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Applied Stochastic Models (SS 09)

Problem Set 8

Problem 1

Consider a series system with two identical components with gamma distributed lifetimes with $\alpha = 5$ and arbitrary λ. Obtain the mean system lifetime.

Solution:
If $X \sim \Gamma(\lambda, \alpha)$ then its density is given by

$$f^X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \mathbb{1}_{(0,\infty)}(x).$$

In case $\alpha = 5$, this yields for the cdf

$$F^X(x) = 1 - e^{-\lambda x} \sum_{j=0}^{4} \frac{(\lambda x)^j}{j!}, \quad x > 0.$$

If T denotes the system lifetime and X_1, X_2 the lifetimes of the individual components, we have

$$P(T > t) = P(X_1 > t, X_2 > t) = P(X_1 > t)^2 = (1 - F^{X_1}(x))^2,$$

and hence for the expected mean system lifetime

$$\begin{aligned} \mathbb{E}T &= \int_0^\infty (1 - F(t))^2 dt = \int_0^\infty e^{-2\lambda t} \sum_{i,j=0}^{4} \frac{(\lambda t)^{i+j}}{i!j!} dt \\ &= \frac{1}{\lambda} \int_0^\infty e^{-2u} \sum_{i,j=0}^{4} \frac{u^{i+j}}{i!j!} du. \end{aligned}$$

Since

$$\mathcal{L}\left(t \mapsto \frac{t^n}{n!} \mathbb{1}_{(0,\infty)}(t)\right)(s) = \frac{1}{s^{n+1}}$$

we may proceed as follows:

$$\begin{aligned} \mathbb{E}T &= \frac{1}{\lambda} \mathcal{L}\left(t \mapsto \sum_{i,j=0}^{4} \binom{i+j}{i} \frac{t^{i+j}}{(i+j)!}\right)(2) \\ &= \frac{1}{\lambda} \sum_{i,j=0}^{4} \binom{i+j}{i} \frac{1}{2^{i+j+1}} \\ &= \frac{965}{256} \frac{1}{\lambda}. \end{aligned}$$

Problem 2

Give an example to show that a k-out-of-n-system with independent and non-identical IFR lifetime components need not be IFR.

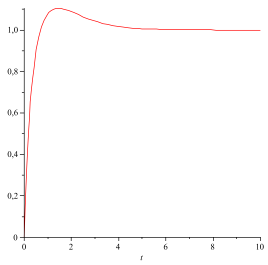
Solution: Consider a parallel system of two components (1-out-of-2-system) where $X_1 \sim Exp(1)$ and $X_2 \sim Exp(2)$. Then their respective failure rates are constant with respect to time and in particular increasing. The life-time of the system is determined by

$$\overline{F}(t) = 1 - (1 - e^{-t})(1 - e^{-2t}) = e^{-t} + e^{-2t} - e^{-3t},$$

which implies that the system failure rate is given by

$$\lambda(t) = \frac{f(t)}{\overline{F}(t)} = \frac{2e^{-2t} + e^{-t} - 3e^{-3t}}{e^{-2t} + e^{-t} - e^{-3t}}.$$

Here is the graph of λ:



The derivative is given by

$$\lambda'(t) = \frac{e^{-t}}{(1 + e^{-t} - 2e^{-2t})^2}(-1 + 4e^{-t} + e^{-2t})$$

where $-1 + 4e^{-t} + e^{-2t}$ is obviously negative for large t.

Problem 3

(a) It costs more to produce an item with a large expected life length than one with a small life expectancy. Suppose that the cost C of producing an item is given as $C = 3\mu^2$ where μ is the mean time to failure. Assume that a profit of D is realized for every hour the item is in service. Determine the maximum expected profit per item.

(b) Suppose there is a penalty cost K involved per unit time the unit fails before t_0. Specifically, the random profit $P(T)$ per item in terms of the random lifetime T is given by

$$P(T) = \begin{cases} DT - 3\mu^2, & T > t_0, \\ DT - 3\mu^2 - K(t_0 - T), & T < t_0 \end{cases}.$$

Explain how the maximum expected profit can be found in this case assuming exponential lifetime T.

Solution: (a) As a function of the random life-time T the profit P per item is given as

$$P(T) = DT - 3\mu^2.$$

The expected profit is therewith given as

$$\mathbb{E}[P(T)] = D\mu - 3\mu^2.$$

One easily verifies that the maximum of this function is attained for $\mu = \frac{D}{6}$ and equals $\frac{D^2}{12}$.
(b) Assuming exponential life-time T, i.e.

$$f^T(t) = \frac{1}{\mu} e^{-\frac{t}{\mu}},$$

we find

$$\begin{aligned}
\mathbb{E}[P(T)] &= \mathbb{E}[(DT - 3\mu^2)\mathbb{1}_{\{T>t_0\}}] + \mathbb{E}[(DT - 3\mu^2 - K(t_0 - T))\mathbb{1}_{\{T<t_0\}}] \\
&= \int (DT(\omega) - 3\mu^2)\mathbb{1}_{\{T(\omega)>t_0\}} P(d\omega) + \int (DT(\omega) - 3\mu^2 - K(t_0 - T(\omega)))\mathbb{1}_{\{T(\omega)<t_0\}} P(d\omega) \\
&= \int (Dt - 3\mu^2)\mathbb{1}_{\{t>t_0\}} P^T(dt) + \int (Dt - 3\mu^2 - K(t_0 - t))\mathbb{1}_{\{t<t_0\}} P^T(dt) \\
&= \int_{t_0}^{\infty} (Dt - 3\mu^2)\frac{1}{\mu} e^{-\frac{t}{\mu}} dt + \int_0^{t_0} (Dt - 3\mu^2 - K(t_0 - t))\frac{1}{\mu} e^{-\frac{t}{\mu}} dt \\
&= \int_0^{\infty} (Dt - 3\mu^2)\frac{1}{\mu} e^{-\frac{t}{\mu}} dt - \int_0^{t_0} (K(t_0 - t))\frac{1}{\mu} e^{-\frac{t}{\mu}} dt \\
&= \int_0^{\infty} (Dt - 3\mu^2)\frac{1}{\mu} e^{-\frac{t}{\mu}} dt - \int_0^{t_0} (K(t_0 - t))\frac{1}{\mu} e^{-\frac{t}{\mu}} dt \\
&= D\mu - 3\mu^2 - Kt_0 + K\mu[1 - e^{-\frac{t_0}{\mu}}].
\end{aligned}$$

Here numerical methods are useful to determine the maximum, for example apply Newtons method to the first derivative.

Problem 4

(a) Draw a sample of size n from Weibull distributed random variables and denote the smallest drawn value with $X_{(1)}$. Prove that $X_{(1)}$ also follows a Weibull distribution.
(b) Draw a sample of size n from random variables each with invertible cdf $F(t)$. Show that $n(1 - F(X_{(n)}))$ tends to an exponential distribution.

Solution: (a) A $Wei(\lambda, \alpha)$ distributed random variable has cfd

$$F(t) = (1 - e^{-(\lambda t)^\alpha})\mathbb{1}_{(0,\infty)}(t).$$

Now

$$\begin{aligned}
P(X_{(1)} > t) &= P(X_1 > t, ..., X_n > t) = \prod_{i=1}^{n} P(X_i > t) = \prod_{i=1}^{n} e^{-(\lambda t)^\alpha} \\
&= e^{-n\lambda^\alpha t^\alpha} = e^{-(n^{\frac{1}{\alpha}}\lambda t)^\alpha}.
\end{aligned}$$

Hence

$$X_{(1)} \sim Wei((n^{\frac{1}{\alpha}}\lambda), \alpha).$$

(b) We have

$$\begin{aligned} P(n(1-F(X_{(n)}))\geq y) &= P(F(X_{(n)})\leq 1-\frac{y}{n}) = P(X_{(n)}\leq F^{-1}(1-\frac{y}{n})) \\ &= \prod_{i=1}^{n} P(X_i \leq F^{-1}(1-\frac{y}{n})) = (1-\frac{y}{n})^n \to e^{-y}. \end{aligned}$$

Hence

$$n(1-F(X_{(n)})) \xrightarrow{d} Exp(1).$$