Karlsruhe Institute of Technology

Institute for Stochastics

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Stochastic Processes
Problem sheet 3

Problem 1
In Example 1.7 we studied a ruin game between two players. Use theorem 2.21 to calculate the probability that player 1 gets ruined if his initial capital is $i \in \{0, \dots, 2B\}$ (i.e. player 2's initial capital is $2B - i$) given that $p \neq \frac{1}{2}$.

Problem 2
Consider a deck of N cards. We model the card shuffling by a homogenous Markov chain $(X_n)_{n\in\mathbb{N}}$ with values in the symmetric group of permutations $S_N := \{\varpi : \{1, \dots, N\} \to \{1, \dots, N\}\}$.
The shuffling procedure is called admissible if for all $\varpi_1, \varpi_2 \in S_N$ there exists $n \in \mathbb{N}$ with $P(X_n = \varpi_1 | X_0 = \varpi_2) > 0$.
The shuffling procedure is called symmetric if the transition matrix $(p_{\varpi\tau})_{\varpi,\tau}$ satisfies the condition $p_{\varpi\tau} = p_{\tau\varpi}$ for all $\varpi, \tau \in S_N$.

a) Show that for any admissible, symmetric shuffling procedure there is a stationary distribution and determine it.

b) Show that the following shuffling procedures are admissible and symmetric:

  (i) In one step we randomly choose two positions i and j with probability α_i and α_j, respectively. Then we interchange the cards in position i and j. Note that $i = j$ is possible. The probabilities $\alpha_1, \dots, \alpha_N$ are strictly positive.

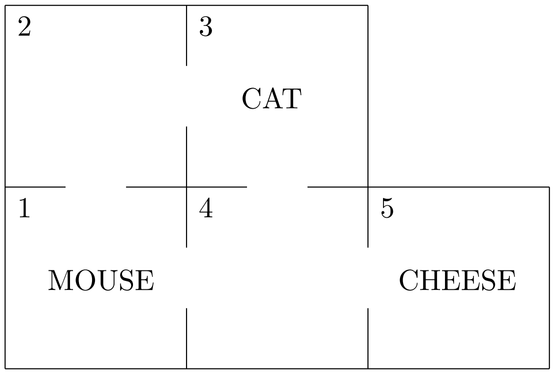
  (ii) In one step we randomly choose one position between 1 and N with probability $\frac{1}{N}$ (uniform distribution) and take the card from this position. Then we randomly choose a value j uniformly from $\{0, 1, \dots, N-1\}$ and put the card we took earlier under the j-th card. For $j = 0$ we put the card on top of the deck.

Problem 3 (The Urn of Ehrenfest)
There are N particles that can be either in compartment A or in compartment B. Suppose that at time $n \geq 0$, $X_n = i$ particles are in A. One then chooses a particle at random, and this particle is moved at time $n + 1$ from where it is to the other compartment.

a) Determine the transition matrix of the Markov chain (X_n).

b) Determine a stationary distribution.

Problem 4 (Cat Eats Mouse Eats Cheese)
A merry mouse moves in a maze. If it is at time n in a room with k adjacent rooms, it will be at time $n+1$ in one of the k adjacent rooms, choosing one at random, each with probability $\frac{1}{k}$. A fat lazy cat remains all the time in a given room, and a piece of cheese waits for the mouse in another room (see figure below). The cat is not completely lazy: If the mouse enters the room inhabited by the cat, the cat will eat it. What is the probability that the mouse ever gets to eat the cheese when starting from room 1, the cat and the cheese being in rooms 3 and 5, respectively?



This sheet will be discussed in the problem sessions on the 5th of May.