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Applied Stochastic Models (SS 09)

Problem Set 3

Problem 1

Let T_1 and T_2 be independent and uniformly distributed on $[-1, 1]$.
Show that, conditional on the event that $R := \sqrt{T_1^2 + T_2^2} \leq 1$,

$$X := \frac{T_1}{R}\sqrt{-2\ln R^2}, \quad Y := \frac{T_2}{R}\sqrt{-2\ln R^2}$$

are independent standard normal random variables.

Problem 2

Let $X_1, X_2, \cdots X_n$ be independent and exponentially distributed with parameter λ. Show that

$$Y_1 = nX_{(1)}, Y_r = (n+1-r)(X_{(r)} - X_{(r-1)}), 1 < r \leq n$$

are also independent and have the same joint distribution as the X_i.

Problem 3

Let $X_1, X_2, \cdots X_n$ be independent and uniformly distributed in $[0, 1]$, with order statistics $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$. Show that $-\ln(X_{(k)})$ has the same distribution as $\sum_{i=k}^{n} Y_i/i$, where the Y_i are independent exponential random variables with parameter 1.

Problem 4

Let $X_1, X_2, X_3, \cdots X_n$ be independent and exponentially distributed with parameter 1. Show that

$$\lim_{n\to\infty} \mathbb{P}(X_{(n)} - \ln n \leq x) = \exp(-e^{-x}).$$

Using this, show that $\int_0^\infty [1 - \exp(-e^{-x})]dx = \gamma$, where γ denotes Eulers constant ($\gamma = \lim_{n\to\infty}[\sum_{k=1}^{n} \frac{1}{k} - \ln(n)]$).