INSTITUTE FOR APPLIED AND NUMERICAL MATHEMATICS 3 — TOMÁŠ DOHNAL
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Introduction to the Numerics of PDEs — WS 2011/12

Problem Sheet 4

due date: 7.12.2011

Problem 1: Prove stability and conclude the order of convergence in the l^∞-norm of the second order centered 5-point stencil finite difference discretization of the 2D Poisson problem with Dirichlet boundary conditions

$$\begin{aligned} -\Delta u(x) &= f(x) \text{ for } x \in \Omega := (0,a) \times (0,b) \text{ with } a, b > 0 \\ u(x) &= g(x) \text{ for } x \in \partial\Omega \end{aligned}$$

with f and g smooth.

Problem 2: Prove stability and conclude the order of convergence in the l^∞-norm of the second order centered 3-point stencil finite difference discretization of the 1D Poisson problem with the following Neumann/Dirichlet boundary conditions

$$\begin{aligned} -u''(x) =& f(x) \text{ for } x \in (0,a) \text{ with } a > 0 \\ u'(0) =& 0, u(a) = 0 \end{aligned}$$

with f smooth and with the first derivative in the Neumann condition discretized by the formula $u'(x_j) \approx (u(x_{j+1}) - u(x_j))/h$.

Problem 3: (optional) Laplace equation on the unit disk with a re-entrant corner.

Consider

$$\begin{aligned} -\Delta u(x) &= 0 \text{ for } x \in \Omega = \{x \in \mathbb{R}^2 : |x| < 1 \text{ and } x \notin [0,1] \times [-1,0]\} \\ u(x) &= r^{2/3} \sin(\tfrac{2}{3}\phi) \text{ for } x \in \partial\Omega, \end{aligned} \tag{0.1}$$

where $r = r(x) = \sqrt{x_1^2 + x_2^2}$ and $\phi = \phi(x) = \arg(x_1 + ix_2)$.

(a) Check that $u(x) = r(x)^{2/3} \sin(\frac{2}{3}\phi(x))$ solves (0.1).

(b) Solve (0.1) numerically in polar coordinates using second order 3-point stencil formulas for all the derivative terms.

Study the convergence of the l^∞-error and comment on the results.

Hint: All boundary conditions in both r and ϕ are of Dirichlet type.