Problem Sheet 2

due date: 9.11.2011

Problem 1: Derive a

(a) centered finite difference approximation D of fourth order in h for $\partial^2/\partial x^2$, i.e.

$$|Du(x_j) - \tfrac{\partial^2 u}{\partial x^2}(x_j)| \leq ch^4$$

with a constant c independent of h. Use the 5-point stencil $x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}$.

(b) centered finite difference approximation D of fourth order in h for $\partial/\partial x$, i.e.

$$|Du(x_j) - \tfrac{\partial u}{\partial x}(x_j)| \leq ch^4$$

with a constant c independent of h. Use the 5-point stencil $x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}$.

Problem 2: Solve the Poisson problem

$$-\Delta u = \cos\left(\tfrac{\pi}{2}x\right)\cos\left(\tfrac{3\pi}{4}y\right) \text{ for } (x,y) \in \Omega := (-1,1)\times(-2,2)$$
$$u = 0 \text{ for } (x,y) \in \partial\Omega$$

(a) using the second order centered finite difference formula $D^{2,2}$ with the 5-point stencil for the 2D-Laplacian. The discrete grid is now (x_i, y_j) with $i \in \{1,\dots,n_1\}, j \in \{1,\dots,n_2\}$. Store the matrix K_h in the sparse format and compute the approximate solution $\tilde{U} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n)^T$ with $n = n_1 n_2$ for $h_1 = h_2 = 1/20, 1/40, 1/80$, and $1/160$. Find the exact solution $u = u(x,y)$ and plot the l^∞-norm of the error on the grid as a function of h and check that the error is $O(h^2)$.

(b) using the fourth order centered discretization formula of Δ obtained by using D from problem 1(a) in each dimension. This formula can be used everywhere except at those grid points which are neighbors of $\partial\Omega$. Here use the formula $D^{2,2}$. Study the convergence of the l^∞-error.

Problem 3:

COnsider the Poisson problem on the unit disk

$$\begin{aligned} -\Delta u(x) &= J_0(|x|) \text{ for } |x| < 1, x \in \mathbb{R}^2 \\ u(x) &= 0 \text{ for } |x| = 1, \end{aligned} \tag{0.1}$$

where $J_0(z)$ is the Bessel function of zeroth order given by

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{z}{2}\right)^{2k} \text{ for all } z \in \mathbb{C}.$$

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(a) Show that

$$z^2 J_0''(z) + zJ_0'(z) + z^2 J_0(z) = 0 \text{ for all } z \in \mathbb{C}.$$

(b) Define the polar coordinates $r(x) = |x| = \sqrt{x_1^2 + x_2^2}$ and $\phi(x) = \arg(x_1 + ix_2)$ and show that $u(x) = \tilde{u}(r(x), \phi(x))$ with $\tilde{u}(r, \phi) = J_0(r) - J_0(1)$ solves (0.1).

Hint: Rewrite the Laplacian in polar coordinates.

(c) Solve (0.1) numerically in polar coordinates via a second order accurate finite difference discretization. Check the convergence of the l^∞ error.

Hints: The problem is ϕ-independent. The boundary condition at $r = 0$ is of Neumann type.