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Applied Stochastic Models (SS 09)

Problem Set 1

Problem 0

Let $X_1,\, X_2 \sim U(\theta - 1/2, \theta + 1/2)$ be independent. Show that the probability density function (pdf) of $X_1 - X_2$ is independent of θ.

Problem 1

Let X and Y be random variables with distribution functions F and G respectively. We say that X is *stochastically dominated* or *dominated in distribution* by Y if

$$F(x) \geq G(x), \quad x \in \mathbb{R}.$$

Denote this relation by $X \overset{d}{\leq} Y$. Further, if F is a distribution function, its generalized inverse $F^{-1} : (0,1) \to \mathbb{R}$ is defined by

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}, \quad u \in (0,1).$$

(a) Prove that $F^{-1}(u) \leq x$ if and only if $u \leq F(x)$.

(b) Prove that if $U \sim U[0,1]$, then $X \overset{d}{=} F^{-1}(U)$.

(c) Prove that $X \overset{d}{\leq} Y$ if and only if there exist two random variables $\hat{X}$ and $\hat{Y}$ such that $\hat{X} \overset{d}{=} X$ and $\hat{Y} \overset{d}{=} Y$ (i.e. two *copies* of X and Y) with the property $\hat{X} \leq \hat{Y}$.

Problem 2

Suppose X is distributed according to a Poisson distribution with random parameter Λ which is $\Gamma(a,c)-$distributed $(a > 0, c > 0)$. Assume in addition $c \in \mathbb{N}$. Find the distribution of X.

Problem 3

Let $X, Y \sim U(0,1)$ be independent. Write $U := \min\{X,Y\}$ and $V := \max\{X,Y\}$. Find $\mathbb{E}[U], \mathbb{E}[V]$ and calculate $\mathrm{Cov}(U,V)$.

Problem 4

Let $n \in \mathbb{N}$ and $Y \sim \mathrm{Bin}(n, X)$, where X is a random variable following a beta distribution on $(0,1)$ (with parameters $p, q > 0$). Find the distribution of Y. What happens if X is uniform on $(0,1)$?

Solutions:

Problem 0

Let X_1, $X_2 \sim U(\theta - 1/2, \theta + 1/2)$ be independent. Show that the probability density function (pdf) of $X_1 - X_2$ is independent of θ.

Solution: Obviously, if we set $Y_1 := X_1 - \theta, Y_2 := X_2 - \theta$, then $Y_1, Y_2 \sim U(-1/2, 1/2)$, which is independent of θ. Hence the distribution (in particular the pdf) of $X_1 - X_2 = Y_1 - Y_2$ is independent of θ.

Problem 1

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Denote this relation by $X \overset{d}{\leq} Y$. Further, if F is a distribution function, its generalized inverse $F^{-1} : (0, 1) \to \mathbb{R}$ is defined by

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(c) Prove that $X \overset{d}{\leq} Y$ if and only if there exist two random variables $\hat{X}$ and $\hat{Y}$ such that $\hat{X} \overset{d}{=} X$ and $\hat{Y} \overset{d}{=} Y$ (i.e. two *copies* of X and Y) with the property $\hat{X} \leq \hat{Y}$.

Solution:

(a) If $F^{-1}(u) \leq x$, then by monotonicity of F, we have $F(F^{-1}(u)) \leq F(x)$. The definition of F^{-1} and right-continuity of F imply on the other hand that $F(F^{-1}(u)) \geq u$, which yields together $u \leq F(x)$.

Conversely, if $F(x) \geq u$, then $x \in \{x' \in \mathbb{R} : F(x') \geq u\}$, hence

$$x \geq \inf\{x' \in \mathbb{R} : F(x') \geq u\} = F^{-1}(u).$$

(b) We have

$$P(F^{-1}(U) \leq x) \overset{(a)}{=} P(U \leq F(x)) = F(x) = P(X \leq x)$$

which is equivalent to $F^{-1}(U) \overset{d}{=} X$.

(c) First assume the existence of $\hat{X}, \hat{Y}$ such that $\hat{X} \overset{d}{=} X$, $\hat{Y} \overset{d}{=} Y$ and $\hat{X} \leq \hat{Y}$. Then

$$\{\hat{Y} \leq x\} \subset \{\hat{X} \leq x\}, \quad x \in \mathbb{R},$$

hence $P(\hat{Y} \leq x) \leq P(\hat{X} \leq x)$, thus $G(x) \leq F(x), x \in \mathbb{R}$, meaning $X \overset{d}{\leq} Y$.

Conversely, if $X \overset{d}{\leq} Y$, then $G(x) \leq F(x), x \in \mathbb{R}$, which implies for $u \in [0,1]$ that

$$\{x \in \mathbb{R} : G(x) \geq u\} \subset \{x \in \mathbb{R} : F(x) \geq u\}.$$

Taking the infimum on both sides, this gives

$$F^{-1}(u) \leq G^{-1}(u), \quad u \in [0,1]. \tag{1}$$

Now take a $U(0,1)-$distributed random variable U and set $\hat{X} := F^{-1}(U)$, $\hat{Y} := G^{-1}(U)$. Then according to (b) these are copies of X and Y and (1) implies $\hat{X} \leq \hat{Y}$.

Problem 2

Suppose X is distributed according to a Poisson distribution with random parameter Λ which is $\Gamma(a,c)-$distributed ($a > 0, c > 0$). Assume in addition $c \in \mathbb{N}$. Find the distribution of X.

Solution:

The density of Λ is given by

$$f^\Lambda(x) = \frac{a^c}{\Gamma(c)} x^{c-1} e^{-ax} \mathbb{1}_{(0,\infty)}(x), \quad x \in \mathbb{R}.$$

Since $X \in \mathbb{N}_0$, it is enough to compute $P(X = n), n \in \mathbb{N}_0$. We find

$$\begin{aligned}
P(X=n) &= \mathbb{E}[P(X=n|\Lambda)] = \int f^\Lambda(\lambda) P(X=n|\Lambda=\lambda) d\lambda \\
&= \int f^\Lambda(\lambda) e^{-\lambda} \frac{\lambda^n}{n!} d\lambda \\
&= \int_0^\infty \frac{a^c}{\Gamma(c)} \lambda^{c-1} e^{-a\lambda} e^{-\lambda} \frac{\lambda^n}{n!} d\lambda \\
&= \frac{a^c}{\Gamma(c) n!} \int_0^\infty \lambda^{c+n-1} e^{-(a+1)\lambda} d\lambda \\
&= \frac{a^c}{\Gamma(c) n! (a+1)} \int_0^\infty \left(\frac{x}{a+1}\right)^{c+n-1} e^{-x} dx \\
&= \frac{a^c}{\Gamma(c) n! (a+1)^{c+n}} \Gamma(c+n) \\
&= \frac{(c+n-1)\ldots(c+1)c}{n!} \left(\frac{a}{a+1}\right)^c \left(\frac{1}{a+1}\right)^n.
\end{aligned}$$

Hence X has negative-binomial distribution with parameters $p = \frac{a}{a+1}$ and $r = c$. Using the formula

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^\infty \binom{n+k}{k} x^k, \quad |x| < 1,$$

we can check that this is a distribution indeed:

$$\sum_{n \geq 0} P(X=n) = \left(\frac{a}{a+1}\right)^c \frac{1}{(1-\frac{1}{a+1})^c} = \left(\frac{a}{a+1}\right)^c \left(\frac{a+1}{a}\right)^c = 1.$$

Problem 3

Let $X, Y \sim U(0,1)$ be independent. Write $U := \min\{X,Y\}$ and $V := \max\{X,Y\}$. Find $\mathbb{E}[U], \mathbb{E}[V]$ and calculate $Cov(U,V)$.

Solution:

We have

$$\begin{aligned} U &= \frac{1}{2}(X+Y) - \frac{1}{2}|X-Y|, \\ V &= \frac{1}{2}(X+Y) + \frac{1}{2}|X-Y|. \end{aligned}$$

One calculates the pdf of $X - Y$ as follows:

$$\int \mathbb{1}_{(0,1)}(x-y)\mathbb{1}_{(0,1)}(y)dy = \begin{cases} 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \end{cases} .$$

If f is the pdf of some random variable Z, then $(f(x)+f(-x))\mathbb{1}_{(0,\infty)}(x)$ is the pdf of $|Z|$. Hence $|X-Y|$ has pdf $(2-2x)\mathbb{1}_{(0,1)}(x)$. We then find

$$\mathbb{E}|X-Y| = \int_0^1 x(2-2x)dx = \left[x^2 - \frac{2}{3}x^3\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

Hence

$$\mathbb{E}[U] = \mathbb{E}\frac{1}{2}(X+Y) - \mathbb{E}\frac{1}{2}|X-Y| = \frac{1}{2} - \frac{1}{6} = \frac{1}{3},$$

which implies $\mathbb{E}[V] = \frac{2}{3}$.

For the covariance, we have

$$\begin{aligned} Cov(U,V) &= \mathbb{E}[(U-\mathbb{E}[U])(V-\mathbb{E}[V])] = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V] \\ &= \mathbb{E}[XY] - \mathbb{E}[U]\mathbb{E}[V] = \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[U]\mathbb{E}[V] = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}. \end{aligned}$$

Problem 4

Let $n \in \mathbb{N}$ and $Y \sim Bin(n, X)$, where X is a random variable following a beta distribution on $(0,1)$ (with parameters $p, q > 0$). Find the distribution of Y. What happens if X is uniform on $(0,1)$?

Solution:

X has pdf

$$f^X(x) = \frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}\mathbb{1}_{(0,1)}(x), \text{ where } B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

Hence for $k \in \{0, 1, ..., n\}$ we have

$$\begin{aligned} P(Y=k) &= \mathbb{E}(P(Y=k|X)) = \mathbb{E}\left[\binom{n}{k}X^k(1-X)^{n-k}\right] = \binom{n}{k}\mathbb{E}\left[X^k(1-X)^{n-k}\right] \\ &= \binom{n}{k}\int_0^1 x^k(1-x)^{n-k}\frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}dx \\ &= \frac{\binom{n}{k}}{B(p,q)}B(k+p, n-k+q). \end{aligned}$$

Since

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

we get

$$P(Y=k) = \binom{n}{k} \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{\Gamma(k+p)\Gamma(n-k+q)}{\Gamma(n+p+q)}.$$

The formula $\Gamma(x+1) = x\Gamma(x)$ yields, after some cancelations,

$$P(Y=k) = \binom{n}{k} \frac{(k-1+p)...p(n-k-1+q)...q}{(n-1+p+q)...(p+q)}.$$

Finally, the case that X is uniform on $(0,1)$ is the special choice of $p = 1 = q$. In this case we have

$$P(Y=k) = \binom{n}{k} \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1}.$$

This means Y is uniform on $\{0,1,...,n\}$.