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Applied Stochastic Models (SS 08)

Problem Set 9

Problem 1

Consider a system of two nonindependent components. Three types of shocks occur at times U_1, U_2, U_{12} following exponential variables with different parameters $\lambda_1, \lambda_2, \lambda_{12}$. Shock of type I destroys component 1, of type II destroys component 2 and of type III destroys both. Let $X = \min(U_1, U_{12})$ and $Y = \min(U_2, U_{12})$. Show that

(a) $P(X > x, Y > y) = e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12} \max(x,y))}$,
(b) $P(\min(X,Y) > t) = e^{-(\lambda_1 + \lambda_2 + \lambda_{12})t}$,
(c) $P(X > x + t, Y > y + t \mid X > t, Y > t) = P(X > x, Y > y)$.
(d) Find the mean and variance of X.

Solution:
(a) We have

$$\begin{aligned} P(X > x, Y > y) &= P(U_1 > x, U_{12} > x, U_2 > y, U_{12} > y) = P(U_1 > x, U_2 > y, U_{12} > \max(x,y)) \\ &= P(U_1 > x) P(U_2 > y) P(U_{12} > \max(x,y)) = e^{-\lambda_1 x} e^{-\lambda_2 y} e^{-\lambda_{12} \max(x,y)} \\ &= e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12} \max(x,y))}. \end{aligned}$$

(b) Put $x = y := t$ in (a).
(c)

$$\begin{aligned} P(X > x + t, Y > y + t \mid X > t, Y > t) &= \frac{P(X > x + t, Y > y + t, X > t, Y > t)}{P(X > t, Y > t)} \\ &= \frac{P(X > x + t, Y > y + t)}{P(X > t, Y > t)} \\ &= \frac{e^{-(\lambda_1 (x+t) + \lambda_2 (y+t) + \lambda_{12}(\max(x,y)+t))}}{e^{-(\lambda_1 + \lambda_2 + \lambda_{12})t}} \\ &= e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12} \max(x,y))} = P(X > x, Y > y). \end{aligned}$$

(d) Since $X \sim Exp(\lambda_1 + \lambda_{12})$ (part (a) for $y = 0$) we have $\mathbb{E}X = \frac{1}{\lambda_1 + \lambda_{12}}$ and $Var X = \frac{1}{(\lambda_1 + \lambda_{12})^2}$.

Problem 2

An aircraft has four engines, each of which has a failure rate λ. For a successful flight at least two engines should be operating.

(a) Find the reliability $R(t)$ and expected lifetime of the aircraft.

(b) Find these if the aircraft needs at least one operating engine on either side for a successful flight.

Solution:

Let X denote the lifetime of one of the engines, $X \sim Exp(\lambda)$. All engines are distributed according to X and are independent.

(a) Since at least 2 out of the 4 engines need to be operating we are dealing with a 2-out-of-4-system. Let T denote the lifetime of the system. Then

$$R(t) = P(T > t) = \sum_{r=2}^{4} \binom{4}{r} (e^{-\lambda t})^r (1 - e^{-\lambda t})^{4-r}.$$

The expected lifetime is then given by

$$\begin{aligned}
\int_0^\infty R(t)dt &= \sum_{r=2}^{4} \binom{4}{r} \int_0^\infty (e^{-\lambda t})^r (1 - e^{-\lambda t})^{4-r} dt \\
&= \sum_{r=2}^{4} \binom{4}{r} \int_0^1 u^r (1-u)^{4-r} \frac{1}{\lambda u} du \\
&= \frac{1}{\lambda} \left(6 \int_0^1 u(1-u)^2 du + 4 \int_0^1 u^2(1-u) du + \int_0^1 u^3 du \right) \\
&= \frac{1}{\lambda} \left(\frac{6}{12} + \frac{4}{12} + \frac{1}{4} \right) \\
&= \frac{13}{12\lambda}.
\end{aligned}$$

(b) Here we are dealing with a series arrangement of two 1-out-of-2-systems. Hence the reliability of the system is given by

$$R(t) = P(T > t) = (1 - P(X < t)^2)^2 = (1 - (1 - e^{-\lambda t})^2)^2 = 4e^{-2\lambda t} - 4e^{-3\lambda t} + e^{-4\lambda t}.$$

The mean lifetime on the other hand is

$$\begin{aligned}
\mathbb{E}T &= \int_0^\infty R(t)dt = \int_0^\infty 4e^{-2\lambda t} - 4e^{-3\lambda t} + e^{-4\lambda t} dt \\
&= \frac{1}{\lambda}\left(\frac{4}{2} - \frac{4}{3} + \frac{1}{4}\right) = \frac{11}{12\lambda}.
\end{aligned}$$

Problem 3

Prove:

(a) If $0 \le \alpha, \lambda \le 1$, then

$$h(y) = (\lambda x)^\alpha + (1 - \lambda^\alpha) y^\alpha - (\lambda x + (1-\lambda) y)^\alpha \ge 0, \quad y \le x.$$

(*Hint*: Note that $f(t) = t^\alpha$ is a concave function, so that $f(t+h) - f(t)$ is decreasing in t.)

(b) Deduce that $r(\mathbf{p}^\alpha) \ge [r(\mathbf{p})]^\alpha, 0 \le \alpha \le 1$.

Solution:
(a) The map $x \mapsto x^\alpha$ is concave for $x \geq 0, 0 \leq \alpha \leq 1$, hence

$$(u_1 + h)^\alpha - (u_1)^\alpha \geq (u_2 + h)^\alpha - u_2^\alpha, \quad 0 \leq u_1 \leq u_2, h \geq 0.$$

Now set $h = \lambda(x - y), u_1 = \lambda y, u_2 = y$ to derive

$$(\lambda y + \lambda(x - y))^\alpha - (\lambda y)^\alpha \geq (y + \lambda(x - y))^\alpha - y^\alpha,$$

i.e.

$$(\lambda x)^\alpha + (1 - \lambda^\alpha) y^\alpha - (\lambda x + (1 - \lambda) y)^\alpha \geq 0.$$

(b) We will prove this part by induction on the number n of components in the system. In case $n = 1$ there are only the trivial cases $r(p) = 0, r(p) = 1$ (if the component is used in such a way that it does not matter for the system) and $r(p) = p$ (if the component is necessary for the system to work). In each case the assumption is trivially fulfilled. Now assume that in any (monotone - as we always assume) system consisting of $n - 1$ components the above inequality holds and consider a system of n components, each with reliability p_i^α. By conditioning on the n^{th} component we get

$$r(p_1^\alpha, ..., p_n^\alpha) = p_n^\alpha r(p_1^\alpha, ..., p_{n-1}^\alpha, 1) + (1 - p_n^\alpha) r(p_1^\alpha, ..., p_{n-1}^\alpha, 0).$$

By induction hypothesis we have $r(p_1^\alpha, ..., p_{n-1}^\alpha, 1) \geq r(p_1, ..., p_{n-1}, 1)^\alpha$ and $r(p_1^\alpha, ..., p_{n-1}^\alpha, 0) \geq r(p_1, ..., p_{n-1}, 0)^\alpha$ which yields

$$r(p_1^\alpha, ..., p_n^\alpha) \geq p_n^\alpha r(p_1, ..., p_{n-1}, 1)^\alpha + (1 - p_n^\alpha) r(p_1, ..., p_{n-1}, 0)^\alpha.$$

Applying part (a) with $\lambda = p_n, x = r(p_1, ..., p_{n-1}, 1), y = r(p_1, ..., p_{n-1}, 0)$ (note that by monotonicity indeed $y \leq x$) implies

$$r(\mathbf{p}^\alpha) \geq [p_n r(p_1, ..., p_{n-1}, 1) + (1 - p_n) r(p_1, ..., p_{n-1}, 0)]^\alpha = r(\mathbf{p})^\alpha.$$

Problem 4
We say that ζ is a p-quantile of the distribution F if $F(\zeta) = p$. Show that if ζ is a p-quantile of the IFRA distribution F, then

$$\bar{F}(x) \leq e^{-\theta x}, \quad x \geq \zeta,$$
$$\bar{F}(x) \geq e^{-\theta x}, \quad x \leq \zeta,$$

where $\theta = \frac{-\ln(1-p)}{\zeta}$.

Solution:
As shown in class an IFRA distribution F satisfies

$$\overline{F}(\alpha x) \geq \overline{F}(x)^\alpha, \quad 0 \leq \alpha \leq 1, x \geq 0.$$

Now if $\zeta \leq x$ we can write

$$1 - p = \overline{F}(\zeta) = \overline{F}(x \frac{\zeta}{x}) \geq \overline{F}(x)^{\frac{\zeta}{x}},$$

which is equivalent to

$$\overline{F}(x) \leq (1 - p)^{\frac{x}{\zeta}} = e^{-\theta x}.$$

For $\zeta \geq x$ note that by the same inequality as above $\overline{F}(x) = \overline{F}(\zeta \frac{x}{\zeta}) \geq \overline{F}(\zeta)^{\frac{x}{\zeta}} = e^{-\theta x}$.