University of Karlsruhe
Institute of Stochastics

Prof. Dr. P. R. Parthasarathy
Dipl.-Math. D. Gentner

Stochastic Methods in Industry I (WS 07/08)

Solutions for Problem Set 4

Problem 1

Insurance claims are made at times distributed according to a Poisson process with rate λ. The successive amounts are iid with mean μ and are independent of the claim arrival times. Let S_i denote the time and C_i denote the amount of the i-th claim. The total discounted cost of all claims made up to time t equals

$$D(t) = \sum_{i=1}^{N(t)} e^{-\alpha S_i} C_i.$$

Here α is the discount rate and $N(t)$ the number of claims made up to time t. Find the mean of $D(t)$.

Solution Problem 1

By conditioning by $N(t)$, we get

$$\mathbb{E}[D(t)] = \sum_{n=0}^{\infty} \mathbb{E}[D(t)|N(t)=n] e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \tag{1}$$

By definition we have

$$\mathbb{E}[D(t)|N(t)=n] = \mathbb{E}[\sum_{i=1}^{N(t)} e^{-\alpha S_i} C_i | N(t)=n].$$

Note that the S_i are not independent to the random upper bound $N(t)$. Instead, as shown in class, the S_i follow an Erlang distribution with parameters λ and i respectively, while the $S_1|N(t), S_2|N(t), ..., S_{N(t)}|N(t)$ are distributed respectively as the ordered values $U_{(1)} < U_{(2)} < ... < U_{(N(t))}$ of $N(t)$ independent and on $(0,t)$ uniformly distributed random variables $U_1, U_2, ..., U_{N(t)}$. Hence

$$\begin{aligned}
\mathbb{E}[D(t)|N(t)=n] &= \mathbb{E}[\sum_{i=1}^{n} e^{-\alpha S_i} C_i | N(t)=n] = \sum_{i=1}^{n} \mathbb{E}[e^{-\alpha U_{(i)}} C_i] \\
&= \sum_{i=1}^{n} \mathbb{E}[e^{-\alpha U_{(i)}}]\mathbb{E}[C_i] = \mu \sum_{i=1}^{n} \mathbb{E}[e^{-\alpha U_{(i)}}] \\
&= \mu \mathbb{E}[\sum_{i=1}^{n} e^{-\alpha U_{(i)}}] = \mu \mathbb{E}[\sum_{i=1}^{n} e^{-\alpha U_i}] \\
&= n\mu \mathbb{E}[e^{-\alpha U_1}] = n\mu \frac{1}{t} \int_0^t e^{-\alpha x} dx \\
&= \frac{n\mu}{\alpha t}(1 - e^{-\alpha t})
\end{aligned}$$

Now insert this into (1) to derive

$$\mathbb{E}[D(t)] = \frac{\lambda\mu}{\alpha}(1 - e^{-\alpha t}).$$

Problem 2

Mr. Müller runs a hot dog stall that opens at 8 AM. Arrivals occur according to a non-homogeneous Poisson process with time-dependent rate $\lambda(t)$, where

$$\lambda(t) = \begin{cases} 5 + 5t, & 0 \leq t \leq 3, \\ 20, & 3 \leq t \leq 5, \\ 20 - 2(t - 5), & 5 \leq t \leq 9, \end{cases}$$

taking 8 AM as origin. Find the probability that no customer arrives between 8:30 AM and 8:40 AM and the expected number of arrivals in this period.

Solution Problem 2

Let X denote the number of arrivals between 8:30 and 8:40. According to Sheet 3, Problem 2, we have that X is Poisson distributed with parameter $\int_{1/2}^{2/3} 5 + 5t dt = \frac{95}{72}$. Hence

$$\mathbb{P}(X = 0) = e^{-\frac{95}{72}}$$

and

$$\mathbb{E}(X) = \frac{95}{72}.$$

Problem 3

Consider a Poisson process $\{N(t)\}$ with parameter λ, where λ follows a gamma density with mean m/α and variance m/α^2. Find $P(N(t) = n)$, mean and variance of $N(t)$.

Solution Problem 3

For a better understanding of the process described in Problem 3, think in two steps: First someone 'rolls' the parameter λ according to a $\Gamma(m, \alpha)$-distribution, and takes in a second step this λ as the (with respect to time constant) parameter of a homogeneous Poisson process. (If you interpreted the process to be a non-homogeneous Poisson process with time-dependent parameter $\lambda(t)$, where $\lambda(t)$ is given by the Gamma-density function, you will get full points, since the formulation was misleading indeed. Sorry!)
The density of the random variable λ is given by

$$f_\lambda(x) = \frac{\alpha}{\Gamma(m)}(\alpha x)^{m-1}e^{-\alpha x}\mathbb{1}_{[0,\infty)}(x).$$

Hence we get by conditioning with λ that

$$\begin{aligned}
\mathbb{P}(N(t)=n) &= \mathbb{E}\left[\mathbb{P}(N(t)=n|\lambda)\right] = \mathbb{E}\left[e^{-\lambda t}\frac{(\lambda t)^n}{n!}\right] \\
&= \int_0^\infty e^{-xt}\frac{(xt)^n}{n!}\frac{\alpha}{\Gamma(m)}(\alpha x)^{m-1}e^{-\alpha x}dx \\
&= \frac{t^n\alpha^n}{n!\Gamma(m)(\alpha+t)^{n+m}}\int_0^\infty y^{n+m-1}e^{-y}dy \\
&= \left(\frac{t}{\alpha+t}\right)^n\left(\frac{\alpha}{\alpha+t}\right)^m\frac{(m+n-1)...(m+1)m}{n!}.
\end{aligned}$$

Further we have for the expectation

$$\mathbb{E}[N(t)] = \mathbb{E}[\mathbb{E}[N(t)|\lambda]] = \mathbb{E}[\lambda t] = t\mathbb{E}[\lambda] = \frac{tm}{\alpha},$$

and for the variance

$$\begin{aligned}
Var(N(t)) &= \mathbb{E}[N(t)^2] - \mathbb{E}[N(t)]^2 = \mathbb{E}[\mathbb{E}[N(t)^2|\lambda]] - (\frac{tm}{\alpha})^2 \\
&= \mathbb{E}[\lambda t + (\lambda t)^2] - (\frac{tm}{\alpha})^2 = t\frac{m}{\alpha} + t^2(\frac{m}{\alpha^2} + \frac{m^2}{\alpha^2}) - (\frac{tm}{\alpha})^2 \\
&= t\frac{m}{\alpha} + t^2\frac{m}{\alpha^2}.
\end{aligned}$$

Problem 4

For a birth and death process, find the average time it takes to enter state $n+1$, starting from state n.

Solution Problem 4

Define

- $T_{n,n+1} :=$ time a particle starting at n needs to reach $n+1$,
- $\tau_n :=$ waiting time for a particle at state n until it moves (up or down),
- $\kappa_n := \begin{cases} 1 & ; \text{ particle at state } n \text{ moves up} \\ -1 & ; \text{ particle at state } n \text{ moves down} \end{cases}$.

From class, we know that

$$\begin{aligned}
\tau_n &\sim Exp(\lambda_n + \mu_n), \\
\tau_n|\kappa_n = 1 &\sim Exp(\lambda_n), \\
\tau_n|\kappa_n = -1 &\sim Exp(\mu_n), \\
\mathbb{P}(\kappa_n = 1) &= \frac{\lambda_n}{\lambda_n + \mu_n}.
\end{aligned}$$

Using this, we get

$$\begin{aligned} \mathbb{E}[T_{n,n+1}] &= \mathbb{E}[T_{n,n+1}|\kappa_n = 1]\mathbb{P}(\kappa_n = 1) + \mathbb{E}[T_{n,n+1}|\kappa_n = -1]\mathbb{P}(\kappa_n = -1) \\ &= \mathbb{E}[\tau_n|\kappa_n = 1]\frac{\lambda_n}{\lambda_n + \mu_n} + \mathbb{E}[T_{n-1,n+1}]\frac{\mu_n}{\lambda_n + \mu_n} \\ &= \frac{1}{\lambda_n} \cdot \frac{\lambda_n}{\lambda_n + \mu_n} + \mathbb{E}[T_{n-1,n} + T_{n,n+1}]\frac{\mu_n}{\lambda_n + \mu_n}. \end{aligned}$$

Simplifying yields

$$\mathbb{E}[T_{n,n+1}] = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n}\mathbb{E}[T_{n-1,n}],$$

and iterating this (or using induction), we get

$$\mathbb{E}[T_{n,n+1}] = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n}\frac{1}{\lambda_{n-1}} + ... + \frac{\mu_n ... \mu_1}{\lambda_n ... \lambda_1}\frac{1}{\lambda_0}.$$

Due date Friday, November 23th 2007, 14:00 o'clock. This week, since the session next Friday needs to be canceled, the solutions must be turned in by 14:00 o'clock in front of the office of the Institute of Stochastics, room number 231, where a box is set up labeled "Übungsblätter: Stochastic Methods in Industry". Please put your **name** and **student id number** on each sheet you turn in and staple the sheets.