot \nabla \psi dx + \iint_\Omega (2+q) \nabla w \cdot \nabla \psi dx, \end{aligned}$$

which can be transformed to

$$\iint_{B \setminus \overline{\Omega}} \nabla w \cdot \nabla \psi dx - \iint_\Omega \nabla w \cdot \nabla \psi dx = \iint_\Omega \frac{2+q}{q} \nabla v \cdot \nabla \psi dx$$

for all $\psi \in H_\diamond^1(B)$. This weak formulation for w and some given v with $\nabla v \in \overline{\mathcal{R}(\mathcal{A})}$ corresponds to the transmission problem to find $w \in \mathcal{R}(SL)$ (compare (8)) and

$$-\partial_\nu w|_+ - \partial_\nu w|_- = \frac{2+q}{q} \partial_\nu v|_- \quad \text{on } \partial\Omega.$$

As already done in the preceding proofs, we represent w by a single layer potential $w = SL\varphi$ with a density $\varphi \in H_\diamond^{-\frac{1}{2}}(\partial\Omega)$ that solves the integral equation

$$-\left(\frac{1}{2}\varphi + D^*\varphi\right) - \left(-\frac{1}{2}\varphi + D^*\varphi\right) = \frac{2+q}{q} \partial_\nu v|_-,$$

which can be simplified to

$$-2D^*\varphi = \frac{2+q}{q} \partial_\nu v|_-.$$

Now we return to the operator $K : \overline{\mathcal{R}(\mathcal{A})} \rightarrow \overline{\mathcal{R}(\mathcal{A})}$, $\nabla u \mapsto \nabla v$, and show that it is a composition of several bounded operators out of which at least one is compact. Let the auxiliary operator $\tilde{T} : \overline{\mathcal{R}(\mathcal{A})} \rightarrow \mathcal{R}(SL)$ be defined by $\nabla u \mapsto w$, then the considered composition of operators is as follows:

$$K : \nabla u \xrightarrow{\tilde{T}} w \xrightarrow{SL^{-1}} \varphi \xrightarrow{D^*} -\frac{2+q}{2q} \partial_\nu v|_- \xrightarrow{} \nabla v.$$

The map \tilde{T} is bounded, and the single layer potential $SL : H_\diamond^{-\frac{1}{2}}(\partial\Omega) \rightarrow \mathcal{R}(SL)$ is bounded, bijective and has a bounded inverse $SL^{-1} : \mathcal{R}(SL) \rightarrow H_\diamond^{-\frac{1}{2}}(\partial\Omega)$. Hence the map $w \mapsto \varphi$ is also bounded. Since $\partial\Omega$ is of class C^2 , $D^* : H_\diamond^{-\frac{1}{2}}(\partial\Omega) \rightarrow H_\diamond^{-\frac{1}{2}}(\partial\Omega)$ is a compact operator. The operator that maps the Neumann boundary value $\frac{2+q}{2q} \partial_\nu v|_- \in H_\diamond^{-\frac{1}{2}}(\partial\Omega)$ to the corresponding weak solution $v \in H_\diamond^1(\Omega)$ of the Laplace equation in Ω as well as the weak derivative $v \mapsto \nabla v \in \overline{\mathcal{R}(\mathcal{A})}$ are again bounded, which completes the proof. \square

Since \mathcal{T} is self-adjoint we conclude that also K is self-adjoint. The spectral theorem for compact, self-adjoint operators applied to K yields the following corollary.

Corollary 3.4. *There is an orthonormal system of eigenfunctions of \mathcal{T} in $\overline{\mathcal{R}(\mathcal{A})}$ with corresponding eigenvalues λ_n ($n \in \mathbb{N}$) converging towards $\lambda^* = \frac{2q}{2+q}$. The spectrum $\sigma(\mathcal{T})$ consists of these eigenvalues λ_n ($n \in \mathbb{N}$) and their limit point λ^* .*

In the following subsection we carry this result over to the case in which q is constant on each component of Ω with different values on the components.

3.2 Several disjoint Inclusions

Now we consider the case of N inclusions. The corresponding problem setting is formulated in the following assumption.

Assumption 3.5. Let $\Omega_1, \dots, \Omega_N$ be N separated C^2 -domains in B , that is $\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset$ ($i, j = 1, \dots, N, i \neq j$). Let the conductivity distribution γ satisfy

$$\gamma(x) = \begin{cases} 1 + q_j, & x \in \Omega_j \quad (j = 1, \dots, N), \\ 1, & \text{otherwise.} \end{cases}$$

The constants q_j are such that $1 + q_j > 0$ for all $j = 1, \dots, N$. By Ω we denote the union $\Omega = \Omega_1 \cup \dots \cup \Omega_N$, while we denote $(\gamma - 1)|_{\Omega}$ by q . We also assume without loss of generality that the conductivity contrasts are mutually different from each other since for $q_j = q_k$ we can subsume Ω_j and Ω_k under one inclusion. As before, $B \setminus \overline{\Omega}$ is assumed to be connected.

The operators of our factorization (7) now have the following mapping properties:

$$\begin{aligned} \mathcal{A} : L^2_{\diamond}(\partial B) &\rightarrow L^2(\Omega_1, \mathbb{R}^2) \times \dots \times L^2(\Omega_N, \mathbb{R}^2), \\ f &\mapsto (\nabla u_0|_{\Omega_1}, \dots, \nabla u_0|_{\Omega_N})^{\top}, \end{aligned}$$

where $u_0 \in H^1_{\diamond}(B)$ solves the background direct problem (2). For the adjoint \mathcal{A}^* we obtain

$$\begin{aligned} \mathcal{A}^* : L^2(\Omega_1, \mathbb{R}^2) \times \dots \times L^2(\Omega_N, \mathbb{R}^2) &\rightarrow L^2_{\diamond}(\partial B), \\ (h_1, \dots, h_N)^{\top} &\mapsto w|_{\partial B}, \end{aligned}$$

and $v \in H^1_{\diamond}(B)$ is the solution to

$$\iint_B \nabla w \cdot \nabla \psi \, dx = \sum_{j=1}^N \iint_{\Omega_j} h_j \cdot \nabla \psi \, dx \quad \text{for all } \psi \in H^1_{\diamond}(B).$$

For the middle operator we obtain

$$\begin{aligned} \mathcal{T} : L^2(\Omega_1, \mathbb{R}^2) \times \dots \times L^2(\Omega_N, \mathbb{R}^2) &\rightarrow L^2(\Omega_1, \mathbb{R}^2) \times \dots \times L^2(\Omega_N, \mathbb{R}^2) \\ (h_1, \dots, h_N)^{\top} &\mapsto (q_1 (h_1 - \nabla w|_{\Omega_1}), \dots, q_N (h_N - \nabla w|_{\Omega_N}))^{\top}, \end{aligned}$$

where $w \in H_\diamond^1(B)$ solves

$$\iint_B \nabla w \cdot \nabla \psi \, dx + \sum_{j=1}^N \iint_{\Omega_j} q_j \nabla w \cdot \nabla \psi \, dx = \sum_{j=1}^N \iint_{\Omega_j} q_j h_j \cdot \nabla \psi \, dx$$

for all $\psi \in H_\diamond^1(B)$.

Our goal is to obtain information about the structure of the spectrum of the operator \mathcal{T} similarly to Corollary 3.4. To this end, the next step is to decouple \mathcal{T} according to the different inclusions, that is to find a representation for \mathcal{T} of the type

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}^{(11)} & \dots & \mathcal{T}^{(1N)} \\ \vdots & \ddots & \vdots \\ \mathcal{T}^{(N1)} & \dots & \mathcal{T}^{(NN)} \end{pmatrix}$$

such that $\mathcal{T}^{(ij)} : L^2(\Omega_j, \mathbb{R}^d) \rightarrow L^2(\Omega_i, \mathbb{R}^d)$.

It is easy to check that $\mathcal{T}^{(ii)}$ is given by $h_i \mapsto q_i(h_i - \nabla w_i|_{\Omega_i})$ and that $\mathcal{T}^{(ij)}$ for $i \neq j$ is defined by $h_j \mapsto -q_i \nabla w_j|_{\Omega_i}$. Here, for $h_j \in L^2(\Omega_j, \mathbb{R}^d)$ and $j = 1, \dots, N$ the potential $w_j \in H_\diamond^1(B)$ is defined as the solution of

$$\iint_B \nabla w_j \cdot \nabla \psi \, dx + \sum_{k=1}^N \iint_{\Omega_k} q_k \nabla w_j \cdot \nabla \psi \, dx = \iint_{\Omega_j} q_j h_j \cdot \nabla \psi \, dx \quad (12)$$

for all $\psi \in H_\diamond^1(B)$. The following lemma shows a connection to the previously considered case of only one inclusion. We therefore define $\mathcal{T}^{(i)} : L^2(\Omega_i, \mathbb{R}^d) \rightarrow L^2(\Omega_i, \mathbb{R}^d)$ to be the operator \mathcal{T} that corresponds to the case of only a single inclusion Ω_i , that is, $\mathcal{T}^{(i)} : h_i \mapsto q_i(h_i - \nabla \tilde{w}_i|_{\Omega_i})$, and $\tilde{w}_i \in H_\diamond^1(B)$ solves

$$\iint_B \nabla \tilde{w}_i \cdot \nabla \psi \, dx + \iint_{\Omega_i} q_i \nabla \tilde{w}_i \cdot \nabla \psi \, dx = \iint_{\Omega_i} q_i h_i \cdot \nabla \psi \, dx \quad (13)$$

for all $\psi \in H_\diamond^1(B)$.

Lemma 3.6. *Let the operators $\mathcal{T}^{(ij)}$ and $\mathcal{T}^{(i)}$ be defined as above for $i, j = 1, \dots, N$, then:*

(a) *for $i \neq j$ the operator $\mathcal{T}^{(ij)}$ is compact ($i, j = 1, \dots, N$),*

(b) *the operator $\mathcal{T}^{(ii)} - \mathcal{T}^{(i)}$ is compact ($i = 1, \dots, N$).*

Proof. Part a): $\mathcal{T}^{(ij)}$ maps $h_j \mapsto -q_i \nabla w_j|_{\Omega_i}$, where w_j is the solution of (12). Let $\tilde{\Omega}_i \subset B$ be a simply connected C^2 -domain such that $\bar{\Omega}_i \subset \tilde{\Omega}_i$ and $\bar{\tilde{\Omega}}_i \cap \bar{\Omega}_j = \emptyset$ for all $j \neq i$. Then we can decompose $\mathcal{T}^{(ij)}$ into $\mathcal{T}^{(ij)} = -q_i \tilde{S} \circ \hat{S}$ where $\hat{S} : L^2(\Omega_j, \mathbb{R}^2) \rightarrow H_\diamond^{-\frac{1}{2}}(\partial\tilde{\Omega}_i)$ maps h_j to the trace $\partial_\nu w_j|_{\partial\tilde{\Omega}_i}$ and $\tilde{S} : H_\diamond^{-\frac{1}{2}}(\partial\tilde{\Omega}_i) \rightarrow L^2(\Omega_i, \mathbb{R}^2)$ maps $g \mapsto \nabla v|_{\Omega_i}$ where $v \in H_\diamond^1(\tilde{\Omega}_i)$ solves

$$\begin{aligned} \Delta v &= 0 \text{ in } \tilde{\Omega}_i \setminus \partial\Omega_i, \\ \partial_\nu v &= g \text{ on } \partial\tilde{\Omega}_i, \\ (1 + q_i) \partial_\nu v|_+ - \partial_\nu v|_- &= 0 \text{ on } \partial\Omega_i \end{aligned}$$

in the weak sense. Both partial operators are bounded and \hat{S} is compact by Theorem 8.8 in [11]. Together with the trace theorem this yields that $\partial_\nu w|_{\partial\tilde{\Omega}_i} \in H^{\frac{1}{2}}(\partial\tilde{\Omega}_i)$ which proves the first assertion.

Part b): The weak formulation for w_i is (12), while the weak formulation for \tilde{w}_i corresponding to $\mathcal{T}^{(i)}$, that is for the case of the only inclusion Ω_i is (13). Setting $v_i := w_i - \tilde{w}_i$ we deduce that v_i solves

$$\iint_B \nabla v_i \cdot \nabla \psi \, dx + \iint_{\Omega_i} q_i \nabla v_i \cdot \nabla \psi \, dx = - \sum_{\substack{k=1 \\ k \neq i}}^N \iint_{\Omega_k} q_k \nabla w_i \cdot \nabla \psi \, dx$$

for all $\psi \in H^1_\diamond(B)$. The map $\mathcal{T}^{(ii)} - \mathcal{T}^{(i)}$ may now be decomposed as follows:

$$h|_{\Omega_i} \mapsto \nabla w_i|_{\Omega \setminus \Omega_i} \mapsto \nabla v_i|_{\Omega_i} \mapsto -q_i \nabla v_i|_{\Omega_i},$$

where we can show analogously to part a) that the first map in this decomposition is compact. Furthermore, it is easy to see that the other maps are bounded. \square

Lemma 3.6 now yields the following representation:

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \mathcal{T}^{(N)} \end{pmatrix} + K,$$

with a compact and self-adjoint operator K .

Furthermore, this representation can now be used to investigate the spectrum of \mathcal{T} as it is done in the following theorem.

Theorem 3.7. *In the case of N inclusions $\Omega_1, \dots, \Omega_N$ with conductivity contrasts q_1, \dots, q_N the spectrum of \mathcal{T} consists of a countable set of eigenvalues and the points $\lambda_1^*, \dots, \lambda_N^*$ that are defined by $\lambda_j^* = \frac{2q_j}{2+q_j}$ ($j = 1, \dots, N$). The points λ_j^* ($j = 1, \dots, N$) are the only possible accumulation points in $\sigma(\mathcal{T})$.*

Proof. From Theorem 3.3 we know that $\mathcal{T}^{(i)} = \lambda_i^* I + K_i$ ($i = 1, \dots, N$), and the K_i are self-adjoint and compact operators. Hence we can represent \mathcal{T} as

$$\mathcal{T} = \begin{pmatrix} \lambda_1^* I & & 0 \\ & \ddots & \\ 0 & & \lambda_N^* I \end{pmatrix} + \tilde{K}, \quad (14)$$

and \tilde{K} is a compact operator.

Now we make use of the concept of the essential spectrum (see [16] or [22]). The essential spectrum consists of all λ for which $\mathcal{T} - \lambda I$ is not semi-Fredholm, that is, for these λ neither the nullspace $\mathcal{N}(\mathcal{T} - \lambda I)$ nor the defect $\overline{\mathcal{R}(\mathcal{A})} / \mathcal{R}(\mathcal{T} - \lambda I)$ is finite-dimensional. This definition implies immediately that the first part in representation (14), namely the operator

$$\begin{pmatrix} \lambda_1^* I & & 0 \\ & \ddots & \\ 0 & & \lambda_N^* I \end{pmatrix},$$

has the essential spectrum $\{\lambda_1^*, \dots, \lambda_N^*\}$. Theorem 5.35 in Chapter IV of [16] states that the compact perturbation \tilde{K} has no effect on the essential spectrum and thus that $\sigma_{ess}(\mathcal{T}) = \{\lambda_1^*, \dots, \lambda_N^*\}$.

From Theorem 5.33 in Chapter IV of [16] we now obtain that $\sigma(\mathcal{T})$ consists of $\sigma_{ess}(\mathcal{T})$ and a countable set of eigenvalues. In addition, these eigenvalues are isolated eigenvalues, which means that none of them is an accumulation point in $\sigma(\mathcal{T})$. □

The results from Corollary 3.4 and Theorem 3.7 imply that in order to find the conductivity contrast q we have to compute an approximation of the spectrum $\sigma(\mathcal{T})$, identify the accumulation points in the spectrum and compute q from them. In the following section we present a projection method to obtain an approximation of $\sigma(\mathcal{T})$ from $\Lambda_0 - \Lambda$ and the knowledge about Ω .

4 Approximation of the spectrum

Our method to approximate the spectrum $\sigma(\mathcal{T})$ is based on the generalized eigenvalue problem

$$(\Lambda_0 - \Lambda)f = \lambda \mathcal{A}^* \mathcal{A}f \quad \text{for } f \in L_\diamond^2(\partial B). \quad (15)$$

Since we assume the inclusion Ω to be known, all the operators in (15) are known. In the following considerations we investigate a discrete version of (15).

Let Y_n be an n -dimensional subspace of $L_\diamond^2(\partial B)$ such that $\bigcup_{n=1}^{\infty} Y_n$ is dense in $L_\diamond^2(\partial B)$. Let $X_n = \mathcal{A}(Y_n)$, then X_n is also n -dimensional since \mathcal{A} is injective and $\bigcup_{n=1}^{\infty} X_n$ is dense in $\overline{\mathcal{R}(\mathcal{A})}$. We define the maps $P_n : L_\diamond^2(\partial B) \rightarrow Y_n$, $Q_n : L^2(\Omega, \mathbb{R}^d) \rightarrow X_n$ to be the corresponding orthogonal projections.

Now we consider the discrete version of (15), namely

$$P_n(\Lambda_0 - \Lambda)f_n = \lambda_n P_n \mathcal{A}^* \mathcal{A}f_n \quad \text{for } f_n \in Y_n. \quad (16)$$

This n -dimensional generalized eigenvalue problem has n real-valued eigenvalues $\lambda_n^{(1)}, \dots, \lambda_n^{(n)}$ and corresponding eigenfunctions $f_n^{(1)}, \dots, f_n^{(n)}$, since injectivity of \mathcal{A} implies injectivity of $P_n \mathcal{A}^* \mathcal{A}|_{Y_n}$, and $\Lambda_0 - \Lambda$ is self-adjoint which also implies self-adjointness of $P_n(\Lambda_0 - \Lambda)|_{Y_n}$. In addition, the operator $P_n \mathcal{A}^* \mathcal{A}|_{Y_n}$ is self-adjoint and positive definite independently of the conductivity.

We investigate the behavior of $\lambda_n^{(j)}$ and $f_n^{(j)}$ for $n \rightarrow \infty$ and arbitrary $j \in \mathbb{N}$. In order to keep our notations simple we only write λ_n, f_n instead of $\lambda_n^{(j)}$ and $f_n^{(j)}$, where λ_n denotes $\lambda_n^{(j)}$ for some $j \in \mathbb{N}$.

In the following lemma we show that the discrete eigenvalue problem (16) can be used to approximate the spectrum of \mathcal{T} . We restrict ourselves to the case of one constant conductivity contrast and we denote $\lambda^* = \frac{2q}{2+q}$ as in Corollary 3.4.

Lemma 4.1. *Let q be constant and let λ_n be eigenvalues according to the discrete generalized eigenvalue problem (16) for $n \in \mathbb{N}$. Then there exist accumulation points of $(\lambda_n)_{n \in \mathbb{N}}$ and every accumulation point is an eigenvalue of \mathcal{T} or equal to λ^* .*

In addition, if the sequence $(\lambda_n)_{n \in \mathbb{N}}$ itself converges to some λ , then the limit λ is an eigenvalue of \mathcal{T} or equal to λ^* .

Proof. We first derive an equivalent formulation of the eigenvalue problem (16) that can be used to show convergence of eigenvalues towards eigenvalues of \mathcal{T} and the relationship to the infinite-dimensional case. By the factorization (7) and Theorem 3.3, (16) can be written as

$$\lambda^* P_n \mathcal{A}^* \mathcal{A} f_n + P_n \mathcal{A}^* K \mathcal{A} f_n = \lambda_n P_n \mathcal{A}^* \mathcal{A} f_n.$$

Setting $g_n := \mathcal{A} f_n \in X_n$ and assuming $\|g_n\|_{L^2(\Omega, \mathbb{R}^d)} = 1$ without loss of generality we write

$$(\lambda^* - \lambda_n) P_n \mathcal{A}^* g_n + P_n \mathcal{A}^* K g_n = 0,$$

which is equivalent to

$$\langle (\lambda^* - \lambda_n) \mathcal{A}^* g_n, \psi_n \rangle_{L^2(\partial B)} + \langle \mathcal{A}^* K g_n, \psi_n \rangle_{L^2(\partial B)} = 0 \quad \text{for all } \psi_n \in Y_n.$$

Using the duality of \mathcal{A} and \mathcal{A}^* we obtain

$$\langle (\lambda^* - \lambda_n) g_n, \mathcal{A} \psi_n \rangle_{L^2(\Omega, \mathbb{R}^d)} + \langle K g_n, \mathcal{A} \psi_n \rangle_{L^2(\Omega, \mathbb{R}^d)} = 0 \quad \text{for all } \psi_n \in Y_n$$

and thus

$$\langle (\lambda^* - \lambda_n) g_n, \phi_n \rangle_{L^2(\Omega, \mathbb{R}^d)} + \langle K g_n, \phi_n \rangle_{L^2(\Omega, \mathbb{R}^d)} = 0 \quad \text{for all } \phi_n \in X_n.$$

Using the orthogonal projection Q_n this is equivalent to

$$(\lambda^* - \lambda_n) g_n + Q_n K g_n = 0. \tag{17}$$

Let λ_n, g_n be eigenpairs according to (17) for $n \in \mathbb{N}$. Since the g_n are uniformly bounded there is a subsequence g_{n_k} that converges weakly to some $g \in L^2(\Omega, \mathbb{R}^d)$ for $k \rightarrow \infty$. We know by the compactness of K and the denseness of $\bigcup_{n=1}^{\infty} X_n$ in $\overline{\mathcal{R}(\mathcal{A})}$ that $Q_{n_k} K g_{n_k} \rightarrow K g$ ($k \rightarrow \infty$). This means that $(\lambda^* - \lambda_{n_k}) g_{n_k} \rightarrow -K g$ ($k \rightarrow \infty$) in $L^2(\Omega, \mathbb{R}^d)$. This implies

$$(\lambda^* - \lambda_{n_k}) \langle g_{n_k}, \phi \rangle_{L^2(\Omega, \mathbb{R}^d)} \rightarrow -\langle K g, \phi \rangle_{L^2(\Omega, \mathbb{R}^d)} \quad (k \rightarrow \infty) \tag{18}$$

for all $\phi \in L^2(\Omega, \mathbb{R}^d)$ and

$$|\lambda^* - \lambda_{n_k}|^2 = |\lambda^* - \lambda_{n_k}|^2 \|g_{n_k}\|_{L^2(\Omega, \mathbb{R}^d)}^2 \rightarrow \|K g\|_{L^2(\Omega, \mathbb{R}^d)}^2 \quad (k \rightarrow \infty). \tag{19}$$

This means in particular that $(\lambda_{n_k})_{k \in \mathbb{N}}$ is a bounded sequence and thus that $\lambda_{n_k} \rightarrow \lambda$ ($k \rightarrow \infty$) for a further subsequence. From (18) we now obtain $(\lambda^* - \lambda)g + K g = 0$. We have to distinguish between two different cases: If $g \neq 0$, then this implies immediately that λ is an eigenvalue of \mathcal{T} . If $g = 0$, then (19) yields $\lambda = \lambda^*$.

For the second assertion assume that $\lambda_n \rightarrow \lambda$ ($n \rightarrow \infty$) and $\lambda \neq \lambda^*$. Then we can again assume without loss of generality that $\|g_n\|_{L^2(\Omega, \mathbb{R}^d)} = 1$ for $n \in \mathbb{N}$ and thus that there exists a weakly convergent subsequence $(g_{n_k})_{k \in \mathbb{N}}$. As above we obtain from (19) that $g \neq 0$ and from (18) that $(\lambda^* - \lambda)g + K g = 0$ and thus that λ is an eigenvalue of \mathcal{T} , which completes the proof. \square

This result means that for sufficiently large n at least some of the generalized eigenvalues λ_n of the discrete eigenvalue problem (16) lie arbitrarily close to eigenvalues of \mathcal{T} or to λ^* . In particular, we observe that they accumulate in λ^* .

A very similar result can be shown for the case of N different conductivity contrast (compare Lemma 3.3.5 in [23]). However, in the case of N inclusions it remains open which conductivity contrast belongs to which inclusion. In the following section we present some numerical examples with this projection method.

5 Numerical Experiments

In this section we present some numerical examples with our new method to approximate the conductivity inside anomalies. We first explain how the operators $\Lambda_0 - \Lambda$ and $\mathcal{A}^* \mathcal{A}$ are computed numerically. Afterwards we present some examples concerning exact and inexact data $\Lambda_0 - \Lambda$ as well as exact and inexact inclusion boundaries.

In all our examples the domain B is the unit disc in \mathbb{R}^2 . At first we need to compute discrete versions of $\Lambda_0 - \Lambda$ and of $\mathcal{A}^* \mathcal{A}$. Since we restrict ourselves to piecewise constant conductivities these operators can be computed easily using boundary integral equation methods.

Let us fix some arbitrary current pattern $f \in L^2_\diamond(\partial B)$. Now define $v \in H^1_\diamond(B)$ by $v := u_0 - u$ where $u \in H^1_\diamond(B)$ solves the direct problem (1) with the conductivity $\gamma(x) = 1 + q\chi_\Omega(x)$ and $u_0 \in H^1_\diamond(B)$ solves (2). Then for v we obtain the transmission problem

$$\begin{aligned} \Delta v &= 0 \text{ in } B \setminus \partial\Omega, \\ \partial_\nu v &= 0 \text{ on } \partial B, \\ (1+q) \partial_\nu v|_- - \partial_\nu v|_+ &= q \partial_\nu u_0 \text{ on } \partial\Omega. \end{aligned} \tag{20}$$

For B being the unit disc, u_0 may be calculated explicitly, and (20) can be transformed to a boundary integral equation of second type. By setting $v = SL\varphi$ we obtain the following integral equation on $\partial\Omega$ for the density $\varphi \in H^{-\frac{1}{2}}_\diamond(\partial\Omega)$:

$$\left(1 + \frac{q}{2}\right) \varphi + qD^* \varphi = q \partial_\nu u_0.$$

For $\mathcal{A}^* \mathcal{A}$ the corresponding transmission problem is (4) with $v = u_0$. Here it is not even necessary to solve a boundary integral equation as for $\Lambda_0 - \Lambda$ since the solution is $v = SL\varphi$ with $\varphi = \partial_\nu u_0|_{\partial\Omega}$.

We apply these integral equation methods for the following orthogonal current patterns:

$$\{1/\sqrt{\pi} \cos(k\theta), 1/\sqrt{\pi} \sin(k\theta) : k = 1, 2, \dots\},$$

where θ denotes the argument of points on the boundary ∂B in polar coordinates. We thus obtain the discrete versions $L_n := P_n(\Lambda_0 - \Lambda)|_{Y_n}$ and $A_n := P_n \mathcal{A}^* \mathcal{A}|_{Y_n}$ of $\Lambda_0 - \Lambda$ and $\mathcal{A}^* \mathcal{A}$, respectively, where Y_n is the n -dimensional subspace of $L^2_\diamond(\partial B)$ spanned by the basis of current patterns up to degree n .

For our experiments we use the following test model: there are two inclusions, where Ω_1 is an elliptic inclusion in the upper right part of B having the

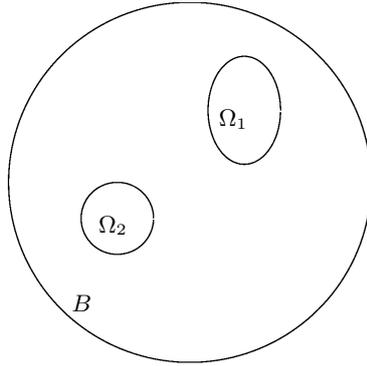


Figure 1: The test model with corresponding inclusions

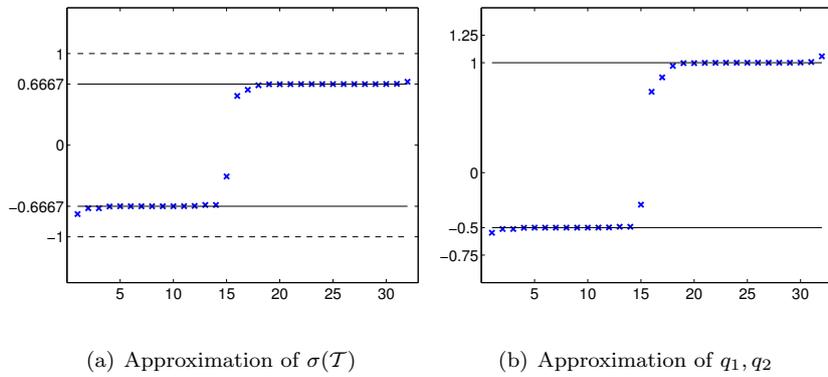


Figure 2: Approximation of $\sigma(\mathcal{T})$ and the conductivity contrasts q_1, q_2 (blue ‘x’) for exact data. continuous lines: accumulation points λ_1^*, λ_2^* (part (a)), and q_1, q_2 (part(b)), dashed lines: upper and lower bound for $\sigma(\mathcal{T})$

conductivity contrast $q_1 = 1$, while Ω_2 is a circle located in the lower left part of B with $q_2 = -0.5$. The inclusions are illustrated in Figure 1. We make use of the generalized eigenvalue problem (16) that can be rewritten as

$$L_n f_n = \lambda_n A_n f_n \quad \text{for } f_n \in Y_n. \quad (21)$$

Such a generalized eigenvalue problem is solved using the Cholesky factorization for the operator A_n on the right hand side. In our first example we solve (21) for $n = 32$ to obtain an approximation of $\sigma(\mathcal{T})$ and compute the corresponding approximated values of q_1, q_2 by using the correspondence $\lambda_i^* = \frac{2q_i}{2+q_i}$ ($i = 1, 2$) between the the accumulation points λ_1^*, λ_2^* in $\sigma(\mathcal{T})$ and the conductivity contrasts q_1, q_2 .

In Figure 2 the corresponding results are shown. We observe that in part (a) the approximated eigenvalues accumulate in λ_1^*, λ_2^* that are indicated by the continuous black line (compare Theorem 3.7). We also observe that the eigenvalues from (21) are all lying between the bounds for $\sigma(\mathcal{T})$ that are indicated by dashed black lines (compare Lemma 3.1). Moreover, the approximations for

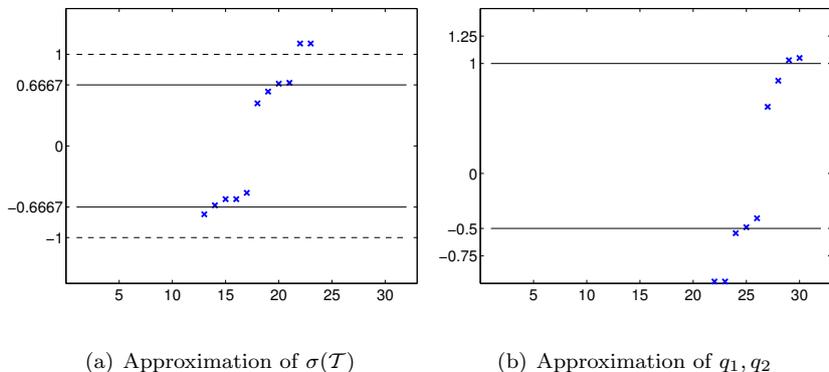


Figure 3: Approximation of $\sigma(\mathcal{T})$ and the conductivity contrasts q_1, q_2 (blue ‘x’) for 0.1% of white noise added to $\Lambda_0 - \Lambda$. continuous lines: accumulation points λ_1^*, λ_2^* (part (a)), and q_1, q_2 (part(b)), dashed lines: upper and lower bound for $\sigma(\mathcal{T})$

q_1, q_2 in part (b) of Figure 2 are quite accurate approximations for the exact values of q_1, q_2 that are indicated by continuous black lines.

The previous test was conducted for unperturbed data $\Lambda_0 - \Lambda$ and under the assumption that the inclusion boundaries are known exactly. In the subsequent examples we show some numerical test for perturbed $\Lambda_0 - \Lambda$ as well as for perturbed inclusion boundaries. We start by computing the approximation of the conductivity contrasts where $\Lambda_0 - \Lambda$ is perturbed by 0.1% of white noise.

Figure 3 shows the corresponding results. We observe in part (a) that for both test models there are only few computed eigenvalues lying in the neighborhood of λ_1^* and λ_2^* , while the others have very large absolute values. This affects the approximations of q as well. In addition, we can only obtain estimates of q_1, q_2 from part (b) of Figure 3 since there are no distinct accumulation points as in the previous noiseless examples.

These results show that our method is quite sensitive to noise which can also be observed in the generalized eigenvalue problem (21): both $\Lambda_0 - \Lambda$ and $\mathcal{A}^* \mathcal{A}$ are compact operators and thus the discrete versions L_n and A_n are ill-conditioned. The algorithm we used to solve the generalized eigenvalue problem (21) performs a Cholesky factorization of A_n : $A_n = C_n C_n^\top$, where C_n is a lower triangular matrix. Now (21) is transformed to an ordinary eigenvalue problem by multiplication with the inverses of the ill-conditioned matrices C_n and C_n^\top on both sides.

In the following example we investigate the effect of an inexactly known inclusion boundary $\partial\Omega$ on the approximations of $\sigma(\mathcal{T})$ and q . Figure 4 shows the exact boundary of $\partial\Omega$ (black lines) as well as the perturbed boundary (red lines). In Figure 5 the corresponding approximations of $\sigma(\mathcal{T})$ and q_1, q_2 are shown. We observe that the approximated eigenvalues don’t exhibit any clear accumulation points. The same holds for the approximations of q_1, q_2 . However, we can still observe that there are two distinct values of q_1 and q_2 whose values can be read out roughly.

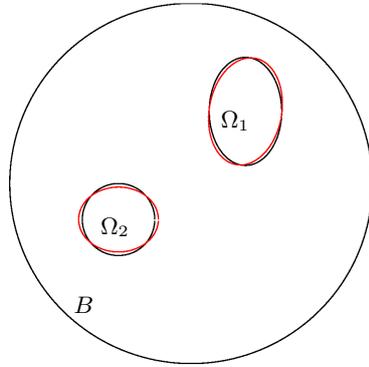


Figure 4: The test model with perturbed inclusion boundaries

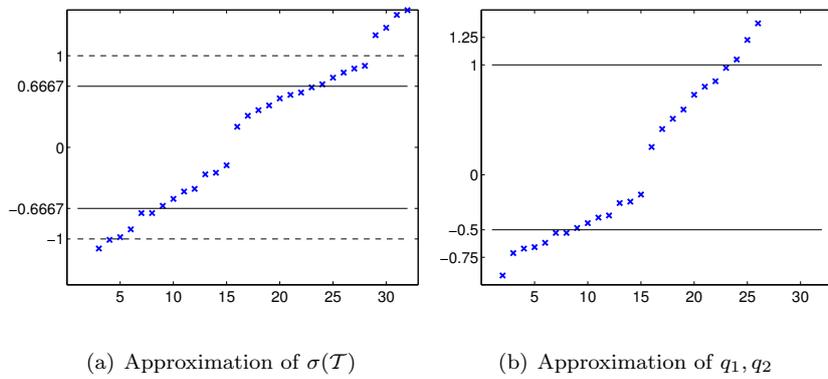


Figure 5: Approximation of $\sigma(\mathcal{T})$ and the conductivity contrasts q_1, q_2 (blue 'x') for perturbed inclusion boundaries. continuous lines: accumulation points λ_1^*, λ_2^* (part (a)), and q_1, q_2 (part(b)), dashed lines: upper and lower bound for $\sigma(\mathcal{T})$

Conclusions

In our theoretical derivations we presented a factorization of the operator $\Lambda_0 - \Lambda$ and observed that the desired information about the conductivity contrast q is contained in the middle operator. We investigated the structure of its spectrum $\sigma(\mathcal{T})$ and showed a connection to boundary integral operators. In particular, we showed that that q is closely connected to the accumulation points in $\sigma(\mathcal{T})$ and presented a projection method to compute $\sigma(\mathcal{T})$ numerically.

The numerical tests show that our method of approximating q via the spectrum of \mathcal{T} works quite well in the case of exact data. In the case where $\Lambda_0 - \Lambda$ is perturbed we can still obtain a rough estimate of the conductivity contrast. The same holds for the case of an imperfectly known boundary of Ω . However, in order to improve the stability of our method towards errors in the data, it would be desirable to find a reasonable regularization strategy for generalized eigenvalue problems with compact operators on both sides.

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