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Applied Stochastic Models (SS 09)

Problem Set 2

Problem 1

Let X, Y, Z be independent exponential variables with respective parameters λ, μ, ν. Find

$$P(X < Y < Z).$$

Solution:

First note that

$$\int_0^t e^{-\alpha x} dx = \frac{1}{\alpha}(1 - e^{-\alpha t}), \quad \alpha \in \mathbb{R}.$$

Since X, Y, Z are independent, their joint density is given by

$$f^{(X,Y,Z)}(x, y, z) = f^X(x) f^Y(y) f^Z(z) = \lambda\mu\nu e^{-\lambda x - \mu y - \nu z}, \quad x, y, z > 0.$$

Hence, if $D := \{(x, y, z) \in \mathbb{R}^3 : z > y > x\}$, we have

$$\begin{aligned}
P(Z > Y > X) &= \int_D f^{(X,Y,Z)}(x, y, z) d(x, y, z) = \int_0^\infty \int_0^z \int_0^y \lambda\mu\nu e^{-\lambda x} e^{-\mu y} e^{-\nu z} dx dy dz \\
&= \int_0^\infty \int_0^z (1 - e^{-\lambda y}) \mu e^{-\mu y} \nu e^{-\nu z} dy dz \\
&= \int_0^\infty \int_0^z (\mu e^{-\mu y} - \mu e^{-(\lambda+\mu) y}) \nu e^{-\nu z} dy dz \\
&= \int_0^\infty ((1 - e^{-\mu z}) - \frac{\mu}{\mu + \lambda}(1 - e^{-(\lambda+\mu) z})) \nu e^{-\nu z} dz \\
&= \int_0^\infty ((\nu e^{-\nu z} - \nu e^{-(\mu+\nu) z}) - \frac{\mu\nu}{\mu + \lambda}(e^{-\nu z} - e^{-(\lambda+\mu+\nu) z})) dz \\
&= 1 - \frac{\nu}{\mu + \nu} - \frac{\mu\nu}{\mu + \lambda}(\frac{1}{\nu} - \frac{1}{\lambda + \mu + \nu}) \\
&= \frac{\mu}{\mu + \nu} \frac{\lambda}{\lambda + \mu + \nu}.
\end{aligned}$$

Problem 2

Let (X,Y) have the 2-dimensional standard normal distribution with zero means, unit variances and correlation coefficient ρ. Write $Z := \max\{X,Y\}$. Show that $\mathbb{E}(Z) = \sqrt{(1-\rho)/\pi}$ and $\mathbb{E}(Z^2) = 1$.

Solution:

We have

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

Hence

$$\begin{aligned} X+Y &\sim \mathcal{N}\left(\begin{pmatrix} 1 , 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 , 1 \end{pmatrix}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \mathcal{N}(0, 2+2\rho), \\ X-Y &\sim \mathcal{N}\left(\begin{pmatrix} 1 , -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 , -1 \end{pmatrix}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \mathcal{N}(0, 2-2\rho). \end{aligned}$$

In addition, if $\xi \sim \mathcal{N}(0,\sigma^2)$, then

$$f^{|\xi|}(x) = f^{\xi}(x) + f^{\xi}(-x), \quad x > 0,$$

and hence

$$\mathbb{E}|\xi| = \int_0^\infty x f^{|\xi|}(x)dx = \int_0^\infty \frac{2x}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{-2\sigma^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\Big|_0^\infty = \frac{\sqrt{2\sigma^2}}{\sqrt{\pi}}.$$

Applying this to $\xi := X-Y$ yields

$$\mathbb{E}(Z) = \mathbb{E}[\frac{1}{2}(X+Y) + \frac{1}{2}|X-Y|] = \frac{1}{2}\mathbb{E}[X+Y] + \frac{1}{2}\mathbb{E}[|X-Y|] = 0 + \frac{1}{2}\frac{\sqrt{2(2-2\rho)}}{\sqrt{\pi}} = \sqrt{\frac{1-\rho}{\pi}}.$$

Furthermore, if $\xi \sim \mathcal{N}(0,\sigma^2)$, then $\mathbb{E}[\xi^2] = \sigma^2$, and hence

$$\mathbb{E}(Z^2) = \frac{1}{4}\mathbb{E}[(X+Y)^2 + 2(X+Y)|X-Y| + (X-Y)^2] = \frac{1}{4}((2+2\rho) + 2\cdot 0 + (2-2\rho)) = 1.$$

Problem 3

Let $X \sim \Gamma(\lambda,\alpha)$ and $Y \sim \Gamma(\lambda,\beta)$ be independent. Show that $X+Y$ and $X/(X+Y) =: Z$ are independent and that Z follows a beta distribution with parameters α,β.

Solution: The transformation theorem for Lebesgue-densities states that if T is a transformation of some measurable subset $A \subset \mathbb{R}^n$ to $T(A) \subset \mathbb{R}^n$ and if $X = (X_1, ... X_n)$ is a random vector in $\mathbb{R}^n$ with density f^X with support in A, then $Y := T(X)$ has density

$$f^Y(y) = \frac{f^X(T^{-1}(y))}{|\Delta(T^{-1}(y))|}, \tag{1}$$

where $\Delta(x) := \det T'(x)$ is the determinant of the Jacobian of T at point x. In our case, a suitable transformtion is $T : (0,\infty)^2 \to (0,\infty)\times(0,1), (x,y) \mapsto (x+y, \frac{x}{x+y})$. Here $|\Delta(x,y)| = \frac{1}{x+y}$ and $T^{-1}(u,v) = (uv, u-uv)$, yielding $|\Delta(T^{-1}(u,v))| = \frac{1}{u}$. Denoting $U := X+Y, V := \frac{X}{X+Y}$, we have $(U,V) = T(X,Y)$ and hence

$$f^{(U,V)}(u,v) = f^{(X,Y)}(uv, u-uv)u.$$

The joint density of X, Y is the product of f^X and f^Y by independence, hence

$$\begin{aligned} f^{(U,V)}(u,v) &= f^X(uv)f^Y(u-uv)u = \frac{\lambda^\alpha}{\Gamma(\alpha)}(uv)^{\alpha-1}e^{-\lambda uv}\frac{\lambda^\beta}{\Gamma(\beta)}(u-uv)^{\beta-1}e^{-\lambda u-uv}u \\ &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)}u^{\alpha+\beta-1}e^{-\lambda u}v^{\alpha-1}(1-v)^{\beta-1}. \end{aligned}$$

Using $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, this gives

$$f^{(U,V)}(u,v) = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)}u^{\alpha+\beta-1}e^{-\lambda u}\frac{1}{B(\alpha,\beta)}v^{\alpha-1}(1-v)^{\beta-1},$$

and thus reveals the joint density of U, V as the product of the two densities (!)

$$\begin{aligned} f^{\Gamma(\alpha+\beta)}(u) &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)}u^{\alpha+\beta-1}e^{-\lambda u}, \\ f^{(\alpha,\beta)}(v) &= \frac{1}{B(\alpha,\beta)}v^{\alpha-1}(1-v)^{\beta-1}. \end{aligned}$$

This shows that $X+Y$ and $\frac{X}{X+Y}$ are independent, and it also shows, that the latter is Beta-distributed with paramters α, β.

Problem 4

Let U be uniform on $[0,1]$ and $0 < q < 1$. Show that $X := 1 + [\ln(U)/\ln(q)]$ has a geometric distribution.

Solution: Clearly $X \in \mathbb{N}$, hence it will be enough to compute $P(X = k), k \in \mathbb{N}$. The calculation

$$\begin{aligned} P(X=k) &= P([\ln(U)/\ln(q)] = k-1) = P(k-1 \leq \ln(U)/\ln(q) < k) \\ &= P((k-1)\ln(q) \geq \ln(U)/ > k\ln(q)) = P(q^{k-1} \geq U > q^k) \\ &= q^{k-1} - q^k = q^{k-1}(1-q) \end{aligned}$$

shows that X is indeed geometrically distributed.