Karlsruher PDE-Seminar

Plane Wave Discontinuous Galerkin Methods

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Joint work with C. Gittelson (Aarau), A. Moiola (Reading), and I. Perugia (Wien)

This presentation reviews the development of convergence theory for a special class of Trefftz-type discontinuous Galerkin (TDG) methods that rely on plane waves for approximating solutions of the homogeneous Helmholtz equation $-\Delta u - \omega^2 u = 0$ locally. These methods have been designed as a cure for the notorious pollution effect that haunts standard low-order Galerkin schemes for the simulation of wave propagation.

The development started with the so-called ultra-weak variational formulation (UWVF) due to Cessenat and Despres [2, 3], which was introduced in the form of a variational problem for functions on the mesh skeleton. More than a decade passed until it was realized in [1,6] that this method can be viewed as a rather standard discontinuous Galerkin (DG) method using local trial spaces spanned by plane waves, a plane wave discontinuous Galerkin method (PWDG). This paved the way for a comprehensive convergence analysis of the h-version of the method. Unfortunately, the h-version still suffers from the pollution effect [5].

The analysis of the p-version of PWDG could be advanced in [7], based on techniques borrowed from least squares methods [18]. Of course, here p counts the number of local plane waves. Together with new approximation estimates for plane waves [16, 17], this allowed detailed a priori predictions of convergence. This initial theory covered only convex domains and could not accommodate locally refined meshes, which is very unfortunate, because numerical experience [11–13] suggests that PWDG should be used on such meshes. Sloppily speaking, the the sophisticated hp-refinement strategy that ensures exponential convergence (in the number of degrees of freedom) for classical polynomial Galerkin finite element approximation of second-order elliptic boundary value problem should also be adopted for PWDG.

Until recently, in the DG context, only polynomial theory could cover this setting [4, 14], but it remained outside the scope of existing TDG theory. Only in [9] asymptotic quasi-optimality of PWDG solutions could be established assuming merely shape-regular families of meshes. Still, these estimates were too weak to yield exponential convergence. It took sophisticated approximation theory for harmonic polynomials [10], analytic elliptic regularity theory developed by M. Melenk [15], and the clever use of weighted norms to accomplish the proof of exponential convergence of *hp*-PWDG for the Helmholtz equation in 2D on domains with piecewise analytic boundaries [8].

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Ort: 1C-03, Allianz-Gebäude 05.20
Gastgeber: Die Dozenten des Schwerpunkts Partielle Differentialgleichungen

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