Stochastic Processes
Problem sheet 4

Problem 1
Let the transition matrix P of an irreducible Markov chain $(X_n)_{n\in\mathbb{N}}$ be doubly stochastic, i.e. $\sum_{i\in S} p_{ij} = 1$ for all $j \in S$. Show:

(a) If S is finite, $(X_n)_{n\in\mathbb{N}}$ is positive recurrent. In this case, determine the stationary distribution.

(b) If S is countably infinite, $(X_n)_{n\in\mathbb{N}}$ is either transient or null recurrent.

Problem 2
Let $X = (X_n)_{n\in\mathcal{N}_0}$ be an irreducible Markov chain with state space S and $i_0 \in S$. Show that X is transient if and only if there is a bounded function $h : S \to \mathbb{R}$, $h \neq 0$ satisfying

$$h(i) = \sum_{j\neq i_0} p_{ij} h(j), \quad i \neq i_0.$$

Hint: You can use without proof: If (X_n) is an irreducible recurrent Markov chain with state space S and $h : S \to \mathbb{R}$ a bounded subharmonic function, then h is constant.

Problem 3
Consider a Birth and Death process in discrete time, i.e. the irreducible Markov chain $X = (X_n)_{n\in\mathcal{N}_0}$ on $S = \mathbb{N}_0$ with transition probabilities

$$\begin{aligned} &p_{00} + p_{01} = 1,\ p_{01} > 0,\\ &p_{i,i-1} + p_{ii} + p_{i,i+1} = 1,\ p_{i,i-1}, p_{i,i+1} > 0, \quad i \in \mathbb{N}. \end{aligned}$$

Use problem 2 to show that X is transient if and only if

$$\sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{p_{k,k-1}}{p_{k,k+1}} < \infty \tag{1}$$

holds. How can you simplify this result given $p_{01} = p$ and $p_{i,i-1} = 1 - p = 1 - p_{i,i+1}$, $i \in \mathbb{N}$, for some $p \in (0, 1)$?

Problem 4

Consider a web server which hosts N websites on a hard disk. In addition there is a cache which can save M frequently visited sites ($M < N$). Assume that the users visit a website independently according to a *Zipf* distribution, i.e.:

$$\alpha_i := P(\text{next user visits site } i) = \frac{\frac{1}{i}}{\sum_{k=1}^{N} \frac{1}{k}}, \quad i = 1, \ldots, N$$

In the case that the cache is filled and a user visits a site which is not in the cache, a site from the cache gets deleted according to a uniform distribution (*random policy*).

For sake of simplicity consider the case $N = 4$, $M = 2$.

a) Model a Markov chain which describes the state of the cache and determine the transition matrix.

b) Determine the stationary distribution and use this result to calculate the expected hit ratio of the cache. Generalize the stationary distribution to arbitrary N and M.

This sheet will be discussed in the problem sessions on the 12th of May.