University of Karlsruhe
Institute of Stochastics

Prof. Dr. P. R. Parthasarathy
Dipl.-Math. D. Gentner

Applied Stochastic Models (SS 09)

Problem Set 4

Problem 1

Find the mean and variance of the Weibull distribution with hazard rate (or failure rate)

$$r(t) = \frac{f(t)}{1 - F(t)} = \lambda\alpha(\lambda t)^{\alpha-1}, \lambda > 0, \alpha > 0, t \geq 0.$$

Soltuion: The hazard rate function $\lambda(t) = \frac{f(t)}{1-F(t)}, t > 0$, of some random lifetime T uniquely determines the distribution of T, since necessarily

$$F^T(t) = 1 - e^{-\int_0^t \lambda(u)du}, \quad f^T(t) = \lambda(t)e^{-\int_0^t \lambda(u)du}, \quad t > 0.$$

A random life time T with the above failure rate

$$r(t) = \lambda\alpha(\lambda t)^{\alpha-1}, \lambda > 0, \alpha > 0, t \geq 0,$$

thus has the density

$$f^T(t) = r(t)e^{-\int_0^t r(u)du} = \lambda\alpha(\lambda t)^{\alpha-1}e^{-(\lambda t)^\alpha}, \lambda > 0, \alpha > 0, t \geq 0.$$

A direct calculation yields

$$\begin{aligned} \mathbb{E}T &= \int_0^\infty t\lambda\alpha(\lambda t)^{\alpha-1}e^{-(\lambda t)^\alpha}dt \\ &= \alpha\int_0^\infty (\lambda t)^\alpha e^{-(\lambda t)^\alpha}dt. \end{aligned}$$

Substituting $u := (\lambda t)^\alpha$ yields

$$\mathbb{E}T = \frac{1}{\lambda}\int_0^\infty u^{\frac{1}{\alpha}}e^{-u}du = \frac{1}{\lambda}\Gamma(\frac{1}{\alpha} + 1).$$

A similar calculation yields for the second moment

$$\mathbb{E}T^2 = \int_0^\infty t^2\lambda\alpha(\lambda t)^{\alpha-1}e^{-(\lambda t)^\alpha}dt,$$

where the same substitution as above yields

$$\mathbb{E}T^2 = \frac{1}{\lambda^2}\int_0^\infty u^{\frac{2}{\alpha}}e^{-u}du = \frac{1}{\lambda^2}\Gamma(\frac{2}{\alpha} + 1).$$

This means that the variance of T is given by

$$Var(T) = \mathbb{E}T^2 - (\mathbb{E}T)^2 = \frac{1}{\lambda^2}(\Gamma(\frac{2}{\alpha}+1) - \Gamma(\frac{1}{\alpha}+1)^2).$$

Problem 2

Let $N(t)$ be a Poisson process, and let $Y_1, Y_2, Y_3, \ldots$ be independent and identically distributed random variables. Find the mean and variance of

$$\sum_{i=1}^{N(t)} Y_i.$$

Solution:

Conditioning by $N(t)$ yields for the expectation

$$\mathbb{E}\sum_{i=1}^{N(t)} Y_i \;=\; \mathbb{E}\left[\mathbb{E}[\sum_{i=1}^{N(t)} Y_i | N(t)]\right] = \mathbb{E}[N(t)\mathbb{E}Y_1] = \mathbb{E}[N(t)]\mathbb{E}[Y_1] = \left(\int_0^t \lambda(u)du\right)\mathbb{E}[Y_1],$$

while the same procedure yields for the second moment

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{N(t)} Y_i\right)^2 &= \mathbb{E}\left[\mathbb{E}[(\sum_{i=1}^{N(t)} Y_i)^2 | N(t)]\right] \\
&= \mathbb{E}\left[\mathbb{E}[\sum_{i,j=1}^{N(t)} Y_i Y_j | N(t)]\right] = \mathbb{E}\left[\mathbb{E}[\sum_{i=1}^{N(t)} Y_i^2 + \sum_{i \neq j} Y_i Y_j | N(t)]\right] \\
&\overset{Y_i \text{ indep}}{=} \mathbb{E}[N(t)]\mathbb{E}[Y_1^2] + \mathbb{E}[N(t)^2 - N(t)](\mathbb{E}Y_1)^2 = \mathbb{E}N(t)VarY_1 + \mathbb{E}N(t)^2(\mathbb{E}Y_1)^2.
\end{aligned}$$

Hence the variance is given by

$$\begin{aligned}
Var(\sum_{i=1}^{N(t)} Y_i) &= \mathbb{E}N(t)VarY_1 + VarN(t)(\mathbb{E}Y_1)^2 \\
&= \left(\int_0^t \lambda(u)du\right)(VarY_1 + (\mathbb{E}Y_1)^2) = \left(\int_0^t \lambda(u)du\right)\mathbb{E}Y_1^2.
\end{aligned}$$

Problem 3

Consider a homogeneous Poisson process $N(t)$ with random (and with respect to time constant) intensity λ which takes two values λ_1, λ_2 with equal probabilities. (This means that first, someone rolls the intensity according to a fair coinflip, and then generates the Poisson process with this intensity.) Find the probability generating function of $N(t)$.

Solution:

If $N(t)$ is a homogeneous Poisson process with deterministic rate λ, then its pgf is given by

$$\phi(z) = \mathbb{E}[z^{N(t)}] = \sum_{k\geq 0} P(N(t) = k)z^k = \sum_{k\geq 0} e^{-\lambda t}\frac{(\lambda t)^k}{k!}z^k = e^{-\lambda t(1-z)}.$$

If the intensity is given by a Bernoulli experiment, we may derive by conditioning that

$$\phi(z) = \mathbb{E}[z^{N(t)}] = \mathbb{E}[\mathbb{E}[z^{N(t)}|\lambda]] = \frac{1}{2}\left(e^{-\lambda_1 t(1-z)} + e^{-\lambda_2 t(1-z)}\right).$$

Problem 4

Let the times between the events of a renewal process $N(t)$ be uniformly distributed on $(0, 1)$. Find the mean and variance of $N(t)$ for $0 < t < 1$.

Solution:

Let $0 < t < 1$ and consider the pgf of $N(t)$:

$$G(z, t) := \mathbb{E}\left[z^{N(t)}\right], \quad z \in \mathbb{R}.$$

Let $T_1, T_2, ...$ denote the times between the events of the renewal process, which are iid and $U(0, 1)$-distributed. Then with $S_n := T_1 + T_2 + ...T_n$ we may write

$$N(\omega, t) = \sum_{n\geq 1} \mathbb{1}\{S_n \leq t\}.$$

We derive a differential equation for $G(z, t)$ as follows:

$$\begin{aligned}
G(z,t) &= \mathbb{E}\left[z^{N(t)}\right] = \int z^{N(\omega,t)}\mathbb{P}(d\omega) = \int z^{N(\omega,t)}\mathbb{1}\{T_1 > t\}\mathbb{P}(d\omega) + \int z^{N(\omega,t)}\mathbb{1}\{T_1 < t\}\mathbb{P}(d\omega) \\
&= \int z^0\mathbb{1}\{T_1 > t\}\mathbb{P}(d\omega) + \int z^{\sum_{n\geq 1}\mathbb{1}\{S_n\leq t\}}\mathbb{1}\{T_1 < t\}\mathbb{P}(d\omega) \\
&= (1-t) + \int z^{\sum_{n\geq 1}\mathbb{1}\{t_1+...+t_n\leq t\}}\mathbb{1}\{t_1 < t\}\mathbb{P}^{(T_1,T_2,...)}(d(t_1, t_2, ...)) \\
&= (1-t) + \iint z^{\sum_{n\geq 1}\mathbb{1}\{t_1+...+t_n\leq t\}}\mathbb{1}\{t_1 < t\}\mathbb{P}^{T_1}(dt_1)\mathbb{P}^{(T_2,T_3,...)}(d(t_2, t_3, ...)) \\
&= (1-t) + \int\int_0^t z^{1+\sum_{n\geq 2}\mathbb{1}\{t_2...+t_n\leq t-t_1\}}\mathbb{P}^{T_1}(dt_1)\mathbb{P}^{(T_2,T_3,...)}(d(t_2, t_3, ...)) \\
&\overset{Fubini}{=} (1-t) + \int_0^t\int z^{1+\sum_{n\geq 2}\mathbb{1}\{t_2...+t_n\leq t-t_1\}}\mathbb{P}^{(T_2,T_3,...)}(d(t_2, t_3, ...))\mathbb{P}^{T_1}(dt_1) \\
&= (1-t) + \int_0^t\int z z^{\sum_{n\geq 2}\mathbb{1}\{T_2...+T_n\leq t-t_1\}}\mathbb{P}(d\omega)\mathbb{P}^{T_1}(dt_1).
\end{aligned}$$

Since $(T_2, T_3, ...) \overset{d}{=} (T_1, T_2, ...)$, this means that

$$G(z,t) = (1-t) + z\int_0^t G(z, t-t_1)dt_1 = (1-t) + z\int_0^t G(z, u)du.$$

Differentiating with respect to t yields

$$\frac{d}{dt}G(z,t) = -1 + zG(z,t).$$

With respect to t this is a first order linear differential equation with constant coefficients. The homogeneous solutions are obviously given by $const \cdot e^{zt}$, while a particular solution is the constant function $t \mapsto \frac{1}{z}$. Since necessarily $G(z,0) = 1$, this yields

$$G(z,t) = \frac{1-(1-z)e^{zt}}{z}. \tag{1}$$

To find the mean and variance of $N(t)$, we need to investige the first and second derivative of its pgf G with respect to z. Equation (1) is equivalent to

$$zG(z,t) = 1-(1-z)e^{zt},$$

and differentiating this twice yields (' denotes differentiation with respect to z)

$$zG'(z,t) + G(z,t) = (1-(1-z)t)e^{zt}, \tag{2}$$

and

$$zG''(z,t) + 2G'(z,t) = t(1-(1-z)t)e^{zt} + te^{zt}. \tag{3}$$

Evaluating (2) at $z = 1$ yields

$$\mathbb{E}[N(t)] = e^t - 1, \quad 0 < t < 1,$$

while evaluating (3) at $z = 1$ yields

$$\mathbb{E}[N(t)(N(t)-1)] + 2\mathbb{E}[N(t)] = 2te^t, \quad 0 < t < 1,$$

which means for the variance that

$$Var(N(t)) = \mathbb{E}[N(t)(N(t)-1)] + E[N(t)] - (E[N(t)])^2 = 2te^t - (e^t-1) - (e^t-1)^2, \quad 0 < t < 1.$$

For $t > 1$ the situation becomes pretty messy, since the differential equation for $G(z,t)$ takes a different form, and its solution seems to require a recursive consideration of the intervals $(k, k+1), k \in \mathbb{N}$ where at each step the solution on $(k, k+1)$ becomes more and more complicated. Another approach is to use the renewal equation and its Laplace transform, which yields a nice closed form of $\hat{m}(s)$. But then again, the inversion of the transform is difficult.