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Applied Stochastic Models (SS 08)

Problem Set 3

Problem 1

Let T_1 and T_2 be independent and uniformly distributed on $[-1, 1]$. Show that, conditional on the event that $R := \sqrt{T_1^2 + T_2^2} \leq 1$,

$$X := \frac{T_1}{R}\sqrt{-2\ln R^2}, \quad Y := \frac{T_2}{R}\sqrt{-2\ln R^2}$$

are independent standard normal random variables.

Solution:
First, we investigate the conditional distributions of $\Theta := \arccos(\frac{T_1}{R}) = \arcsin(\frac{T_2}{R})$ and R^2, given that $R \leq 1$. We have $\Theta \in (0, 2\pi)$, $R^2 \in (0, 1)$, and thus for $\theta \in (0, 2\pi)$ and $r \in (0, 1)$

$$\mathbb{P}(R^2 \leq r | R \leq 1) = \frac{\mathbb{P}(R^2 \leq r, R \leq 1)}{\mathbb{P}(R \leq 1)} = \frac{\mathbb{P}(R \leq \sqrt{r})}{\mathbb{P}(R \leq 1)} = \frac{\frac{\pi r}{4}}{\frac{\pi}{4}} = r,$$

$$\mathbb{P}(\Theta \leq \theta | R \leq 1) = \frac{\mathbb{P}(\Theta \leq \theta, R \leq 1)}{\mathbb{P}(R \leq 1)} = \frac{\frac{\pi \frac{\theta}{2\pi}}{4}}{\frac{\pi}{4}} = \frac{\theta}{2\pi}.$$

Hence $R^2 | R \leq 1 =: U \sim U(0, 1)$ and $\Theta | R \leq 1 =: 2\pi\tilde{U}$ where $U, \tilde{U} \sim U(0, 1)$. Using $U, \tilde{U}$, we have

$$X = \cos(2\pi\tilde{U})\sqrt{-2\ln U}, \quad Y = \sin(2\pi\tilde{U})\sqrt{-2\ln U},$$

and using the transformation

$$T : \begin{cases} (0,1)^2 & \to \mathbb{R}^2 \\ (u, \tilde{u}) & \mapsto (\cos(2\pi\tilde{u})\sqrt{-2\ln u}, \sin(2\pi\tilde{u})\sqrt{-2\ln u}) \end{cases},$$

we can write $(X, Y) = T(U, \tilde{U})$. The determinant of the Jacobian of T is easily calculated as

$$\det(T')(u, \tilde{u}) = -\frac{2\pi}{u}.$$

Since this depends only on u and the joint density of $U, \tilde{U}$ is the constant 1, it will be enough to express u in terms of $(x, y) = T(u, \tilde{u})$. This can be done by observing that $x^2 + y^2 = -2\ln(u)$, and hence $u = e^{-\frac{x^2+y^2}{2}}$. Thus the transformation formula for densities yields

$$f^{(X,Y)}(x, y) = \frac{f^{(U,V)}(T^{-1}(x, y))}{\Delta(T^{-1}(x, y))} = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}.$$

Problem 2

Let $X_1, X_2, \cdots X_n$ be independent and exponentially distributed with parameter λ. Show that

$$Y_1 = nX_{(1)}, Y_r = (n+1-r)(X_{(r)} - X_{(r-1)}), 1 < r \leq n$$

are also independent and have the same joint distribution as the X_i.

Solution: The joint density of the X_i is by independence given by

$$f^{(X_1,...X_n)}(x_1, ..., x_n) = \lambda^n e^{-\lambda x_1} ... e^{-\lambda x_n},$$

while their order statistics $X_{(i)}$ have joint density

$$f^{(X_{(1)},...X_{(n)})}(x_1, ..., x_n) = n! \lambda^n e^{-\lambda x_1} ... e^{-\lambda x_n} \mathbb{1}_D(x_1, ..., x_n),$$

where $D = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 < x_2 < ... < x_n\}$. Using the transformation

$$T_n := x \mapsto A_n x,$$

where

$$A_n := \begin{pmatrix} n & 0 & 0 & ... & 0 \\ -(n-1) & (n-1) & 0 & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & ... & -2 & 2 & 0 \\ 0 & ... & 0 & -1 & 1 \end{pmatrix},$$

we have $(Y_1, ..., Y_n) = T_n(X_{(1)}, ..., X_{(n)})$. One calculates recursively

$$\begin{aligned} X_{(1)} &= \frac{1}{n} Y_1 \\ X_{(2)} &= \frac{1}{n-1} Y_2 + \frac{1}{n} Y_1 \\ X_{(3)} &= \frac{1}{n-2} Y_3 + \frac{1}{n-1} Y_2 + \frac{1}{n} Y_1 \\ &\vdots \\ X_{(n)} &= Y_n + \frac{1}{2} Y_{n-1} + ... + \frac{1}{n} Y_1 \end{aligned}$$

which yields

$$A_n^{-1} = \begin{pmatrix} \frac{1}{n} & 0 & 0 & ... & 0 \\ \frac{1}{n} & \frac{1}{n-1} & 0 & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{1}{n} & \frac{1}{n-1} & ... & \frac{1}{2} & 0 \\ \frac{1}{n} & \frac{1}{n-1} & ... & \frac{1}{2} & 1 \end{pmatrix}.$$

The Jacobian of T_n is just A_n, whose determinant is obviously $n!$, hence

$$\Delta \equiv n!.$$

The transformation theorem for densities thus yields for $y_1, ...y_n \geq 0$

$$f^{(Y_1,...Y_n)}(y_1, ..., y_n) = \frac{n!\lambda^n e^{-\lambda(\frac{1}{n}y_1)} e^{-\lambda(\frac{1}{n}y_1 + \frac{1}{n-1}y_2)} ... e^{-\lambda(\frac{1}{n}y_1 + ... + \frac{1}{2}y_{n-1} + y_n)}}{n!} = \lambda^n e^{-\lambda y_1} ... e^{-\lambda y_n},$$

and hence shows that the Y_i have the same joint distribution as the initial X_i!

Problem 3

Let $X_1, X_2, \cdots X_n$ be independent and uniformly distributed in $[0, 1]$, with order statistics $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$. Show that $-\ln(X_{(k)})$ has the same distribution as $\sum_{i=k}^{n} Y_i/i$, where the Y_i are independent exponential random variables with parameter 1.

Solution: The joint density of the $X_{(i)}$ is given by

$$f^{(X_{(1)},...,X_{(n)})}(x_1, ..., x_n) = n! \mathbb{1}_{[0,1]^n}(x_1, ..., x_n) \mathbb{1}_D(x_1, ..., x_n),$$

where $D = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 < x_2 < ... < x_n\}$. Define

$$\begin{aligned} U_n &:= -n \ln X_{(n)}, \\ U_k &:= -k \ln \frac{X_{(k)}}{X_{(k+1)}}, \quad 1 \leq k < n. \end{aligned}$$

Then

$$\begin{aligned} X_{(n)} &= e^{-\frac{U_n}{n}}, \\ X_{(k)} &= X_{(k+1)} e^{-\frac{U_k}{k}} = e^{-\sum_{i=k}^{n} \frac{U_i}{i}}, \quad 1 \leq k < n. \end{aligned}$$

Let T denote the above transformation, i.e. $(U_1, ..., U_n) = T(X_{(1)}, ..., X_{(n)})$. Then

$$\Delta(x_1, ..., x_n) = (\det T')(x_1, ..., x_n) = \begin{vmatrix} -\frac{1}{x_1} & \frac{1}{x_2} & 0 & ... & 0 \\ 0 & -\frac{2}{x_2} & \frac{2}{x_3} & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & ... & -\frac{n-1}{x_{n-1}} & \frac{n-1}{x_n} \\ 0 & 0 & ... & 0 & -\frac{n}{x_n} \end{vmatrix} = \frac{(-1)^n n!}{x_1 ... x_n}.$$

Hence, the joint density of the U_i is given by

$$f^{(U_1,...,U_n)}(u_1, ..., u_n) = \frac{f^{(X_{(1)},...,X_{(n)})}(T^{-1}(u_1, ..., u_n))}{|\Delta(T^{-1}(u_1, ..., u_n))|} = \frac{n!}{n!} e^{-(u_1 + ... + u_n)}.$$

This means that the U_i are iid $Exp(1)$. As shown above, we have $-\ln(X_{(k)}) = \sum_{i=k}^{n} \frac{U_i}{i}$.

Problem 4

Let $X_1, X_2, X_3, \cdots X_n$ be independent and exponentially distributed with parameter 1. Show that

$$\lim_{n\to\infty} \mathbb{P}(X_{(n)} - \ln n \leq x) = \exp(-e^{-x}).$$

Using this, show that $\int_0^\infty [1 - \exp(-e^{-x})] dx = \gamma$, where γ denotes Eulers constant ($\gamma = \lim_{n\to\infty} [\sum_{k=1}^{n} \frac{1}{k} - \ln(n)]$).

Solution:
For the first assertion, note that

$$\begin{aligned}\mathbb{P}(X_{(n)} - \ln n \leq x) &= \mathbb{P}(\max\{X_1, \cdots, X_n\} \leq x + \ln n) = \mathbb{P}(X_i \leq x + \ln n, i \in \{1, ..., n\}) \\ &= (1 - e^{-(x+\ln n)})^n = (1 - \frac{e^{-x}}{n})^n \to e^{-e^{-x}}.\end{aligned}$$

The integral can now be expressed as follows:

$$\int_0^\infty [1 - \exp(-e^{-x})] dx = \int_0^\infty \lim_{n\to\infty} \mathbb{P}(X_{(n)} - \ln n > x) dx.$$

Here,

$$\mathbb{P}(X_{(n)} - \ln n > x) = 1 - (1 - \frac{e^{-x}}{n})^n,$$

is monotonically decreasing in n, hence

$$\mathbb{P}(X_{(n)} - \ln n > x) \leq 1 - (1 - \frac{e^{-x}}{1})^1 = e^{-x},$$

which has a finite integral. Hence, we may apply the dominated convergence theorem to interchange limit and integration:

$$\int_0^\infty [1 - \exp(-e^{-x})] dx = \lim_{n\to\infty} \int_0^\infty \mathbb{P}(X_{(n)} - \ln n > x) dx.$$

From here, we proceed as follows:

$$\begin{aligned}\int_0^\infty [1 - \exp(-e^{-x})] dx &= \lim_{n\to\infty} \int_0^\infty \mathbb{P}(X_{(n)} - \ln n > x) dx \\ &= \lim_{n\to\infty} \mathbb{E}[X_{(n)} - \ln n].\end{aligned}$$

For the expectation, we have

$$\begin{aligned}\mathbb{E}[X_{(n)}] &= \int_0^\infty \mathbb{P}(X_{(n)} > x) dx = \int_0^\infty 1 - \mathbb{P}(X_{(n)} \leq x) dx \\ &= \int_0^\infty 1 - \mathbb{P}(X_1, ..., X_n \leq x) dx = \int_0^\infty 1 - (1 - e^{-x})^n dx \\ &= \int_0^\infty 1 - \sum_{i=0}^n \binom{n}{i} (-e^{-x})^i dx \\ &= -\sum_{i=1}^n \binom{n}{i} (-1)^i \int_0^\infty e^{-ix} dx \\ &= \sum_{i=1}^n \binom{n}{i} (-1)^{i-1} \frac{1}{i}.\end{aligned}$$

We need to show that this equals $\sum_{i=1}^n \frac{1}{i}$. Using the generating function

$$f(z) := \sum_{i=1}^n \binom{n}{i} (-1)^{i-1} \frac{z^i}{i}$$

of the above series, we find for its derivative that

$$\begin{aligned} f'(z) &= \frac{1}{z}\sum_{i=1}^{n}\binom{n}{i}(-1)^{i-1}z^i = \frac{-1}{z}\sum_{i=1}^{n}\binom{n}{i}(-z)^i = \frac{1-(1-z)^n}{1-(1-z)} \\ &= 1+(1-z)+(1-z)^2+...+(1-z)^{n-1}. \end{aligned}$$

Integrating this from 0 to z yields

$$f(z)-f(0) = z - \frac{(1-z)^2}{2} - \frac{(1-z)^3}{3} - ... - \frac{(1-z)^n}{n} + 0 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n},$$

and hence

$$f(1) = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}.$$

Finally, this means that

$$\int_0^\infty [1-\exp(-e^{-x})]dx = \lim_{n\to\infty}[\sum_{i=1}^{n}\frac{1}{i} - \ln n] = \gamma.$$