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Applied Stochastic Models (SS 08)

Problem Set 10

Problem 1

Describe how the integrals

$$(a) \quad \int_0^1 e^{e^x} dx,$$
$$(b) \quad \int_0^\infty \frac{x}{1+x^2} dx,$$
$$(c) \quad \int_0^\infty \int_0^x e^{-(x+y)} dy dx$$

can be approximated by simulation.

Solution: If $(U_1^k, ..., U_n^k) \sim U((0,1)^n), k \in \mathbb{N}$, then by the SLLN finite (!) integrals of the form

$$\int_0^1 ... \int_0^1 g(x_1, ..., x_n) dx_1 ... dx_n (= \mathbb{E} g(U_1, ..., U_n))$$

where $g \geq 0$ can be approximated by

$$\sum_{i=1}^N \frac{g(U_1^i, ..., U_n^i)}{N}, \quad N \text{ large.}$$

Hence we only need to check whether the integrals of the problem are finite and then transform them suitably. In (a) the integral is clearly finite and we can apply the method directly without further transformations. In part (b) the integral is infinite and the method fails.
In (c) the integral is clearly finite and by substituting $y = tx$ and $x = -\ln(u)$ it transforms into

$$\int_0^\infty \int_0^x e^{-(x+y)} dy dx = \int_0^1 \int_0^1 u^t (-\ln(u)) dt du.$$

Problem 2

Let U be uniform in $[0,1]$. Use simulation to approximate $Cor(U, \sqrt{1-U^2})$ and $Cor(U^2, \sqrt{1-U^2})$.

Solution: Since $Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$ the task reduces to simulate integrals again:

$$Cor(U, \sqrt{1-U^2}) = \frac{\int_0^1 x\sqrt{1-x^2}dx - \int_0^1 x dx \int_0^1 \sqrt{1-x^2} dx}{\sqrt{\int_0^1 x^2 dx - (\int_0^1 x dx)^2} \sqrt{\int_0^1 (1-x^2) dx - (\int_0^1 \sqrt{1-x^2} dx)^2}},$$

$$Cor(U^2, \sqrt{1-U^2}) = \frac{\int_0^1 x^2\sqrt{1-x^2}dx - \int_0^1 x^2 dx \int_0^1 \sqrt{1-x^2}dx}{\sqrt{\int_0^1 x^4 dx - (\int_0^1 x^2 dx)^2}\sqrt{\int_0^1 (1-x^2)dx - (\int_0^1 \sqrt{1-x^2}dx)^2}}.$$

Problem 3

Use the inverse transform method to generate a random variable having the density function

$$\begin{aligned} (a) \quad f(x) &= e^x/(e-1), \quad 0 \le x \le 1, \\ (b) \quad f(x) &= \frac{2}{\pi\sqrt{1-x^2}}, \quad 0 \le x \le 1. \end{aligned}$$

Use the inverse transform method to generate a random variable having the cfd

$$(c) \quad F(x) = \frac{1}{1+e^{-x}}, \quad -\infty < x < \infty.$$

Solution: (a) Let X have the given pdf. Integration yields $F(x) = \frac{e^x-1}{e-1}, 0 \le x \le 1$. Now $F(X) \sim U$ where $U \sim U(0,1)$, and inverting yields $X = \ln((e-1)U + 1)$.
(b) Let X have the given pdf. Integration yields $F(x) = \frac{2}{\pi}\arcsin(x), 0 \le x \le 1$. Again $F(X) \sim U$ where $U \sim U(0,1)$, and inverting yields $X = \sin(\frac{\pi}{2}U)$
(c) Again $F(X) = U \sim U(0,1)$ which is equivalent to $X = -\ln(\frac{1}{U} - 1)$.

Problem 4

Use the acceptance-rejection method to generate a random variable with pdf

$$f(x) = 20x(1-x)^3, \quad 0 < x < 1.$$

Solution: The acceptance-rejection method works as follows: We want to generate some random variable X with pdf $f(x)$. Consider some random variable Y which can be easily generated with density $g(x)$ and whose density satisfies

$$\frac{f(x)}{g(x)} \le c$$

for some $c > 0$. Then

$$0 \le \frac{f(Y)}{cg(Y)} \le 1$$

and if $U \sim U(0,1)$ then the conditional pdf of Y given $\{U \le \frac{f(Y)}{cg(Y)}\}$ is just $f(x)$. Here we can set

$$g(x) = 1, \quad 0 < x < 1.$$

Then

$$\frac{f(x)}{g(x)} = 20x(1-x)^3.$$

The map $x \mapsto 20x(1-x)^3, 0 < x < 1$, has maximal value $\frac{135}{64} =: c$. Since

$$\frac{f(x)}{cg(x)} = \frac{256}{27}x(1-x)^3,$$

the acceptance-rejection method works as follows:
Step 1: Generate $Y = U_1 \sim U(0,1)$.
Step 2: Generate $U_2 \sim U(0,1)$ and check whether $U_2 \le \frac{f(Y)}{cg(Y)} = \frac{256}{27}U_1(1-U_1)^3$. If this is the case, return $X = Y$, if it is not the case restart at Step 1.