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Applied Stochastic Models (SS 09)

Problem Set 5

Problem 1

The r^{th} point T_r of a Poisson process $N(t)$ of constant intensity λ gives rise to an effect

$$X_r e^{-\alpha(t-T_r)}$$

at time $t \geq T_r$, where the X_r are independent and identically distributed with finite variance. Find the characteristic function (or Laplace transform) of the total effect

$$S(t) = \sum_{r=1}^{N(t)} X_r e^{-\alpha(t-T_r)}$$

and its first two moments in terms of the first two moments of the X_r, and calculate $Cov(S(s), S(t))$. Show that

$$\rho(S(s), S(s+v)) \to e^{-\alpha v} \text{ as } s \to \infty.$$

Solution:

Conditional on $N(t)$ the vector $(T_1, ..., T_{N(t)})$ has the same distribution as the vector of the order statistics $(U_{(1)}, U_{(2)}, ..., U_{(N(t))})$ of $N(t)$ on $(0, t)$ uniformly distributed random variables $U_1, ..., U_{N(t)}$. Let π denote the random permutation that satisfies

$$(U_{\pi(1)}, U_{\pi(2)}, ..., U_{\pi(N(t))}) = (U_{(1)}, U_{(2)}, ..., U_{(N(t))}).$$

Note that since the X_i are iid, we have

$$(X_{\tau(1)}, ..., X_{\tau(N(t))}) \stackrel{d}{=} (X_1, ..., X_{N(t)})$$

for any fixed permutation τ. This is also true for random permutations τ being independent from the X_i. In particular

$$(X_{\pi^{-1}(1)}, ..., X_{\pi^{-1}(N(t))}) \stackrel{d}{=} (X_1, ..., X_{N(t)}). \tag{1}$$

Using this, we derive

$$\begin{aligned}
\mathbb{E}[e^{-sS(t)}] &= \mathbb{E}[\mathbb{E}[e^{-sS(t)}|N(t)]] = \mathbb{E}[\mathbb{E}[\prod_{i=1}^{N(t)} e^{-sX_i e^{-\alpha(t-T_i)}}|N(t)]] \\
&= \mathbb{E}[\mathbb{E}[\prod_{i=1}^{N(t)} e^{-sX_i e^{-\alpha(t-U_{(i)})}}|N(t)]] = \mathbb{E}[\mathbb{E}[\prod_{i=1}^{N(t)} e^{-sX_i e^{-\alpha(t-U_{\pi(i)})}}|N(t)]] \\
&= \mathbb{E}[\mathbb{E}[\prod_{i=1}^{N(t)} e^{-sX_{\pi^{-1}(i)} e^{-\alpha(t-U_i)}}|N(t)]] \stackrel{(1)}{=} \mathbb{E}[\mathbb{E}[\prod_{i=1}^{N(t)} e^{-sX_i e^{-\alpha(t-U_i)}}|N(t)]] \\
&= \mathbb{E}[(\mathbb{E}[e^{-sX_1 e^{-\alpha(t-U_1)}}])^{N(t)}] = \mathbb{E}[(\mathbb{E}[e^{-sX_1 e^{-\alpha U_1}}])^{N(t)}].
\end{aligned}$$

The pgf of a homogenous Poisson process $N(t)$ with parameter λ is given by

$$\mathbb{E}[z^{N(t)}] = e^{\lambda t(z-1)}.$$

Hence, we may proceed as follows:

$$\begin{aligned}\mathbb{E}[e^{-sS(t)}] &= e^{\lambda t(\mathbb{E}[e^{-sX_1e^{-\alpha U_1}}-1])} = e^{\lambda(\int_0^t \mathbb{E}e^{-sX_1e^{-\alpha u}}-1du)} \\ &= e^{\lambda(\int_0^t \phi(se^{-\alpha u})-1du)},\end{aligned}$$

where ϕ denotes the Laplace transform of X_1. This means for the first two moments of $S(t)$:

$$\mathbb{E}[S(t)] = -(\frac{d}{ds}\mathbb{E}[e^{-sS(t)}]|_{s=0}) = -\lambda\int_0^t -(\mathbb{E}X)e^{-\alpha u}du = \frac{\lambda}{\alpha}(\mathbb{E}X)(1-e^{-\alpha t}),$$

$$\mathbb{E}[S(t)^2] = \frac{d^2}{ds^2}\mathbb{E}[e^{-sS(t)}]|_{s=0} = \frac{\lambda}{2\alpha}\mathbb{E}[X^2](1-e^{-2\alpha t}) + \frac{\lambda^2}{\alpha^2}(\mathbb{E}[X])^2(1-e^{-\alpha t})^2.$$

For $s < t$ we can write $S(t)$ as follows:

$$S(t) = S(s)e^{-\alpha(t-s)} + \sum_{r=N(s)+1}^{N(t)} X_r e^{-\alpha(t-T_r)} =: S(s)e^{-\alpha(t-s)} + \hat{S}(s,t).$$

Here $\hat{S}(s,t)$ is clearly independent from $S(s)$ and by linearity of the covariance this yields for $s < t$

$$Cov(S(s),S(t)) = Var(S(s))e^{-\alpha(t-s)} = \frac{\lambda\mathbb{E}[X^2]}{2\alpha}(1-e^{-2\alpha s})e^{-\alpha(t-s)},$$

which means that

$$Cov(S(s),S(s+v)) \longrightarrow \frac{\lambda\mathbb{E}[X^2]}{2\alpha}e^{-\alpha v} \text{ as } s \longrightarrow \infty.$$

In turn, this yields

$$\rho(S(s),S(s+v)) \longrightarrow e^{-\alpha v} \text{ as } s \longrightarrow \infty.$$

Problem 2

Let $N(t)$ be a non-homogeneous Poisson process with intensity function $\lambda(t)$. Show that the joint density function of the first two inter-event times is given by

$$\lambda(x)\lambda(x+y)e^{-\int_0^{x+y}\lambda(u)du}$$

and deduce that they are not in general independent.
Hint: Start with $P(T_1 \le x, T_2 - T_1 > y)$.

Solution: Writing

$$\Lambda(x) := \int_0^x \lambda(u)du,$$

we have

$$P(T_1 \le x, T_2 - T_1 > y) = \iint \mathbb{1}\{t_1 \le x, t_2 - t_1 > y\}\mathbb{P}^{T_2-T_1|T_1=t_1}(dt_2)\mathbb{P}^{T_1}(dt_1).$$

Here

$$\mathbb{P}^{T_2-T_1|T_1=t_1}(T_2-t_1>y)=e^{-\int_{t_1}^{t_1+y}\lambda(u)du}=e^{-(\Lambda(t_1+y)-\Lambda(t_1))}$$

and hence

$$P(T_1\le x, T_2-T_1>y)=\int_0^x \lambda(t_1)e^{-\Lambda(t_1)}e^{-(\Lambda(t_1+y)-\Lambda(t_1))}dt_1=\int_0^x \lambda(t_1)e^{-\Lambda(t_1+y)}dt_1.$$

Hence the joint density of T_1 and T_2-T_1 is given by

$$\begin{aligned}\frac{d^2}{dxdy}(P(T_1\le x)-P(T_1\le x, T_2-T_1>y)) &= \frac{d^2}{dxdy}(1-P(T_1\ge x)-P(T_1\le x, T_2-T_1>y))\\ &= \frac{d^2}{dxdy}\left(1-e^{-\Lambda(x)}-\int_0^x \lambda(t_1)e^{-\Lambda(t_1+y)}dt_1\right)\\ &= \lambda(x)\lambda(x+y)e^{-\Lambda(x+y)}.\end{aligned}$$

Since the density of T_1 is given by

$$\frac{d}{dx}(1-\mathbb{P}(T_1>x))=\lambda(x)e^{-\Lambda(x)},$$

independence of T_1 and T_2-T_1 would necessarily imply

$$f^{T_2-T_1}(y)=\lambda(x+y)e^{\Lambda(x+y)-\Lambda(x)}$$

for any x, which cannot depend on x. This in turn is true iff $\lambda\equiv const$. Hence in any other case T_1 and T_2-T_1 are not independent.

Problem 3

Show that for a structure function ϕ

(a) if $\phi(0,0,...,0)=0, \phi(1,1,...,1)=1$, then min $x_i\le\phi(\mathbf{x})\le$ max x_i,

(b) ϕ(max $(\mathbf{x},\mathbf{y}))\ge$ max $(\phi(\mathbf{x}),\phi(\mathbf{y}))$,

(c) ϕ(min $(\mathbf{x},\mathbf{y}))\le$ min $(\phi(\mathbf{x}),\phi(\mathbf{y}))$.

Solution:

(a) By assumption $\phi(0,0,...,0)=0,\phi(1,1,...,1)=1$. There are the following three cases:

- Case 1: $\min x_i=0, \max x_i=1$. Then clearly

$$0=\min_i x_i\le\phi(x)\le\max_i x_i=1.$$

- Case 2: $\max x_i=0$. Then $x=(0,...,0)$ and by assumption $\phi(x)=0$. The inequality holds trivially.

- Case 3: $\min x_i=1$. Then $x=(1,...,1)$ and by assumption $\phi(x)=1$. The inequality holds trivially.

(b) By monotonicity of a structure function, we have

$$\phi(x) \leq \phi(\max(x,y)), \quad \phi(y) \leq \phi(\max(x,y)),$$

and hence

$$\max\{\phi(x), \phi(y)\} \leq \phi(\max(x,y)).$$

(c) By monotonicity of a structure function, we have

$$\phi(x) \geq \phi(\min(x,y)), \quad \phi(y) \geq \phi(\min(x,y)),$$

and hence

$$\min\{\phi(x), \phi(y)\} \geq \phi(\min(x,y)).$$

Problem 4

For any structure function ϕ, we define the dual structure ϕ^D by

$$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x}), \quad \text{where } \mathbf{1} := (1, 1, ..., 1).$$

(a) Show that the dual of a parallel (series) system is a series (parallel) system.

(b) Show that the dual of the dual structure is the original structure.

(c) What is the dual of a k-out-of-n struture?

(d) Show that a minimal path (cut) set of the dual system is a minimal cut (path) set of the original structure.

Solution:

(a) If ϕ corresponds to a series system, then it has the form

$$\phi(x) = x_1 x_2 ... x_n.$$

The respective dual structure is given by

$$\phi^D(x) = 1 - (1 - x_1)(1 - x_2)...(1 - x_n),$$

which is the structure function of a parallel system. Part (b) implies that the dual of a parallel system is a series system again.

(b) We have

$$\phi^{D^D}(x) = 1 - \phi^D(1 - x) = 1 - (1 - \phi(1 - (1 - x))) = \phi(x).$$

(c) Let $A_1, A_2..., A_l$ denote the $\binom{n}{k}$ different minimal path sets of a k-out-of-n-structure (i.e. $l = \binom{n}{k}$). The structure function can be written as

$$\phi(x) = \max_j \prod_{i \in A_j} x_i = \begin{cases} 1, & \sum x_i \geq k \\ 0, & \sum x_i < k \end{cases},$$

and the dual is then given by

$$\phi^D(x) = 1 - \max_j \prod_{i \in A_j} (1 - x_i) = \begin{cases} 1, & \sum x_i \geq n-k+1 \\ 0, & \sum x_i < n-k+1 \end{cases},$$

which is a $(n-k+1)$-out-of-n structure.
(d) Let A be a minimal path set of ϕ^D and let x be the respective minimal path vector of A in the dual structure. This means

$$\phi^D(x) = 1 \text{ and } \phi^D(y) = 0, \quad y < x.$$

According to part (b), dualizing is self-inverse, which means

$$\phi(x) = 1 - \phi^D(1-x).$$

Then $1-x$ is a minimal cut vector of the original structure: First

$$\phi(1-x) = 1 - \phi^D(x) = 1 - 1 = 0$$

and second for each $y > 1-x$, we have $x > 1-y$ and hence $\phi^D(1-y) = 0$ which means

$$\phi(y) = 1 - \phi^D(1-y) = 1.$$

$1-x$ being a minimal cut vector of ϕ just means that A is a minimal cut set of the original structure. The dual statement can be proved in the same way.