INSTITUTE FOR APPLIED AND NUMERICAL MATHEMATICS 3 — TOMÁŠ DOHNAL
KARLSRUHE INSTITUTE OF TECHNOLOGY — dohnal@kit.edu
Introduction to the Numerics of PDEs — WS 2011/12

Problem Sheet 6

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Problem 1: Define the affine map $F : \hat{K} \mapsto K, F(x) = Bx + \delta$ between the reference simplex element $\hat{K} \subset \mathbb{R}^d$ with vertices $\hat{a}_i, i \in \{1, \dots, d+1\}$ and an arbitrary simplex $K \subset \mathbb{R}^d$ with vertices $a_i, i \in \{1, \dots, d+1\}$, and where $B \in \mathbb{R}^{d\times d}, \delta \in \mathbb{R}^d$ are defined by $F(\hat{a}_i) = a_i$ for all $i \in \{1, \dots, d+1\}$. Show that F maps $\hat{K}$ onto K.

Problem 2: (quadrature on triangles)

Consider the reference triangle $\hat{T} := \{x \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < 1, x_1 + x_2 < 1\}$. And for a set $G \subset \mathbb{R}^2$ denote by $\mathcal{P}_k(G)$ the set of all polynomials of degree less than or equal to k, restricted to G. In other words, consider functions $f : G \mapsto \mathbb{R}$ of the form

$$f(x) = \sum_{|\alpha|\leq k} a_\alpha x^\alpha,$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ is a multiindex and $|\alpha| = \alpha_1 + \alpha_2$, and where a_α is the coefficient of the monomial $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$.

(a) Show that the quadrature formula $Q^{(1)} : C(\hat{T}) \mapsto \mathbb{R}$ given by

$$Q^{(1)} f := \tfrac{1}{2} f\left(\tfrac{1}{3}, \tfrac{1}{3}\right)$$

reproduces the integral over $\hat{T}$ of a function $f \in \mathcal{P}_1(\hat{T})$ exactly.

(b) Show that the quadrature formula $Q^{(2)} : C(\hat{T}) \mapsto \mathbb{R}$ given by

$$Q^{(2)} f := \tfrac{1}{6}\left(f\left(\tfrac{1}{6}, \tfrac{1}{6}\right) + f\left(\tfrac{1}{6}, \tfrac{2}{3}\right) + f\left(\tfrac{2}{3}, \tfrac{1}{6}\right)\right)$$

reproduces the integral over $\hat{T}$ of a function $f \in \mathcal{P}_2(\hat{T})$ exactly.

Problem 3: Solve

$$-u''(x) = 1 \text{ for } x \in \Omega = (0, 1), \qquad u(0) = u(1) = 0$$

in Matlab writing your own program. Use quadratic finite elements on an equidistant grid with $h = 0.1$ and the local shape functions $N_1(x) = 2(x-1/2)(x-1), N_2(x) = 4x(1-x), N_3 = 2x(x-1/2)$ as given in the lecture. Plot the solution together with the exact solution.

Note: You do NOT have to use the sparse format for the stiffness matrix K_h.

Problem 4: (getting acquainted with the PDE Toolbox of Matlab)

Matlab has a Toolbox for solving a large class of PDEs using the linear finite elements in 2D. In this problem we solve an elliptic PDE on a nontrivial geometry:

$$-\Delta u(x) + u(x) = f(x) \text{ for } x \in \Omega = ((-1, 1) \times (-0.4, 0.4)) \setminus D_{1/5}(0)$$
$$u(x) = g(x) \text{ for } x \in \partial\Omega,$$

where $D_{1/5}(0) = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1/5\}$.

(a) Solve the problem using the PDE Toolbox first for $f(x) = 1$ and $g(x) = 0$ and then for $f(x) = -10x_1^2$ and $g(x) = \sin(\theta)$, where $\theta = \arg(x_1 + \mathrm{i}x_2)$. Use a triangulation with $h = 0.04$, where h is the maximal diameter of the triangles. Plot the resulting solution in each case.

Hints: Use heavily the documentation at `http://www.mathworks.com/help/toolbox/pde/pde_product_page.html`.

1 In the Matlab command window start the GUI for the PDE Toolbox by typing “pdetool”.

2 Generate a rectangle by the commands “pderect” and a circle by “pdecirc” in the standard Matlab command window (or by executing a script). In the bar over the figure in the GUI put a minus sign between the symbol for the rectangle and the circle.

3 Using a menu item in the ‘Draw’ category of the GUI, export the geometry description and thus get the variables gd, sf, ns.

4 Decompose the geometry into minimal regions (a necessary step in the Toolbox before meshing) using the command “decsg”.

5 Generate a mesh (triangulation) using “initmesh” setting the parameter $Hmax = 0.04$.

6 Define the boundary conditions using the “Boundary Condition matrix” b, see `http://www.mathworks.com/help/toolbox/pde/ug/assemb.html` for the format of b. Note that the PDE Toolbox divides the boundray $\partial\Omega$ into several segments. These can be viewed by clicking on “Show edge labels” in the GUI. Each of these segments must correspond to a different column of the matrix b. To define the BC $g(x) = \sin(\theta)$, you can use the command Matlab “atan2”.

7 Define f (see the hekp page for the command “assempde” cocerning the format of f).

8 Solve the problem by calling “assempde”.

9 Plot the solution using “pdeplot”.

(b) Play with the PDE Toolbox!