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Applied Stochastic Models (SS 09)

Problem Set 11

Problem 1

Describe three distinct methods to generate a random variable with density

$$f(x) = 6x(1-x), \quad 0 \le x \le 1.$$

Solution: The given density is the density of a $\beta(2,2)$-distributed random variable.
Method 1: From sheet 2, problem 3, we know that if $X \sim \Gamma(\lambda, \alpha), Y \sim \Gamma(\lambda, \beta)$ then $\frac{X}{X+Y} \sim \beta(\alpha, \beta)$. If α, β are natural numbers, we get the special case of Erlang distributed random variables X and Y, hence just sums of exponentials. This is the case here, since $\alpha = 2 = \beta$. For simplicity we may choose $\lambda = 1$. If $U_1, U_2 \sim U(0,1)$ then $-\ln(U_1) \sim Exp(1)$, $-\ln(U_1U_2) = -\ln(U_1) - \ln(U_2) \sim Erl(2,1)$ and hence for $U_1, U_2, U_3, U_4 \sim U(0,1)$ (independent!)

$$\frac{-\ln(U_1U_2)}{-\ln(U_1U_2U_3U_4)} \sim \beta(2,2).$$

Method 2: Let U_1, U_2, U_3 be independent and on $(0,1)$ uniformly distributed random variables. Denote by $U_{(2)}$ the second order statistic. Then

$$\begin{aligned} P(U_{(2)} \le x) &= P(\text{at least 2 of the 3 variables are} \le x) \\ &= 3P(U_1, U_2 \le x, U_3 > x) + P(U_1, U_2, U_3 \le x) \\ &= 3x^2(1-x) + x^3 = 3x^2 - 2x^3, \end{aligned}$$

and hence

$$f^{U_{(2)}}(x) = 6x(1-x).$$

This yields the following method: Generate three independent $U(0,1)$ random variables and return the realization in the middle.
Method 3: Acceptance-Rejection method as done in class.

Problem 2

Let X be a non-negative integer-valued random variable with

$$h(r) = P(X = r | X \ge r).$$

If $U_i, i = 0, 1, 2, 3, ...$ are independent and uniform in $[0,1]$, show that $Z = \min\{n \in \mathbb{N}_0 : U_n \le h(n)\}$ has the same distribution as X.

Solution: A direct calculation shows:

$$\begin{aligned} P(Z=m) &= P(U_1 > h(1), ..., U_{m-1} > h(m-1), U_m \leq h(m)) = \left[\prod_{n=1}^{m-1}(1-h(i))\right] h(m) \\ &= P(X>0)P(X>1|X>0)...P(X>m-1|X>m-2)P(X=m|X>m-1) \\ &= P(X=m). \end{aligned}$$

Problem 3

Suppose it is easy to simulate from the distributions $F_i, i = 1, 2, ..., n$. Give a procedure to simulate

$$F(x) = \sum_{i=1}^{n} p_i F_i(x), \quad p_i > 0, \sum p_i = 1.$$

Then give a method to simulate from

$$F(x) = \begin{cases} \frac{1-e^{-2x}+2x}{3}, & 0 < x < 1, \\ \frac{3-e^{-2x}}{3}, & 1 < x < \infty. \end{cases}$$

Solution: Let $X_i \sim F_i$ and $U \sim U(0,1)$ be independent from the X_i. Then define $P_0 := 0, P_k := \sum_{i=1}^{k} p_i, k = 1..n$ and

$$Y := \sum_{i=1}^{n} \mathbb{1}_{(P_i - P_{i-1})}(U) X_i.$$

One easily verifies that $Y \sim F$. Hence one possible procedure to simulate from F is to generate U, check for which i the inequality $P_{i-1} < U \leq P_i$ is satisfied and return for this i the value of the previously generated X_i.
The given $F(x)$ can be written as $F(x) = \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x)$ where $F_1(x) = 1 - e^{-2x}, x > 0$, and $F_2(x) = x\mathbb{1}_{(0,1)}(x) + \mathbb{1}_{[1,\infty)}(x)$. Hence Y can be simulated as follows: Generate $U_1, U_2, U_3 \sim U(0,1)$ independent. Then return

$$Y = \begin{cases} -\frac{1}{2}\ln(U_2), & \text{if } U_1 < \frac{1}{3} \\ U_3, & \text{else} \end{cases}.$$

Problem 4

For a (non-homogeneous) Poisson process with intensity function $\lambda(t) > 0, t \geq 0$, where

$$\int_0^\infty \lambda(t)dt = \infty,$$

let $X_1, X_2, ...$ denote the sequence of times at which events occur. Show that

$$\int_{X_{i-1}}^{X_i} \lambda(t)dt, \quad i \geq 1,$$

are independent exponential with rate 1 where $X_0 = 0$.

Solution: Set $\Lambda(t) = \int_0^t \lambda(s)ds$ and denote the Poisson process by N. Since $\lambda(t) > 0$ it follows that $\Lambda(t)$ is strictly increasing, hence it is invertible.

For $i = 1$ we have

$$P(\Lambda(X_1) > x) = P(X_1 > \Lambda^{-1}(x)) = P(N(\Lambda^{-1}(x)) = 0) = e^{-\Lambda(\Lambda^{-1}(x))} = e^{-x},$$

while for $i > 1$ we have

$$\begin{aligned} P(\Lambda(X_i) - \Lambda(X_{i-1}) > x) &= \mathbb{E}[P(\Lambda(X_i) - \Lambda(X_{i-1}) > x | X_{i-1})] \\ &= \mathbb{E}[P(\int_{X_{i-1}}^{X_i} \lambda(u) du > x | X_{i-1})] \end{aligned}$$

Here $P(\int_{X_{i-1}}^{X_i} \lambda(u)du > x|X_{i-1}) = P(X_i > c(X_{i-1})|X_{i-1})$ where $\int_{X_{i-1}}^{c(X_{i-1})} \lambda(u)du = x$. Hence

$$\begin{aligned} P(\Lambda(X_i) - \Lambda(X_{i-1}) > x) &= \mathbb{E}[P(X_i > c(X_{i-1})|X_{i-1})] \\ &= \mathbb{E}[P(N(c(X_{i-1})) - N(X_{i-1}) = 0|X_{i-1})] \\ &= \mathbb{E}[P(N(c(X_{i-1})) - N(X_{i-1}) = 0|X_{i-1})] \\ &= \mathbb{E}[e^{-\int_{X_{i-1}}^{c(X_{i-1})} \lambda(u)du}] \\ &= \mathbb{E}[e^{-x}] = e^{-x}. \end{aligned}$$

A similar calculation shows that

$$P(\Lambda(X_i) - \Lambda(X_{i-1}) > x | \Lambda(X_1), \Lambda(X_2) - \Lambda(X_1), ..., \Lambda(X_{i-1}) - \Lambda(X_{i-2}))$$

also equals e^{-x}, which shows independence.