Random Euclidean coverage and connectivity problems

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Consider sample of \( n \) uniform random points in a bounded region \( A \) in \( \mathbb{R}^d, \ d \geq 2 \), having a smooth boundary. The coverage threshold \( T_n \) is the smallest \( r \) such that the union \( Z \) of Euclidean balls of radius \( r \) centred on the sample points covers \( A \). The connectivity threshold \( K_n \) is twice the smallest \( r \) required for \( Z \) to be connected. The two-sample coverage threshold \( S_{n,m} \) is the smallest \( r \) such that \( Z \) covers all the points of a second independent sample of \( m \) points in \( A \). These thresholds are random variables determined by the sample, and are of interest, for example, in wireless communications, set estimation, and topological data analysis.

We discuss new/recent results on the large-\( n \) limiting distributions of \( T_n \), and \( K_n \) and \( S_{n,m} \) (taking \( m = m(n) \sim \tau n \) for some constant \( \tau \)). When \( A \) has unit volume, with \( v \) denoting the volume of the unit ball in \( \mathbb{R}^d \) and \( |dA| \) the perimeter of \( A \), these take the form of weak convergence of \( nvT_n^d - (2 - 2/d)\log n - a_d \log(\log n) \) to a Gumbel-type random variable with cumulative distribution function

\[
F(x) = \exp(-b_d e^{-x} - c_d |dA| e^{-x/2}),
\]

for suitable constants \( a_d, c_d \) with \( b_2 = 1, \ b_d = 0 \) for \( d > 2 \). The corresponding result for \( K_n \) takes the same form with different constants \( a_d, c_d \).

If time permits, we may also discuss extensions and related results concerning (i) the \( k \)-connectivity threshold of the associated geometric graph, and (ii) non-uniform random samples of points.

Some of the work described here is joint work with Frankie Higgs and Xiaochuan Yang.