

## Applied Stochastic Models (SS 09)

### Problem Set 9

#### Problem 1

Consider a system of two nonindependent components. Three types of shocks occur at times  $U_1, U_2, U_{12}$  following exponential variables with different parameters  $\lambda_1, \lambda_2, \lambda_{12}$ . Shock of type *I* destroys component 1, of type *II* destroys component 2 and of type *III* destroys both. Let  $X = \min(U_1, U_{12})$  and  $Y = \min(U_2, U_{12})$ . Show that

- (a)  $P(X > x, Y > y) = e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12} \max(x, y))}$ ,
- (b)  $P(\min(X, Y) > t) = e^{-(\lambda_1 + \lambda_2 + \lambda_{12})t}$ ,
- (c)  $P(X > x + t, Y > y + t \mid X > t, Y > t) = P(X > x, Y > y)$ .
- (d) Find the mean and variance of  $X$ .

#### Solution:

(a) We have

$$\begin{aligned} P(X > x, Y > y) &= P(U_1 > x, U_{12} > x, U_2 > y, U_{12} > y) = P(U_1 > x, U_2 > y, U_{12} > \max(x, y)) \\ &= P(U_1 > x)P(U_2 > y)P(U_{12} > \max(x, y)) = e^{-\lambda_1 x} e^{-\lambda_2 y} e^{-\lambda_{12} \max(x, y)} \\ &= e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12} \max(x, y))}. \end{aligned}$$

- (b) Put  $x = y := t$  in (a).
- (c)

$$\begin{aligned} P(X > x + t, Y > y + t \mid X > t, Y > t) &= \frac{P(X > x + t, Y > y + t, X > t, Y > t)}{P(X > t, Y > t)} \\ &= \frac{P(X > x + t, Y > y + t)}{P(X > t, Y > t)} \\ &= \frac{e^{-(\lambda_1(x+t) + \lambda_2(y+t) + \lambda_{12}(\max(x, y) + t))}}{e^{-(\lambda_1 + \lambda_2 + \lambda_{12})t}} \\ &= e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12} \max(x, y))} = P(X > x, Y > y). \end{aligned}$$

- (d) Since  $X \sim \text{Exp}(\lambda_1 + \lambda_{12})$  (part (a) for  $y = 0$ ) we have  $\mathbb{E}X = \frac{1}{\lambda_1 + \lambda_{12}}$  and  $\text{Var}X = \frac{1}{(\lambda_1 + \lambda_{12})^2}$ .

#### Problem 2

An aircraft has four engines, each of which has a failure rate  $\lambda$ . For a successful flight at least two engines should be operating.

- (a) Find the reliability  $R(t)$  and expected lifetime of the aircraft.

(b) Find these if the aircraft needs at least one operating engine on either side for a successful flight.

**Solution:**

Let  $X$  denote the lifetime of one of the engines,  $X \sim \text{Exp}(\lambda)$ . All engines are distributed according to  $X$  and are independent.

(a) Since at least 2 out of the 4 engines need to be operating we are dealing with a 2-out-of-4-system. Let  $T$  denote the lifetime of the system. Then

$$R(t) = P(T > t) = \sum_{r=2}^4 \binom{4}{r} (e^{-\lambda t})^r (1 - e^{-\lambda t})^{4-r}.$$

The expected lifetime is then given by

$$\begin{aligned} \int_0^\infty R(t) dt &= \sum_{r=2}^4 \binom{4}{r} \int_0^\infty (e^{-\lambda t})^r (1 - e^{-\lambda t})^{4-r} dt \\ &= \sum_{r=2}^4 \binom{4}{r} \int_0^1 u^r (1 - u)^{4-r} \frac{1}{\lambda u} du \\ &= \frac{1}{\lambda} \left( 6 \int_0^1 u(1 - u)^2 du + 4 \int_0^1 u^2(1 - u) du + \int_0^1 u^3 du \right) \\ &= \frac{1}{\lambda} \left( \frac{6}{12} + \frac{4}{12} + \frac{1}{4} \right) \\ &= \frac{13}{12\lambda}. \end{aligned}$$

(b) Here we are dealing with a series arrangement of two 1-out-of-2-systems. Hence the reliability of the system is given by

$$R(t) = P(T > t) = (1 - P(X < t))^2 = (1 - (1 - e^{-\lambda t}))^2 = 4e^{-2\lambda t} - 4e^{-3\lambda t} + e^{-4\lambda t}.$$

The mean lifetime on the other hand is

$$\begin{aligned} \mathbb{E}T &= \int_0^\infty R(t) dt = \int_0^\infty (4e^{-2\lambda t} - 4e^{-3\lambda t} + e^{-4\lambda t}) dt \\ &= \frac{1}{\lambda} \left( \frac{4}{2} - \frac{4}{3} + \frac{1}{4} \right) = \frac{11}{12\lambda}. \end{aligned}$$

**Problem 3**

Prove:

(a) If  $0 \leq \alpha, \lambda \leq 1$ , then

$$h(y) = (\lambda x)^\alpha + (1 - \lambda^\alpha)y^\alpha - (\lambda x + (1 - \lambda)y)^\alpha \geq 0, \quad y \leq x.$$

(Hint: Note that  $f(t) = t^\alpha$  is a concave function, so that  $f(t + h) - f(t)$  is decreasing in  $t$ .)

(b) Deduce that  $r(\mathbf{p}^\alpha) \geq [r(\mathbf{p})]^\alpha, 0 \leq \alpha \leq 1$ .

**Solution:**

(a) The map  $x \mapsto x^\alpha$  is concave for  $x \geq 0, 0 \leq \alpha \leq 1$ , hence

$$(u_1 + h)^\alpha - (u_1)^\alpha \geq (u_2 + h)^\alpha - u_2^\alpha, \quad 0 \leq u_1 \leq u_2, h \geq 0.$$

Now set  $h = \lambda(x - y), u_1 = \lambda y, u_2 = y$  to derive

$$(\lambda y + \lambda(x - y))^\alpha - (\lambda y)^\alpha \geq (y + \lambda(x - y))^\alpha - y^\alpha,$$

i.e.

$$(\lambda x)^\alpha + (1 - \lambda^\alpha)y^\alpha - (\lambda x + (1 - \lambda)y)^\alpha \geq 0.$$

(b) We will prove this part by induction on the number  $n$  of components in the system. In case  $n = 1$  there are only the trivial cases  $r(p) = 0, r(p) = 1$  (if the component is used in such a way that it does not matter for the system) and  $r(p) = p$  (if the component is necessary for the system to work). In each case the assumption is trivially fulfilled. Now assume that in any (monotone - as we always assume) system consisting of  $n - 1$  components the above inequality holds and consider a system of  $n$  components, each with reliability  $p_i^\alpha$ . By conditioning on the  $n^{\text{th}}$  component we get

$$r(p_1^\alpha, \dots, p_n^\alpha) = p_n^\alpha r(p_1^\alpha, \dots, p_{n-1}^\alpha, 1) + (1 - p_n^\alpha) r(p_1^\alpha, \dots, p_{n-1}^\alpha, 0).$$

By induction hypothesis we have  $r(p_1^\alpha, \dots, p_{n-1}^\alpha, 1) \geq r(p_1, \dots, p_{n-1}, 1)^\alpha$  and  $r(p_1^\alpha, \dots, p_{n-1}^\alpha, 0) \geq r(p_1, \dots, p_{n-1}, 0)^\alpha$  which yields

$$r(p_1^\alpha, \dots, p_n^\alpha) \geq p_n^\alpha r(p_1, \dots, p_{n-1}, 1)^\alpha + (1 - p_n^\alpha) r(p_1, \dots, p_{n-1}, 0)^\alpha.$$

Applying part (a) with  $\lambda = p_n, x = r(p_1, \dots, p_{n-1}, 1), y = r(p_1, \dots, p_{n-1}, 0)$  (note that by monotonicity indeed  $y \leq x$ ) implies

$$r(\mathbf{p}^\alpha) \geq [p_n r(p_1, \dots, p_{n-1}, 1) + (1 - p_n) r(p_1, \dots, p_{n-1}, 0)]^\alpha = r(\mathbf{p})^\alpha.$$

**Problem 4**

We say that  $\zeta$  is a  $p$ -quantile of the distribution  $F$  if  $F(\zeta) = p$ . Show that if  $\zeta$  is a  $p$ -quantile of the IFRA distribution  $F$ , then

$$\begin{aligned} \bar{F}(x) &\leq e^{-\theta x}, & x \geq \zeta, \\ \bar{F}(x) &\geq e^{-\theta x}, & x \leq \zeta, \end{aligned}$$

where  $\theta = \frac{-\ln(1-p)}{\zeta}$ .

**Solution:**

As shown in class an IFRA distribution  $F$  satisfies

$$\bar{F}(\alpha x) \geq \bar{F}(x)^\alpha, \quad 0 \leq \alpha \leq 1, x \geq 0.$$

Now if  $\zeta \leq x$  we can write

$$1 - p = \bar{F}(\zeta) = \bar{F}\left(x \frac{\zeta}{x}\right) \geq \bar{F}(x)^{\frac{\zeta}{x}},$$

which is equivalent to

$$\bar{F}(x) \leq (1 - p)^{\frac{x}{\zeta}} = e^{-\theta x}.$$

For  $\zeta \geq x$  note that by the same inequality as above  $\bar{F}(x) = \bar{F}\left(\zeta \frac{x}{\zeta}\right) \geq \bar{F}(\zeta)^{\frac{x}{\zeta}} = e^{-\theta x}$ .