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Applied Stochastic Models (SS 08)

Problem Set 6

Problem 1

(a) Write the structure function corresponding to a system consisting of four components in such a way that the system functions if and only if components 1 and 2 both function and at least one of components 3 and 4 function. Find the minimal path sets and minimal cut sets of this system.

(b) Work the above problem for a system with six components which functions if and only if components 1 and 6 both function, and at either 2 and 4 both function or 3 and 5 both function.

Solution: (a) The structure function is given by

$$\phi(x) = x_1 x_2 (1 - (1 - x_3)(1 - x_4)).$$

The minimal path vectors are given by $(1,1,1,0)$ and $(1,1,0,1)$, hence the minimal path sets are

$$P_1 = \{1,2,3\}, \quad P_2 = \{1,2,4\}.$$

The minimal cut vectors are $(1,1,0,0)$, $(0,1,1,1)$ and $(1,0,1,1)$, hence the minimal cut sets are

$$C_1 = \{3,4\}, \quad C_2 = \{1\}, \quad C_3 = \{2\}.$$

(b) The structure function is given by

$$\phi(x) = x_1 (1 - (1 - x_2 x_4)(1 - x_3 x_5)) x_6.$$

The minimal path vectors are given by $(1,1,0,1,0,1)$ and $(1,0,1,0,1,1)$, hence the minimal path sets are

$$P_1 = \{1,2,4,6\}, \quad P_2 = \{1,3,5,6\}.$$

The minimal cut vectors are $(0,1,1,1,1,1)$, $(1,0,0,1,1,1)$, $(1,0,1,1,0,1)$, $(1,1,0,0,1,1)$, $(1,1,1,0,0,1)$ and $(1,1,1,1,1,0)$, hence the minimal cut sets are

$$C_1 = \{1\}, \quad C_2 = \{2,3\}, \quad C_3 = \{2,5\}, \quad C_4 = \{3,4\}, \quad C_5 = \{4,5\}, \quad C_6 = \{6\}.$$

Problem 2

(a) The minimal path sets of a system are $\{1,2,4\}, \{1,3,5\}$, and $\{5,6\}$. Find the minimal cut sets.

(b) The minimal cut sets of a system are $\{1,2,3\}, \{2,3,4\}$, and $\{3,5\}$. Find the minimal path sets.

Solution:

(a) Any system can be written as a parallel arrangement of its respective minimal paths (whose components are arranged in series). In our case the system must consequently have a structure function of the form

$$\phi(x) = 1 - (1 - x_1x_2x_4)(1 - x_1x_3x_5)(1 - x_5x_6).$$

Now a cut set must contain at least one point of each path set, otherwise it would not cut. Hence since we are looking only for the minimal cut sets, we can restrict our attention to the following candidates:

$$\{1,5\}, \{1,6\}, \{2,1,5\}, \{2,1,6\}, \{2,3,5\}, \{2,3,6\}, \{2,5\}, \{2,1,6\}$$
$$\{2,5,6\}, \{4,1,5\}, \{4,1,6\}, \{4,3,5\}, \{4,3,6\}, \{4,5\}, \{4,5,6\}.$$

From these, only

$$\{1,5\}, \{1,6\}, \{2,3,6\}, \{2,5\}, \{4,3,6\}, \{4,5\}$$

are minimal.
(b) Any system can be written as a series arrangement of its respective minimal cut sets (whose components are arranged parallel). In our case the system must consequently have a structure function of the form

$$\phi(x) = (1 - (1 - x_1)(1 - x_2)(1 - x_3))(1 - (1 - x_2)(1 - x_3)(1 - x_4))(1 - (1 - x_3)(1 - x_5)).$$

A path set must contain at least one point of each cut set, otherwise it would not be a path. Hence we may restrict our attention to the sets

$$\{1,2,3\}, \{1,2,5\}, \{1,3\}, \{1,3,5\}, \{1,4,3\}, \{1,4,5\},$$
$$\{2,3\}, \{2,5\}, \{2,3,5\}, \{2,3\}, \{2,4,3\}, \{2,4,5\},$$
$$\{3,2\}, \{3,2,5\}, \{3\}, \{3,5\}, \{3,4\}, \{3,4,5\}.$$

From these the minimal ones are

$$\{1,4,5\}, \{2,5\}, \{3\} \quad .$$

Problem 3

(a) Find the reliability function r of a 2-out-of-4 system with independent components. If the reliabilities of the components are all equal to 0.2, find $r(0.2)$.
(b) Find the reliability function of a bridge system with independent components. If the reliabilities of the components are all equal to 0.2, find $r(0.2)$.

Solution: (a) The structure function of a 2-out-of-4-system is given by (parallel system of minimal path sets!)

$$\phi(x) = 1 - (1 - x_1x_2)(1 - x_1x_3)(1 - x_1x_4)(1 - x_2x_3)(1 - x_2x_4)(1 - x_3x_4).$$

One could expand this, use the relation $x_i^2 = x_i$ and form the expectation of the result using linearity. Instead we note that one may write any system in terms of its paths sets $\mathcal{P}$ (all path sets - not only the minimal ones) in the form

$$\phi(x) = \sum_{P \in \mathcal{P}} \prod_{i \in P} x_i \prod_{i \in P^c} (1 - x_i).$$

(To see this, plug a path vector and note that all but one term in the sum vanish. If you plug in a cut vector on the other hand, then all terms vanish.) In our case, this yields

$$\begin{aligned}\phi(x) &= x_1x_2(1-x_3)(1-x_4)+x_1x_3(1-x_2)(1-x_4)+x_1x_4(1-x_2)(1-x_3)\\&+ x_2x_3(1-x_1)(1-x_4)+x_2x_4(1-x_1)(1-x_3)+x_3x_4(1-x_1)(1-x_2)\\&+ x_1x_2x_3(1-x_4)+x_1x_2x_4(1-x_3)+x_1x_3x_4(1-x_2)+x_2x_3x_4(1-x_1)\\&+ x_1x_2x_3x_4.\end{aligned}$$

Using the linearity of expectation yields

$$\begin{aligned}r(p) &= p_1p_2(1-p_3)(1-p_4)+p_1p_3(1-p_2)(1-p_4)+p_1p_4(1-p_2)(1-p_3)\\&+ p_2p_3(1-p_1)(1-p_4)+p_2p_4(1-p_1)(1-p_3)+p_3p_4(1-p_1)(1-p_2)\\&+ p_1p_2p_3(1-p_4)+p_1p_2p_4(1-p_3)+p_1p_3p_4(1-p_2)+p_2p_3p_4(1-p_1)\\&+ p_1p_2p_3p_4.\end{aligned}$$

In case $p_i \equiv p$, this takes the form

$$r(p) = \sum_{k=2}^{4} \binom{4}{k} p^k(1-p)^{4-k} = 1-(1-p)^4 - 4p(1-p)^3,$$

and $p = \frac{4}{5}$ yields $r(\frac{4}{5}) = \frac{113}{625}$.
(b) Here we use the parallel representation in terms of minimal path sets to find

$$\phi(x) = 1-(1-x_1x_4)(1-x_1x_3x_5)(1-x_2x_5)(1-x_2x_3x_4)$$

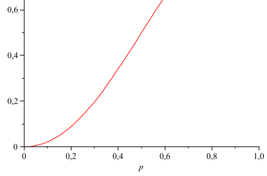
and using the relations $x_i^2 = x_i$ we can expand this as follows:

$$\begin{aligned}\phi(x) &= x_1x_4+x_1x_3x_5+x_2x_5+x_2x_3x_4\\&- x_1x_3x_4x_5 - x_1x_2x_4x_5 - x_1x_2x_3x_4 - x_1x_2x_3x_5 - x_1x_2x_3x_4x_5 - x_2x_3x_4x_5\\&+ 4\cdot x_1x_2x_3x_4x_5\\&- x_1x_2x_3x_4x_5\end{aligned}$$

Taking expectation this yields

$$r(p) = p_1p_4+p_1p_3p_5+p_2p_5+p_2p_3p_4-p_1p_3p_4p_5-p_1p_2p_3p_4-p_1p_2p_4p_5-p_1p_2p_3p_5-p_2p_3p_4p_5+2\cdot p_1p_2p_3p_4p_5.$$

If $p_i \equiv p$, then $r(p) = 2p^2+2p^3-5p^4+2p^5$. For example $r(0.2) = 0.08864...$.
Here is a picture:



Problem 4

(a) Consider a parallel system with identical components each with reliability 0.8. If the reliability of the system is to be at least 0.99, find the minimum number of components in this system.
(b) Consider a series system with identical components each with reliability 0.8. If the reliability of the system is to be at least 0.5, find the maximum number of components in this system.

Solution: (a) The reliability function of a pure parallel system with n iid components is given by

$$r_n(p) = 1 - (1 - p)^n,$$

and for $p = 0.8$ we have to determine n such that $r_n(0.8) \geq 0.99$. This leads to

$$n \geq \left\lfloor \frac{\ln 0.01}{\ln(1 - p)} \right\rfloor + 1 = 3.$$

(a) The reliability function of a pure series system with n iid components is given by

$$r_n(p) = p^n,$$

and for $p = 0.8$ we have to determine n such that $r_n(0.8) \geq 0.5$. This leads to

$$n \leq \left\lfloor \frac{\ln 0.5}{\ln(p)} \right\rfloor = 3.$$