University of Karlsruhe
Institute of Stochastics

Prof. Dr. P. R. Parthasarathy
Dipl.-Math. D. Gentner

Applied Stochastic Models (SS 09)

Problem Set 7

Problem 1

Compute the reliability function of the bridge system by conditioning upon whether or not component 3 is working.

Solution: The minimal path sets of the bridge system are given by

$$\{1,4\}, \{1,3,5\}, \{2,3,4\}, \{2,5\},$$

and rewriting the system as a parallel system its respective minimal paths yields the following form of its structure function:

$$\Phi(x) = 1 - (1 - x_1x_4)(1 - x_1x_3x_5)(1 - x_2x_3x_4)(1 - x_2x_5).$$

The reliability function is given by

$$r(p) = \mathbb{E}[\Phi(X)] = P(\Phi(X) = 1),$$

and conditioning by the third component yields

$$r(p) = P(X_3 = 1)P(\Phi(X) = 1|X_3 = 1) + P(X_3 = 0)P(\Phi(X) = 1|X_3 = 0).$$

Here $P(\Phi(X) = 1|X_3 = 0)$ is simply the reliability function of a parallel system of the two series systems $\{1,4\}$ and $\{2,5\}$. Hence

$$P(\Phi(X) = 1|X_3 = 0) = 1 - (1 - p_1p_4)(1 - p_2p_5).$$

On the other hand

$$\begin{aligned} P(\Phi(X) = 1|X_3 = 1) &= P(X_2 = 0|X_3 = 1)P(\Phi(X) = 1|X_2 = 0, X_3 = 1) \\ &\quad + P(X_2 = 1|X_3 = 1)P(\Phi(X) = 1|X_2 = 1, X_3 = 1), \end{aligned}$$

where

$$\begin{aligned} P(\Phi(X) = 1|X_2 = 0, X_3 = 1) &= P(X_1 = 1, \max\{X_4, X_5\} = 1) = p_1(1 - (1 - p_4)(1 - p_5)) \\ P(\Phi(X) = 1|X_2 = 1, X_3 = 1) &= P(\max\{X_4, X_5\} = 1) = (1 - (1 - p_4)(1 - p_5)). \end{aligned}$$

Plugging in yields

$$\begin{aligned} P(\Phi(X) = 1) &= p_3[(1 - p_2)p_1(1 - (1 - p_4)(1 - p_5)) + p_2(1 - (1 - p_4)(1 - p_5))] \\ &\quad + (1 - p_3)[1 - (1 - p_1p_4)(1 - p_2p_5)]. \end{aligned}$$

Problem 2

Compute the upper and lower bounds from class for the reliability function for the two-out-of-three system and two-out-of-four system. Compare these bounds with the exact reliability when

$$p_i \equiv 0.8, p_i \equiv 0.5 \text{ and } p_i \equiv 0.2.$$

Solution: If $A_1, ..., A_s$ are all minimal path sets and $C_1, ..., C_k$ are all minimal cut sets of a system, then

$$l(p) := \prod_{i=1}^{k} \left[1 - \prod_{j \in C_i} (1 - p_j) \right] \leq r(p) \leq 1 - \prod_{i=1}^{s} \left(1 - \prod_{j \in A_i} p_j \right) =: u(p).$$

In a 2-out-of-3-system the minimal path sets are $\{1,2\}, \{1,3\}, \{2,3\}$, while the minimal cut sets are (also) $\{1,2\}, \{1,3\}, \{2,3\}$. Writing $q_i := 1 - p_i$ this yields the following estimates:

$$(1 - q_1 q_2)(1 - q_1 q_3)(1 - q_2 q_3) \leq r(p) \leq 1 - (1 - p_1 p_2)(1 - p_1 p_3)(1 - p_2 p_3).$$

If $p_i \equiv p$, then $q_i \equiv q$, and the above estimates simplify to

$$(1 - q^2)^3 \leq r(p) \leq 1 - (1 - p^2)^3.$$

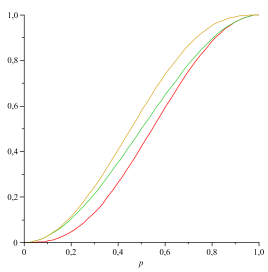
The exact reliability function of a 2-out-of-3-system on the other hand is given by

$$r(p) = \sum_{i=2}^{3} \binom{3}{i} p^i q^{3-i} = 3p^2 q + p^3.$$

This yields the following table:

p	$l(p)$	$r(p)$	$u(p)$
0.2	0.047	0.104	0.115
0.5	0.422	0.500	0.578
0.8	0.885	0.896	0.953

Here is a picture:



Similarly in a 2-out-of-4-system the minimal path sets are $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, while the minimal cut sets are $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$. This yields

$$(1 - q_1 q_2 q_3)(1 - q_1 q_2 q_4)(1 - q_1 q_3 q_4)(1 - q_2 q_3 q_4) \leq r(p)$$
$$\leq 1 - (1 - p_1 p_2)(1 - p_1 p_3)(1 - p_1 p_4)(1 - p_2 p_3)(1 - p_2 p_4)(1 - p_3 p_4).$$

If $p_i \equiv p$, then $q_i \equiv q$, and the above estimates simplify to

$$(1-q^3)^4 \le r(p) \le 1-(1-p^2)^6.$$

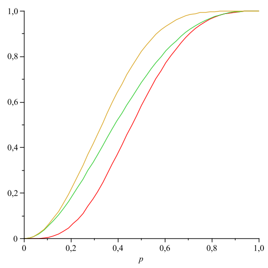
The exact reliability function of a 2-out-of-4-system on the other hand is given by

$$r(p) = \sum_{i=2}^{4} \binom{4}{i} p^i q^{4-i} = 6p^2q^2 + 4p^3q + p^4.$$

This yields the following table:

p	$l(p)$	$r(p)$	$u(p)$
0.2	0.057	0.181	0.217
0.5	0.586	0.688	0.822
0.8	0.968	0.973	0.998

Here is a picture:



Problem 3
(a) Prove that F is an IFR distribution iff $\log(\overline{F}(t))$ is concave.
(b) Deduce that if F is IFR, then $[\overline{F}(t)]^{1/t}$ is decreasing in t.

Solution: (a) F is IFR iff $\frac{f(t)}{\overline{F}(t)}$ is increasing in t. In turn this is clearly equivalent to the convexity of

$$G(t) = \int_0^t \frac{f(u)}{\overline{F}(u)} du = -\log \overline{F}(t) + \log \overline{F}(0) = -\log \overline{F}(t),$$

and hence to the concavity of $\log \overline{F}(t)$.
(b) Let F be an IFR. This implies $\frac{f(t)}{\overline{F}(t)} =: R(t)$ is increasing in t. We need to show that

$$\frac{d}{dt}(\overline{F}(t)^{\frac{1}{t}}) < 0, \quad t \in (0, \infty).$$

Since

$$\frac{d}{dt}(\overline{F}(t)^{\frac{1}{t}}) = (-\frac{1}{t^2}\log \overline{F}(t) - \frac{1}{t}\frac{f(t)}{\overline{F}(t)})e^{\frac{1}{t}\log \overline{F}(t)},$$

this is equivalent to $\frac{1}{t^2}\log \overline{F}(t) + \frac{1}{t}\frac{f(t)}{\overline{F}(t)} > 0$, and hence to

$$G(t) := \frac{tf(t)}{\overline{F}(t)} > -\log \overline{F}(t) =: H(t), \quad t \in (0, \infty).$$

But this is the case, since $G(0) = H(0) = 0$ and for $t \in (0, \infty)$

$$G'(t) \quad = \quad R(t) + tR'(t) > R(t) = H'(t),$$

where we used $R'(t) > 0$ since F is IFR.

Problem 4

Show that $r(\mathbf{p} \cdot \mathbf{p'}) \leq r(\mathbf{p})r(\mathbf{p'})$ (where $(p_1, ..., p_n) \cdot (p'_1, ..., p'_n) := (p_1 p'_1, ..., p_n p'_n)$) and give an interpretation.

Solution: We have $r(p) = \mathbb{E}[\Phi(X)]$, where X is the random state vector of the system components. We consider an independent system of the same form as the first one, but possibly with different components, i.e. a different random state vector X' whose components have possibly different failure probabilites.
Now construct a new system as follows: Replace in the first system each component by a serie of this component and the corresponding component in the second system. The probability that this system works is given by

$$P(\Phi(X_1 X'_1, ..., X_n X'_n) = 1) = P(\Phi(X \cdot X') = 1) = r(p \cdot p').$$

Since $X \cdot X' = \min\{X, X'\}$ we have $\Phi(X \cdot X') \leq \min\{\Phi(X), \Phi(X')\}$, hence the equation $\Phi(X \cdot X') = 1$ implies both $\Phi(X) = 1$ and $\Phi(X') = 1$. Hence

$$\{\Phi(X \cdot X') = 1\} \subset \{\Phi(X) = 1\} \cap \{\Phi(X') = 1\},$$

which means

$$P(\Phi(X \cdot X') = 1) \leq P(\Phi(X) = 1, \Phi(X') = 1) = P(\Phi(X) = 1)P(\Phi(X') = 1) = r(p)r(p').$$